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Observational Coalgebras and Complete Sets of Co-operations

Clemens Kupke[1,2] and Jan Rutten[3]

CWI & Vrije Universiteit
Amsterdam, The Netherlands

Abstract

In this paper we introduce the notion of an observational coalgebra structure and of a complete set of co-operations. We demonstrate in various example the usefulness of these notions, in particular, we show how they give rise to coalgebraic proof and definition principles.

Keywords: Coalgebra, Coinduction, infinite data structures, Hidden Algebra.

1 Introduction

It is well-known that coalgebras provide a framework for studying infinite data structures, such as streams and trees, in a uniform way. The theory of coalgebras is formulated in category theoretic terms. Therefore coalgebras are usually studied "up-to-isomorphism", e.g., one talks about *the* final coalgebra of a functor because it is determined uniquely up-to-isomorphism. When reasoning about a concrete type of coalgebras one then has a certain "canonical" representation of the final coalgebra in mind. For the stream functor $A \times \mathrm{Id}$ the final coalgebra is usually given by the set of infinite A-streams A^ω together with the usual operations head and tail. There are, however, infinitely many ways of turning A^ω into the final stream coalgebra - we will discuss some of them in the paper. The point we want to make is, that each of these representations of the final coalgebra is potentially interesting in its own as each of them yields a different proof and definition principle.

More generally, we consider not only the various representations of a given set X as a final coalgebra of some kind, but also its representations as a subcoalgebra of a

[1] Supported by NWO under FOCUS/BRICKS grant 642.000.502.
[2] Email: kupke@cwi.nl
[3] Email: janr@cwi.nl

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final coalgebra. We call such a representation of a set X an *observational coalgebra*. Any observational coalgebra has two crucial properties: strong extensionality and what we call relative finality. The first property is the basis for a proof principle on observational coalgebras and the second one is the key for the coinductive definition of constants and functions on observational coalgebras.

In the paper we first introduce the notion of an observational coalgebra and then motivate it with various examples. After that, in Section 3, we provide a simpler, syntactic version of the notion of an observational coalgebra by using the terminology from [3] of a *cosignature* and of a *co-operation*. We call a collection of co-operations *complete* for some set if it turns this set into an observational coalgebra. After having defined these notions we turn to the discussion of the proof principle and of the definition scheme.

In Section 4 we discuss the proof principle for a complete set of co-operations and demonstrate with an example that a clever choice of co-operations for the set of streams can simplify proofs. After that, in Section 5 we develop a definition scheme for constants and functions on a given set that is equipped with a complete set of co-operations. The main advantage of this scheme lies in the fact that it works for various types of objects as we demonstrate at the end of Section 5. In particular, our scheme can be applied to sets of objects, that have no "nice", purely coalgebraic representation, such as bi-infinite streams. We conclude our paper in Section 6 by linking our research to related work, in particular, to the field of hidden algebra, and by the discussion of future work.

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Basic terminology

We assume that the reader is familiar with the basic notions from category theory and universal coalgebra. The purpose of the following basic definitions is mainly to fix our notation.

Definition 1.1 We define the range of a function $f : X \to Y$ by putting $\mathrm{range}(f) := \{y \in Y \mid \exists x \in X. f(x) = y\}$.

Definition 1.2 Let $G : \mathbf{Set} \to \mathbf{Set}$ be a functor. A set X together with a function $\gamma : X \to GX$ is a *G-coalgebra*. A function $f : X_1 \to X_2$ is a *G-coalgebra morphism* from $\mathbb{X}_1 = (X_1, \gamma_1 : X \to GX)$ to $\mathbb{X}_2 = (X_2, \gamma_2 : X \to GX)$ if $\gamma_2 \circ f = Gf \circ \gamma_1$. In case the final G-coalgebra exists we denote by $\varphi^{\mathbb{X}_1}$ the unique coalgebra morphism from (X_1, γ_1) into the final G-coalgebra.

A relation $R \subseteq X_1 \times X_2$ is a *G-bisimulation* between (X_1, γ_1) and (X_2, γ_2) if there is a map $\mu : R \to GR$ such that the projection maps $\pi_i : R \to X_i$ are G-coalgebra morphisms $\pi_i : (R, \mu) \to (X_i, \gamma_i)$ for $i = 1, 2$. For G-coalgebra states $x_1 \in X_1$ and $x_2 \in X_2$ we write $x_1 \leftrightarroweq_G x_2$ if there is a G-bisimulation $R \subseteq X_1 \times X_2$

such that $(x_1, x_2) \in R$.

