

Linear systems, coalgebraically

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CALCO 2007

Motivation

Why linear systems, coalgebraically?

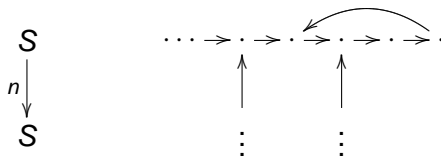
- Very simple **state-based** systems
- Basic model of **memory** and **feedback**
- Uses **vector spaces**, not just sets
- Central role for the notion of **rationality**
- Interplay algebra-coalgebra
- Fundamental role (again) for **streams**
- **Teaching!!!**

Motivation

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Coalgebras of $id : Set \rightarrow Set$



- *Finite* S : not very interesting
- All (S, n) are behaviourally trivial:

$$\begin{array}{ccc}
 S & \xrightarrow{!} & 1 \\
 n \downarrow & & \parallel \\
 S & \xrightarrow{!} & 1
 \end{array}$$

- (But morphisms and bisimulations are *not* trivial.)
 - Cf. Lawvere and Schanuel: *Conceptual Mathematics*.

Adding observations: $A \times - : \mathbf{Set} \rightarrow \mathbf{Set}$

$$\begin{array}{ccc}
 S & \xrightarrow{\exists! h} & A^\omega \\
 \downarrow \langle 0, n \rangle & & \downarrow \cong \\
 A \times S & \longrightarrow & A \times A^\omega
 \end{array}$$

Special case:

$$\begin{array}{ccc}
 S & \xrightarrow{\exists! \tilde{n}} & S^\omega \\
 \downarrow \langle 1, n \rangle & & \downarrow \cong \\
 S \times S & \longrightarrow & S \times S^\omega
 \end{array}$$

Note: $\tilde{n}(s) = (1, n, n^2, \dots)(s) = (s, n(s), n^2(s), \dots)$

Overview

Today we will . . .

- analyse the situation

$$\begin{array}{ccc}
 V & \xrightarrow{\exists! \tilde{F}} & V^\omega \\
 \downarrow \langle 1, F \rangle & & \downarrow \cong \\
 V \times V & \longrightarrow & V \times V^\omega
 \end{array}$$

for the category *Vect* of *vector spaces* and *linear maps*

- using, *first*, plain stream calculus and stream circuits, as in
 - J.J.M.M. Rutten. A tutorial on coinductive stream calculus and signal flow graphs. TCS 343, 2005
- and, *next*, stream calculus of linear maps, as in
 - the present CALCO 2007 paper

Coalgebras of $id : Vect \rightarrow Vect$

- *Vect*: vector spaces (here: over \mathbb{R}) and linear maps.
- Today we concentrate on *finite-dimensional* spaces.
- Our running example is the coalgebra

$$\mathbb{R}^2 \xrightarrow{G} \mathbb{R}^2$$

given, for all $(r_1, r_2) \in \mathbb{R}^2$, by

$$G(r_1, r_2) = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = (-r_2, r_1 + 2r_2)$$

- We'll represent such linear maps by *stream circuits*

Stream circuits

- Simple representations of linear maps
- Four basic types of gates:

