Solutions to Exercise Set 4

Solution 4.1 Since ρ commutes with any power of ρ, we have

$$\mathrm{D}_\alpha(\rho\|\rho) = \frac{1}{\alpha-1}\log\mathrm{tr}\Big(\big(\rho^{\frac{1-\alpha}{2\alpha}}\rho\,\rho^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\Big) = \frac{1}{\alpha-1}\log\mathrm{tr}(\rho) = 0\,.$$

With the help of the previous exercise set, we see that

$$\begin{aligned}\mathrm{D}_\alpha(\rho\otimes\rho',\sigma\otimes\sigma') &= \frac{1}{\alpha-1}\log\mathrm{tr}\Big[\Big((\sigma\otimes\sigma')^{\frac{1-\alpha}{2\alpha}}(\rho\otimes\rho')(\sigma\otimes\sigma')^{\frac{1-\alpha}{2\alpha}}\Big)^\alpha\Big]\\ &= \frac{1}{\alpha-1}\log\mathrm{tr}\Big[\Big(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\,\sigma^{\frac{1-\alpha}{2\alpha}}\Big)^\alpha\otimes\Big(\sigma'^{\frac{1-\alpha}{2\alpha}}\rho'\,\sigma'^{\frac{1-\alpha}{2\alpha}}\Big)^\alpha\Big]\\ &= \frac{1}{\alpha-1}\log\mathrm{tr}\Big[\Big(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\,\sigma^{\frac{1-\alpha}{2\alpha}}\Big)^\alpha\Big] + \frac{1}{\alpha-1}\log\mathrm{tr}\Big[\Big(\sigma'^{\frac{1-\alpha}{2\alpha}}\rho'\,\sigma'^{\frac{1-\alpha}{2\alpha}}\Big)^\alpha\Big]\\ &= \mathrm{D}_\alpha(\rho,\sigma)+\mathrm{D}_\alpha(\rho',\sigma')\,.\end{aligned}$$

Therefore, for a product state $\rho_{AE}=\rho_A\otimes\rho_E$, we have

$$\mathrm{D}_\alpha(\rho_{AE}\,\|\,\mathbb{I}_A\otimes\sigma_E) = \mathrm{D}_\alpha(\rho_A\,\|\,\mathbb{I}_A)+\mathrm{D}_\alpha(\rho_E\,\|\,\sigma_E) \geq -\mathrm{H}_\alpha(A)$$

with equality for $\sigma_E=\rho_E$. Therefore, $\mathrm{H}_\alpha(A|E) = -\min_{\sigma_E}\mathrm{D}_\alpha(\rho_{AE}\,\|\,\mathbb{I}_A\otimes\sigma_E) = \mathrm{H}_\alpha(A)$.

Solution 4.2 By definition, we have

$$F(\rho,\sigma) = \big\|\sqrt{\rho}\sqrt{\sigma}\big\|_{tr} = \mathrm{tr}\big|\sqrt{\rho}\sqrt{\sigma}\big| = \mathrm{tr}\sqrt{\sqrt{\sigma}\rho\,\sqrt{\sigma}}\,.$$

Furthermore, applying Lemma 0.4 with $A=\sqrt{\rho}\sqrt{\sigma}$ ensures that $\sqrt{\sigma}\sqrt{\rho}\,\sqrt{\rho}\sqrt{\sigma}$ and $\sqrt{\rho}\,\sqrt{\sigma}\sqrt{\sigma}\sqrt{\rho}$ have the same spectrum, and thus so have their respective square roots. Thus,

$$\mathrm{tr}\sqrt{\sqrt{\sigma}\rho\,\sqrt{\sigma}} = \mathrm{tr}\sqrt{\sqrt{\sigma}\sqrt{\rho}\,\sqrt{\rho}\sqrt{\sigma}} = \mathrm{tr}\sqrt{\sqrt{\rho}\,\sqrt{\sigma}\sqrt{\sigma}\sqrt{\rho}} = \mathrm{tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}\,,$$

which proves that the fidelity is symmetric.