2 Observational coalgebra structures

In this section we introduce the notion of an observational coalgebra structure. Despite the fact that this is a rather simple notion we hope to demonstrate throughout the remainder of the paper its usefulness.

Definition 2.1 Let X be a set and let $G : \mathbf{Set} \to \mathbf{Set}$ be a functor for which the final G-coalgebra (Ω_G, ω_G) exists. We call $\gamma : X \to GX$ *observational* for X if the unique morphism $\varphi : X \to \Omega_G$ into the final G-coalgebra is injective. In this case the coalgebra (X, γ) will be called *observational*.

Remark 2.2 It should be stressed that the concept of an observational coalgebra is nothing essentially new. Observational coalgebras are merely subcoalgebras of some final coalgebra and, under the condition that the final coalgebra for the functor $G : \mathbf{Set} \to \mathbf{Set}$ exists, observational G-coalgebras are exactly the simple G-coalgebras from [10] or the minimal G-coalgebras from [6]. The novelty of our work lies in the fact, that we focus on the various *observational coalgebra structures* that turn a given set into an observational coalgebra.

In order to motivate this definition we provide a number of examples:

Example 2.3 (i) Let (X, γ) be the final G-coalgebra for a functor $G : \mathbf{Set} \to \mathbf{Set}$. Then γ is observational for X.

(ii) Consider the set $\mathbb{N}$ of natural numbers and let $P : \mathbb{N} \to 1 + \mathbb{N}$ be the predecessor map, i.e. $P(n+1) := n$ and $P(0) := * \in 1$. Then P is observational for $\mathbb{N}$: P turns $\mathbb{N}$ into a coalgebra for the functor $1 + \mathrm{Id}$ and this functor has as final coalgebra the set $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ together with the "extended" predecessor map $\bar{P}$, where $\bar{P}(n) := P(n)$ for all $n \in \mathbb{N}$ and $\bar{P}(\infty) := \infty$. The obvious embedding of $\mathbb{N}$ into $\bar{\mathbb{N}}$ is the injective coalgebra morphism from $(\mathbb{N}, P)$ into the final $1 + \mathrm{Id}$-coalgebra $(\bar{\mathbb{N}}, \bar{P})$.

(iii) Let $p > 0$ be a natural number and let

$$P_p : \mathbb{N} \longrightarrow 1 + \{0, \ldots, p-1\} \times \mathbb{N}$$

be the map defined by

$$P_p(n) := * \qquad \text{if} \quad n = 0$$
$$P_p(n) := (n \bmod p, \lfloor \tfrac{n}{p} \rfloor) \qquad \text{if} \quad n > 0.$$

Then P_p is observational for $\mathbb{N}$ for all $p > 0$. The carrier of the final coalgebra of $G = 1 + \{0, \ldots, p-1\} \times \mathrm{Id}$ is the set $p^\infty = p^* \cup p^\omega$ where $p = \{0, \ldots, p-1\}$. The final map $\varphi : \mathbb{N} \to p^\infty$ maps a natural number n to its p-adic representation starting with the least significant digit. Therefore φ is obviously injective.

(iv) Let A be a set and $A^{\mathbb{Z}}$ be the set of bi-infinite streams over A. Then the map $\langle h, l, r \rangle : A^{\mathbb{Z}} \to A \times A^{\mathbb{Z}} \times A^{\mathbb{Z}}$ is observational for $A^{\mathbb{Z}}$. Here $\langle h, l, r \rangle$ is the function that maps a given bi-infinite stream $\tau = \ldots a_{-3}a_{-2}a_{-1}\underline{a_0}a_1a_2a_3 \ldots$ to its head $h(\tau) = a_0$, its left neighbour $l(\tau) = \ldots a_{-4}a_{-3}a_{-2}\underline{a}_{-1}a_0a_1a_2\ldots$ and its right neighbour $r(\tau) = \ldots a_{-2}a_{-1}a_0\underline{a}_1a_2a_3a_4\ldots$.