r -multiplier: $x \mapsto r \times x$

register: $x \mapsto \boxed{r} \rightarrow \text{first } r \text{ then } x$

$x \mapsto$
 $y \mapsto$

$+$ $\rightarrow x+y$

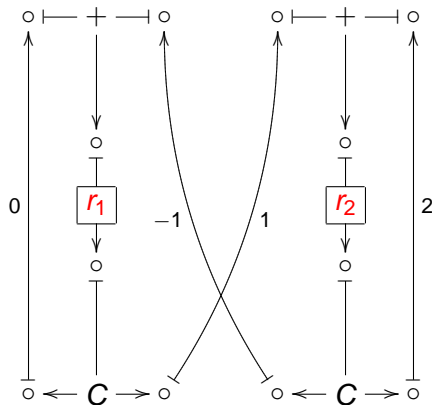
$x \mapsto$

C $\rightarrow x$
 $\rightarrow x$

adder

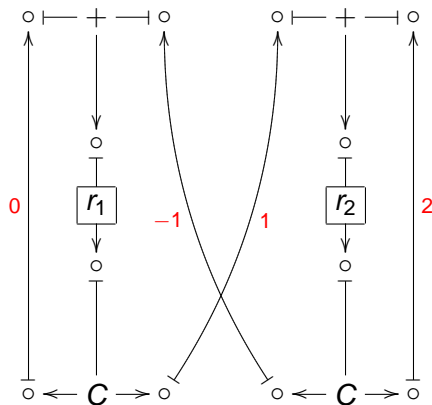
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$$G(r_1, r_2) = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$



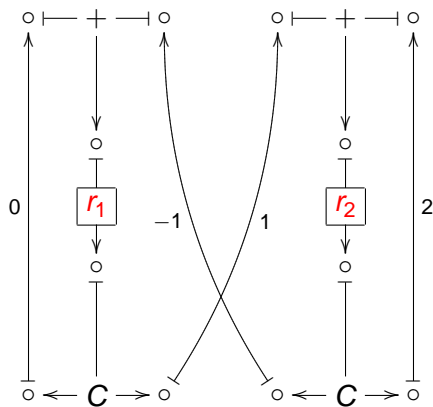
Present state = contents of the registers

$$G(r_1, r_2) = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

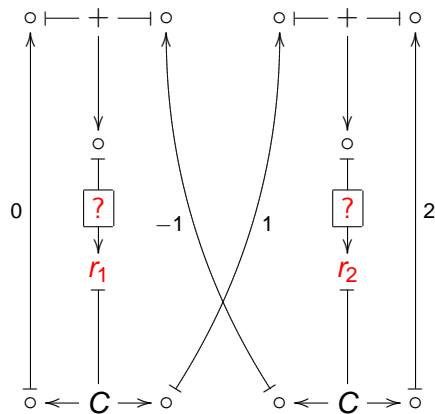


Matrix entries determine values of multipliers (feedback lines)

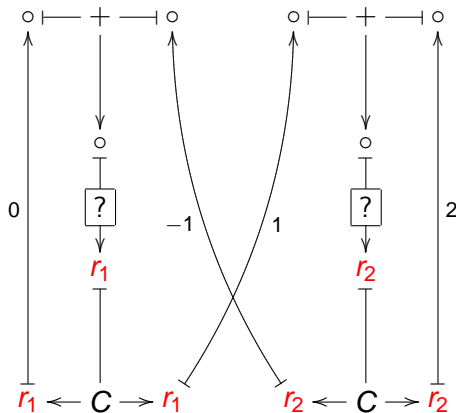
Computing $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$



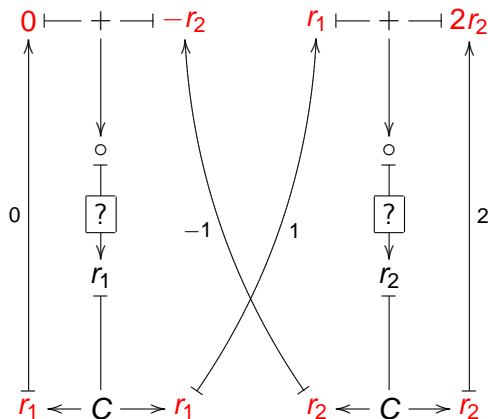
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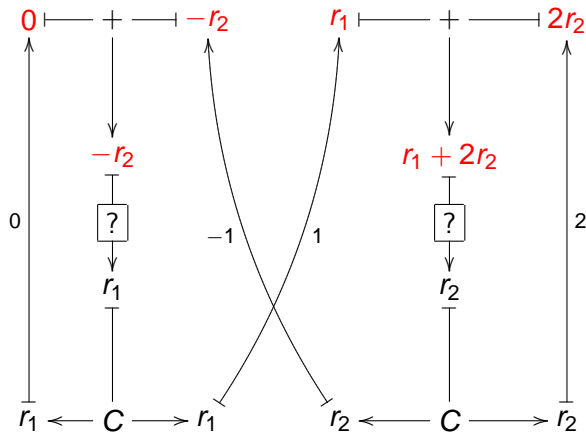
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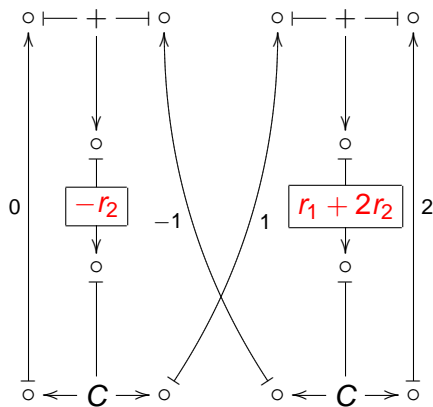
Computing $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$