If $\sigma=|\psi\rangle\langle\psi|$ is pure then $\sqrt{\sigma}=|\psi\rangle\langle\psi|$, and thus

$$F(\rho,\sigma) = \mathrm{tr}\sqrt{\sqrt{\sigma}\rho\,\sqrt{\sigma}} = \mathrm{tr}\sqrt{|\psi\rangle\langle\psi|\rho|\psi\rangle\langle\psi|} = \sqrt{\langle\psi|\rho|\psi\rangle}\;\mathrm{tr}\sqrt{|\psi\rangle\langle\psi|} = \sqrt{\langle\psi|\rho|\psi\rangle}\,.$$

Plugging in $\alpha=1/2$ into the definition of D_α we obtain

$$\mathrm{D}_{1/2}(\rho\|\sigma) = -2\log\mathrm{tr}\sqrt{\sqrt{\sigma}\rho\,\sqrt{\sigma}} = -2\log F(\rho,\sigma)\,.$$

Solution 4.3 By definition, and using that the scalar λ commutes with the operators ρ and σ,

$$\begin{aligned}\mathrm{D}_\alpha(\rho\|\lambda\sigma) &= \frac{1}{\alpha-1}\log\mathrm{tr}\Big(\big((\lambda\sigma)^{\frac{1-\alpha}{2\alpha}}\rho\,(\lambda\sigma)^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\Big) = \frac{1}{\alpha-1}\log\Big(\lambda^{1-\alpha}\,\mathrm{tr}\Big(\big(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\,\sigma^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\Big)\Big)\\ &= \frac{1}{\alpha-1}\log\mathrm{tr}\Big(\big(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\,\sigma^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\Big) - \log\lambda = \mathrm{D}_\alpha(\rho\|\sigma)-\log\lambda\,.\end{aligned}$$

In particular,

$$\mathrm{D}_\alpha(\rho_{AE}\,\|\,\mathbb{I}_A\otimes\sigma_E) = \mathrm{D}_\alpha(\rho_{AE}\,\|\,N\mu_A\otimes\sigma_E) = \mathrm{D}_\alpha(\rho_{AE}\,\|\,\mu_A\otimes\sigma_E) - n$$

and thus

$$\mathrm{H}_\alpha(A|E) = -\min_{\sigma_E} \mathrm{D}_\alpha(\rho_{AE} \,\|\, \mathbb{I}_A \otimes \sigma_E) = n - \min_{\sigma_E} \mathrm{D}_\alpha(\rho_{AE} \,\|\, \mu_A \otimes \sigma_E)\,.$$

Solution 4.4 For $x', y' \in C'$ and $0 \le p \le 1$, using linearily of L and convexity of C,

$$L\big(px' + (1-p)y'\big) = pL(x') + (1-p)L(y') \in C\;.$$

Thus, $px' + (1-p)y' \in C'$ as well by definition of C'. Furthermore, by linearity of L

$$f'\big(px' + (1-p)y'\big) = f\Big(L\big(px' + (1-p)y'\big)\Big) = f\big(pL(x') + (1-p)L(y')\big)$$

which is bounded as

$$\le pf\big(L(x')\big) + (1-p)f\big(L(y')\big) = pf'(x') + (1-p)f'(y')\,.$$

by convexity of f.

Solution 4.5 For 1., we see that

$$g\Big(\sum_i \varepsilon_i x_i\Big) = \max_y f\Big(\sum_i \varepsilon_i x_i, y\Big) \le \max_y \sum_i \varepsilon_i f(x_i, y) \le \sum_i \varepsilon_i \max_{y_i} f(x_i, y_i) = \sum_i \varepsilon_i g(x_i)\,.$$

And, for 2., letting y_i be so that $g(x_i) = f(x_i, y_i)$, we have

$$g\Big(\sum_i \varepsilon_i x_i\Big) = \max_y f\Big(\sum_i \varepsilon_i x_i, y\Big) \ge f\Big(\sum_i \varepsilon_i x_i, \sum_i \varepsilon_i y_i\Big) \ge \sum_i \varepsilon_i f(x_i, y_i) = \sum_i \varepsilon_i g(x_i)\,.$$