(v) Consider the functor $G(X) = \mathbb{R} \times X$. The final coalgebra of G consists of the set $\mathbb{R}^\omega$ of real-valued streams together with the familiar coalgebra map $< h, t >: \mathbb{R}^\omega \to \mathbb{R} \times \mathbb{R}^\omega$ of head and tail:

$$h(\sigma) = \sigma(0), \qquad t(\sigma) = (\sigma(1), \sigma(2), \sigma(3), \ldots)$$

We can also supply $\mathbb{R}^\omega$ with an alternative coalgebra structure as follows. For $\sigma \in \mathbb{R}^\omega$ we define

$$\Delta\sigma = \ (\sigma(1) - \sigma(0), \sigma(2) - \sigma(1), \sigma(3) - \sigma(2), \ldots)$$

(cf. [7,11]). We claim that the coalgebra map

$$< h, \Delta >: \mathbb{R}^\omega \to \mathbb{R} \times \mathbb{R}^\omega \qquad \sigma \mapsto < \sigma(0), \Delta\sigma >$$

is observational for $\mathbb{R}^\omega$. The unique morphism

$$\varphi : (\mathbb{R}^\omega, < h, \Delta >) \to (\mathbb{R}^\omega, < h, t >)$$

is given by

$$\varphi(\sigma) = \ ((\Delta^{(0)}\sigma)(0), (\Delta^{(1)}\sigma)(0), (\Delta^{(2)}\sigma)(0), \ldots)$$

where $\Delta^{(0)}\sigma = \sigma$ and $\Delta^{(n+1)}\sigma = \Delta(\Delta^{(n)}\sigma)$. One can easily verify that φ is injective.

(vi) Here is yet another coalgebra structure on $\mathbb{R}^\omega$. For $\sigma \in \mathbb{R}^\omega$, we define

$$\frac{d\sigma}{dX} = \ (\sigma(1), 2 \cdot \sigma(2), 3 \cdot \sigma(3), \ldots)$$

The coalgebra map $< h, d/dX >: \mathbb{R}^\omega \to \mathbb{R} \times \mathbb{R}^\omega$ is observational for $\mathbb{R}^\omega$ as the unique morphism

$$\varphi : (\mathbb{R}^\omega, < h,\ d/dX >) \to (\mathbb{R}^\omega, < h, t >)$$

which is given by

$$\varphi(\sigma) = \ (\sigma(1), 2! \cdot \sigma(2), 3! \cdot \sigma(3), \ldots)$$

is injective.

(vii) Let $F = G_\Sigma$ for a finite cosignature Σ (cf. Def. 3.3 below) and let (A, α) be the inital F-algebra. Then $\alpha^{-1} : A \to FA$ is observational for A. The claim is a consequence of a more general result in [1]. Note that this example generalizes (ii) above.

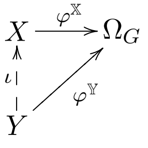
Two properties of observational coalgebras will play a central rôle in our paper: given an observational coalgebra structure $\gamma : X \to GX$ for some set X, we have that the G-coalgebra (X, γ) is strongly extensional (Prop. 2.5) and relatively final (Prop. 2.7). The first property gives rise to a proof principle for elements of observational coalgebras, the second property is the basis of the definition scheme which we develop in Section 5.

Definition 2.4 Let G : **Set** $\to$ **Set** be a functor and let $\mathbb{X} = (X, \gamma)$ be a G-coalgebra. We say $\mathbb{X}$ is *strongly extensional* iff for all $x_1, x_2 \in X$ we have $x_1 \leftrightarrows x_2$ iff $x_1 = x_2$.

Proposition 2.5 *Let $G : \mathbf{Set} \to \mathbf{Set}$ be a functor with final coalgebra and let $\mathbb{X} = (X, \gamma)$ be a G-coalgebra. If γ is observational for X then $\mathbb{X}$ is strongly extensional.*

Proof. Let (X, γ) be observational and suppose $x_1, x_2 \in X$ with $x_1 \leftrightarroweq x_2$. Furthermore let φ be the unique morphism from $\mathbb{X}$ into the final coalgebra. Obviously we have $\varphi(x_1) \leftrightarroweq \varphi(x_2)$. It is well known that final coalgebras are strongly extensional and thus $\varphi(x_1) = \varphi(x_2)$ which implies by the injectivity of φ that $x_1 = x_2$. □