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$$\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = (-r_2, r_1 + 2r_2)$$

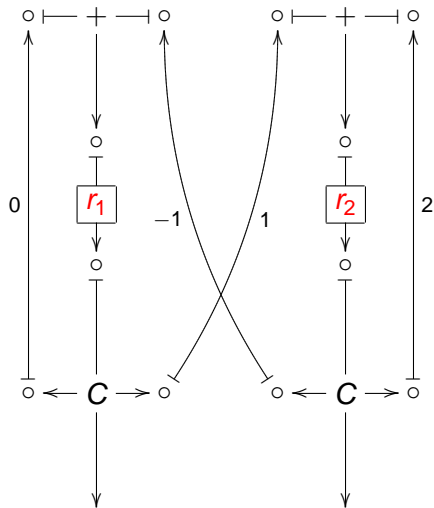


Observing states

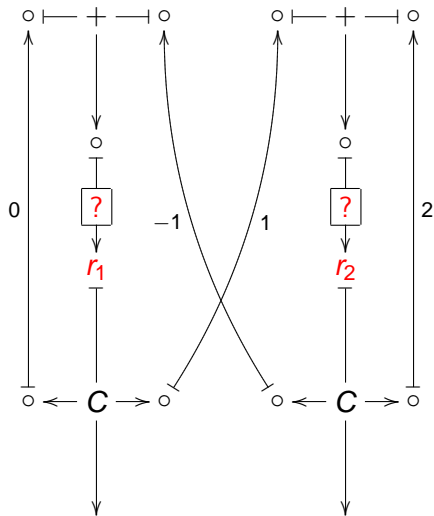
Remember that we want to analyse

$$\begin{array}{c} \mathbb{R}^2 \\ \downarrow \langle 1, G \rangle \\ \mathbb{R}^2 \times \mathbb{R}^2 \end{array}$$

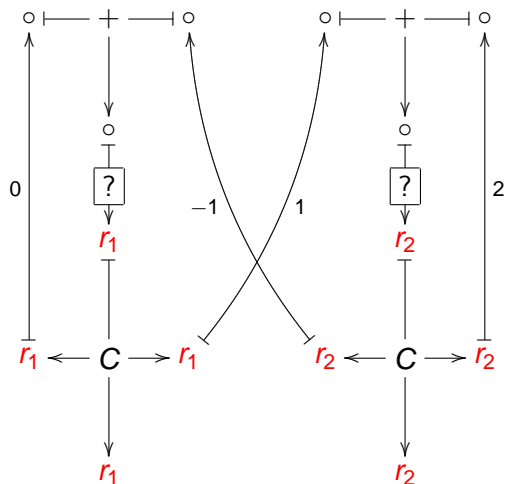
Adding two output ports



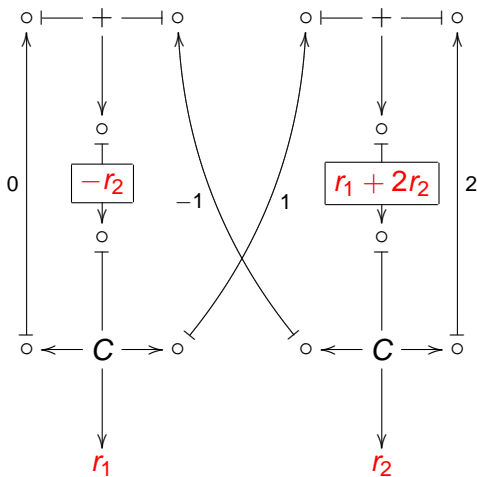
Computing next state: as before



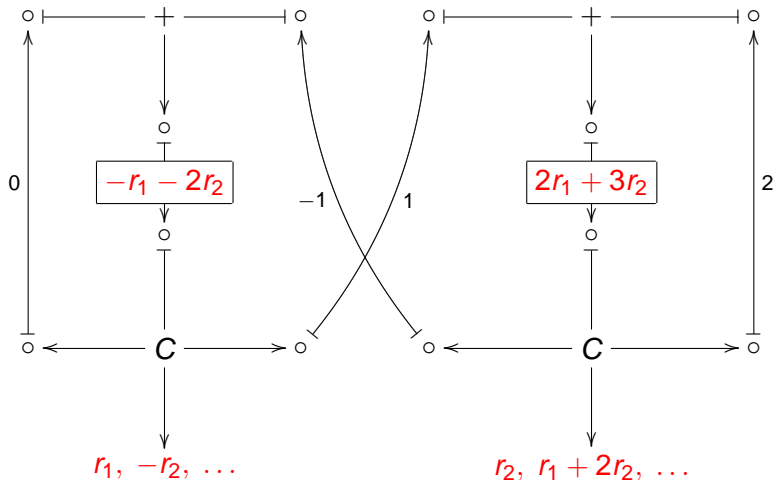
Present state is output



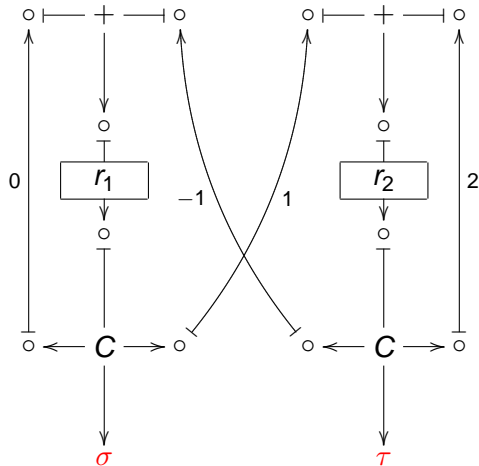
And new state as before



And so on ...



Our next goal: compute streams σ and τ



Stream calculus (on \mathbb{R}^ω)

We use the following constants and operators:

- constants (one for each $r \in \mathbb{R}$): $[r] = (r, 0, 0, 0, \dots)$
- constant (cf. formal variable): $X = (0, 1, 0, 0, 0, \dots)$
- sum: $(\sigma + \tau)(n) = \sigma(n) + \tau(n)$
- convolution (aka Cauchy) product:

$$(\sigma \times \tau)(n) = \sigma(0) \cdot \tau(n) + \dots + \sigma(n) \cdot \tau(0)$$

- (formal) inverse to product: if $\sigma(0) \neq 0$ then $\exists! \frac{1}{\sigma}$ s.t.

$$\sigma \times \frac{1}{\sigma} = (1, 0, 0, 0, \dots)$$

Stream calculus (on \mathbb{R}^ω)

- Note:

$$X \times \sigma = (0, \sigma(0), \sigma(1), \sigma(2), \dots)$$

- Convention:

$$3 \times X^2 = [3] \times X \times X \quad (= (0, 0, 3, 0, 0, 0, \dots))$$

- Polynomial streams: for instance,

$$2 + 3X - 7X^4 \quad (= (2, 3, 0, 0, -7, 0, 0, 0, \dots))$$

- Rational streams: for instance,

$$\frac{1 + X}{1 - 2X + X^2} \quad (= (1, 3, 5, 7, \dots))$$

Semantics of stream circuits

$$r\text{-multiplier: } \sigma \xrightarrow{r} r \times \sigma$$

$$\text{register: } \sigma \xrightarrow{\boxed{r}} r + (X \times \sigma) \quad (= (r, \sigma(0), \sigma(1), \dots))$$

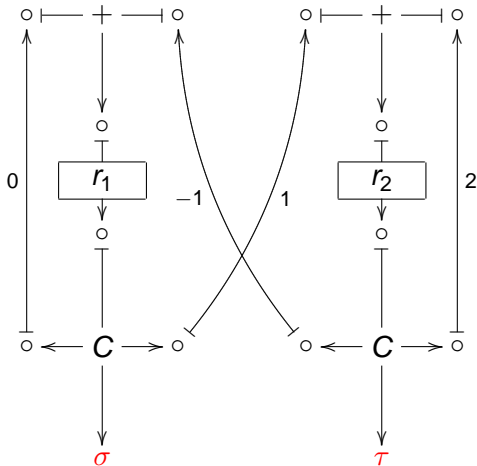
$$\begin{array}{c} \sigma \\ \downarrow \\ \tau \end{array} \xrightarrow{+} \sigma + \tau$$

adder

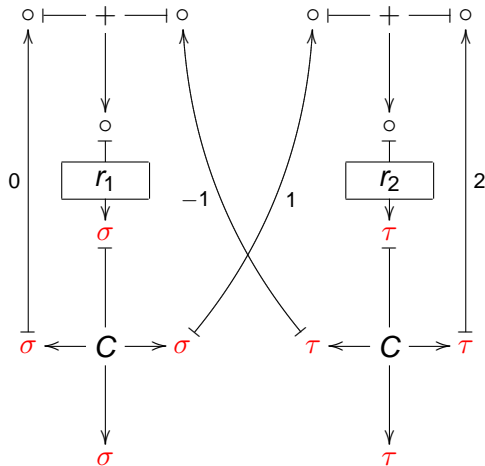
$$\sigma \xrightarrow{C} \begin{array}{c} \sigma \\ \sigma \end{array}$$

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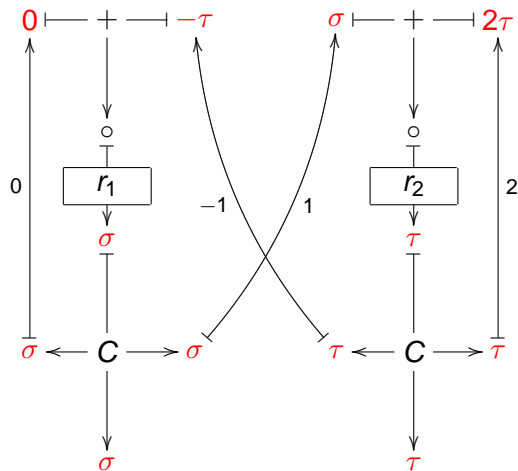
Computing streams σ and τ



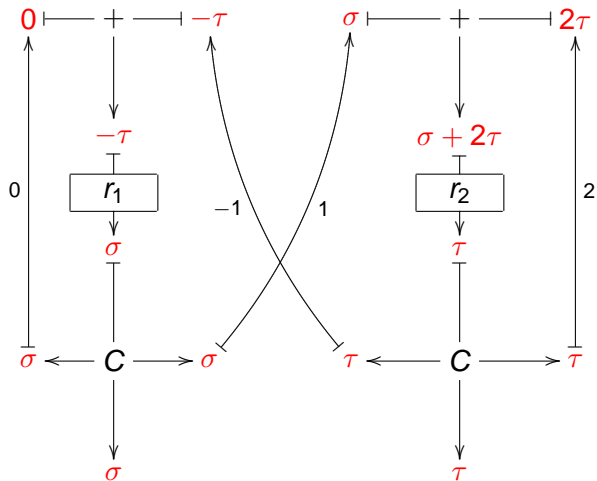
Computing streams σ and τ



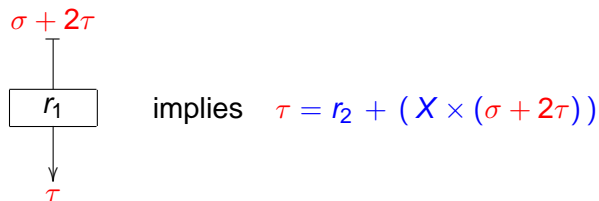
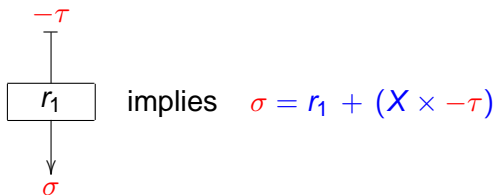
Computing streams σ and τ



Computing streams σ and τ



Using $\rho \vdash \boxed{r} \longrightarrow r + (X \times \rho)$



Computing streams σ and τ

Two equations:

$$\sigma = r_1 + (X \times -\tau)$$

$$\tau = r_2 + (X \times (\sigma + 2\tau))$$

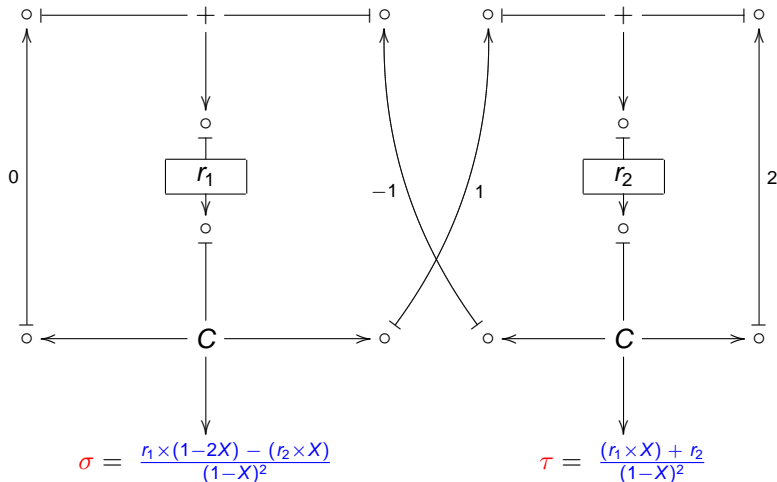
with two unknowns: σ, τ

solved in stream calculus (just compute as usual):

$$\sigma = \frac{r_1 \times (1 - 2X) - (r_2 \times X)}{(1 - X)^2} \quad \tau = \frac{(r_1 \times X) + r_2}{(1 - X)^2}$$

Note: σ and τ are **rational** streams!

Our present goal completed:



Summary sofar:

We have represented the coalgebra

$$\begin{array}{c}
 \mathbb{R}^2 \\
 \downarrow \langle 1, G \rangle \\
 \mathbb{R}^2 \times \mathbb{R}^2
 \end{array}
 \quad \text{with} \quad
 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
 \quad G = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$$

as a stream circuit which computes, for any $\langle r_1, r_2 \rangle \in \mathbb{R}^2$, two *rational* streams $\sigma, \tau \in \mathbb{R}^\omega$:

$$\sigma = \frac{r_1 \times (1 - 2X) - (r_2 \times X)}{(1 - X)^2}
 \quad \tau = \frac{(r_1 \times X) + r_2}{(1 - X)^2}$$

Final semantics

- Stream circuits: computationally intuitive model of linear systems
- Present CALCO paper: same result purely coalgebraically, via *final semantics*:

$$\begin{array}{ccc}
 \mathbb{R}^2 & \xrightarrow{\exists! \tilde{G}} & (\mathbb{R}^2)^\omega \\
 \downarrow \langle 1, G \rangle & & \downarrow \cong \\
 \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \times (\mathbb{R}^2)^\omega
 \end{array}$$

Final semantics \tilde{G}

- First note that

$$\begin{array}{ccccc}
 \mathbb{R}^2 & \xrightarrow{\exists! \tilde{G}} & (\mathbb{R}^2)^\omega & \xleftarrow[\cong]{(-)^T} & (\mathbb{R}^\omega)^2 \\
 \downarrow \langle 1, G \rangle & & \downarrow & & \\
 \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \times (\mathbb{R}^2)^\omega & &
 \end{array}$$

where $(-)^T$ is the *transpose* function.

- So $(\tilde{G}(r_1, r_2))^T$ is a pair of streams, as expected.

Final semantics \tilde{G}

$$\begin{array}{ccc}
 \mathbb{R}^2 & \xrightarrow{\exists! \tilde{G}} & (\mathbb{R}^2)^\omega \\
 \downarrow \langle 1, G \rangle & & \downarrow \\
 \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \times (\mathbb{R}^2)^\omega
 \end{array}$$

Note that

$$\tilde{G}(r_1, r_2) = ((r_1, r_2), G(r_1, r_2), G^2(r_1, r_2), \dots) \in (\mathbb{R}^2)^\omega$$

In other words:

$$\tilde{G} = (1, G, G^2, \dots) \quad \text{is itself a *stream* !!!}$$

$(\mathbb{R}^2 \rightarrow_L \mathbb{R}^2)$ is a ring

- We have a *ring* $(\mathbb{R}^2 \rightarrow_L \mathbb{R}^2, +, \circ, 0, 1)$ with
 - $F + H$: elementwise sum
 - $F \circ H$: function composition or, equivalently
matrix multiplication: $M_{F \circ H} = M_F \times M_H$
- Hence we can do *stream calculus* on $(\mathbb{R}^2 \rightarrow_L \mathbb{R}^2)^\omega$
and compute a nice closed formula for

$$\begin{aligned} \tilde{G} &= (1, G, G^2, G^3, \dots) \\ &= (1, G, G \circ G, G \circ G \circ G, \dots) \in (\mathbb{R}^2 \rightarrow_L \mathbb{R}^2)^\omega \end{aligned}$$

Stream calculus on $(\mathbb{R}^2 \rightarrow_L \mathbb{R}^2)^\omega$

- elements $\sigma = (F_0, F_1, F_2, \dots)$
- constants: $[F] = (F, 0, 0, 0, \dots)$
- constant (cf. formal variable): $X = (0, 1, 0, 0, 0, \dots)$
- sum: $(\sigma + \tau)(n) = \sigma(n) + \tau(n)$
- convolution product:
 $(\sigma \times \tau)(n) = \sigma(0) \circ \tau(n) + \dots + \sigma(n) \circ \tau(0)$
- inverse: if $\sigma(0)$ is *invertible* then $\exists!$ σ^{-1} s.t.

$$\sigma \times \sigma^{-1} = (1, 0, 0, 0, \dots)$$

Computing $\tilde{G} = (1, G, G^2, \dots)$

- Recall from stream calculus on \mathbb{R}^ω :

$$(1 - (c \times X))^{-1} = \frac{1}{1 - (c \times X)} = (1, c, c^2, \dots)$$

- Similarly, on $(\mathbb{R}^2 \rightarrow_L \mathbb{R}^2)^\omega$:

$$(1 - (G \times X))^{-1} = (1, G, G^2, \dots)$$

- Let's *do* this for our $G = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

Please sit back, relax, and have confidence.

Computing \tilde{G} for $G = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

$$\begin{aligned}
 \tilde{G} &= (1, G, G^2, \dots) \\
 &= (1 - (G \times X))^{-1} \\
 &= (1 - \left(\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \times X \right))^{-1} \\
 &= (1 - \begin{pmatrix} 0 & -X \\ X & 2X \end{pmatrix})^{-1} \\
 &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -X \\ X & 2X \end{pmatrix} \right)^{-1} \\
 &= \begin{pmatrix} 1 & X \\ -X & 1 - 2X \end{pmatrix}^{-1}
 \end{aligned}$$

Computing \tilde{G} for $G = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

$$\begin{aligned}\tilde{G} &= (1, G, G^2, \dots) \\ &= (1 - (G \times X))^{-1} \\ &= (1 - \left(\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \times X\right))^{-1} \\ &= (1 - \begin{pmatrix} 0 & -X \\ X & 2X \end{pmatrix})^{-1} \\ &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -X \\ X & 2X \end{pmatrix}\right)^{-1} \\ &= \begin{pmatrix} 1 & X \\ -X & 1 - 2X \end{pmatrix}^{-1}\end{aligned}$$

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Computing $\tilde{G} = (1, G, G^2, \dots)$

$$\begin{aligned}\tilde{G} &= \begin{pmatrix} 1 & X \\ -X & 1 - 2X \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{1-2X}{(1-X)^2} & \frac{-X}{(1-X)^2} \\ \frac{X}{(1-X)^2} & \frac{1}{(1-X)^2} \end{pmatrix} \in (\mathit{Rat} \mathbb{R}^\omega)^{2 \times 2}\end{aligned}$$

(The paper in the proceedings justifies all of the above.)

Computing $\tilde{G} = (1, G, G^2, \dots)$

$$\begin{aligned}\tilde{G} &= \begin{pmatrix} 1 & X \\ -X & 1 - 2X \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{1-2X}{(1-X)^2} & \frac{-X}{(1-X)^2} \\ \frac{X}{(1-X)^2} & \frac{1}{(1-X)^2} \end{pmatrix} \in (\mathit{Rat} \mathbb{R}^\omega)^{2 \times 2}\end{aligned}$$

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Final semantics

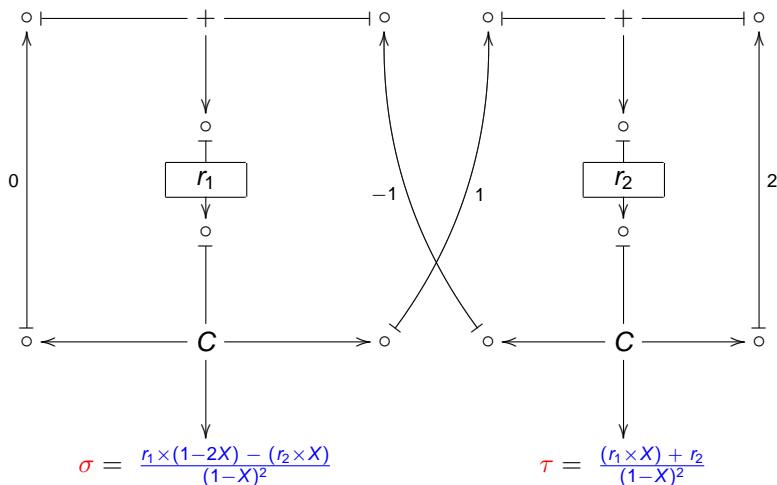
We have computed (omitting transpose):

$$\begin{array}{ccc}
 \mathbb{R}^2 & \xrightarrow{\exists! \tilde{G}} & (\mathbb{R}^\omega)^2 \\
 \downarrow \langle 1, G \rangle & & \downarrow \\
 \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \times (\mathbb{R}^\omega)^2
 \end{array}$$

$$\tilde{G}(r_1, r_2) = \begin{pmatrix} \frac{1-2X}{(1-X)^2} & \frac{-X}{(1-X)^2} \\ \frac{X}{(1-X)^2} & \frac{1}{(1-X)^2} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \in \text{Rat}(\mathbb{R}^\omega)^2$$

This coincides with our previous solution:

Final semantics coincides with our previous solution:



Summarizing

- Final semantics of **finite dimensional** linear system in terms of matrices of **rational** streams.
- Final semantics is basis for **minimization** of linear systems (see paper).
- Final semantics is basis for **realisation** of linear systems (see paper).
- Adding **outputs** (= non-full observations) is trivial (see paper).
- Adding **inputs** is (somewhat surprisingly) trivial too (see paper).

In conclusion

- Simplest model of linear systems up to date . . .
- at least for me, that is . . . ; -)
- Central role for the notion of **rationality**
- Interplay algebra-coalgebra
- Fundamental role (again) for **streams**
- **Teaching!!!**

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