Definition 2.6 Let $G : \mathbf{Set} \to \mathbf{Set}$ be a functor with final coalgebra. A G-coalgebra $\mathbb{X} = (X, \gamma)$ is called *relatively final* if for all G-coalgebras $\mathbb{Y} = (Y, \delta)$ such that range$(\varphi^{\mathbb{Y}}) \subseteq$ range$(\varphi^{\mathbb{X}})$ there is a unique G-coalgebra morphism $\iota : \mathbb{Y} \to \mathbb{X}$ with



Proposition 2.7 *Let $G : \mathbf{Set} \to \mathbf{Set}$ be a functor with final coalgebra and let $\mathbb{X} = (X, \gamma)$ be a G-coalgebra. If γ is observational for X, then $\mathbb{X}$ is relatively final.*

Proof. Let $\mathbb{X}$ be an observational G-coalgebra, let $\mathbb{Y} = (Y, \delta)$ be a G-coalgebra and let $\varphi^{\mathbb{X}}$ and $\varphi^{\mathbb{Y}}$ be the coalgebra morphisms from $\mathbb{X}$ and $\mathbb{Y}$ into the final G-coalgebra. Furthermore we assume that range$(\varphi^{\mathbb{Y}}) \subseteq$ range$(\varphi^{\mathbb{X}})$. We want to show that there is a unique G-coalgebra morphism ι from $\mathbb{Y}$ to $\mathbb{X}$. In order to show the existence of ι we define a function $\iota : Y \to X$ by putting for all $y \in Y$

$$\iota(y) := x \qquad \text{if } \varphi^{\mathbb{Y}}(y) = \varphi^{\mathbb{X}}(x).$$

This function is well-defined because of the injectivity of $\varphi^{\mathbb{X}}$ and the fact that the range of $\varphi^{\mathbb{Y}}$ is contained in the range of $\varphi^{\mathbb{Y}}$. Clearly we have $\varphi^{\mathbb{Y}} = \varphi^{\mathbb{X}} \circ \iota$ which implies that ι is a coalgebra morphism because $\varphi^{\mathbb{X}}$ is injective (cf. [10, Lemma 2.4]). Uniqueness of ι follows also from the injectivity of $\varphi^{\mathbb{X}}$: any $\iota' : \mathbb{Y} \to \mathbb{X}$ has the property that $\varphi^{\mathbb{X}} \circ \iota' = \varphi^{\mathbb{Y}} = \varphi^{\mathbb{X}} \circ \iota$ and thus $\iota = \iota'$. □

Remark 2.8 Both Proposition 2.5 and Proposition 2.7 are easy consequences of the fact that an observational coalgebra structure γ represents a subcoalgebra of a final coalgebra. We provided the easy proofs in order to keep our paper as self-contained as possible.

3 Complete sets of co-operations

The notion of an observational coalgebra is in general too abstract to work with. In this section we define the more concrete notion of a complete set of co-operations. We first introduce the notion of a cosignature and of a co-operation and then state when a given set of co-operations is complete.

3.1 Cosignatures

Definition 3.1 Let $\mathcal{S} := \{S_j\}_{j\in J}$ be a family of sets ("observable sorts"). A basic $\mathcal{S}$-arity α is an element of the set $\mathcal{S}^* \times (\mathcal{S} \cup \{\bullet\})$, i.e. any basic $\mathcal{S}$-arity α is either of the form $(S_1 \ldots S_n, S)$ or of the form $(S_1 \ldots S_n, \bullet)$, where $\bullet$ should be thought of as the "hidden sort". The set $\text{Arity}(\mathcal{S})$ of $\mathcal{S}$-arities is defined as

$$\text{Arity}(\mathcal{S}) = \{\alpha_1 + \ldots + \alpha_m \mid m \in \mathbb{N}, \alpha_i \text{ is a basic } \mathcal{S}\text{-arity}\}.$$ [4]

An $\mathcal{S}$-sorted cosignature consists of a set Σ of "co-operation" symbols and a function $a : \Sigma \to \text{Arity}(\mathcal{S})$ that assigns to each $\sigma \in \Sigma$ its arity $a(\sigma) = \alpha_1 + \ldots + \alpha_m$. We call Σ *basic* if it contains only co-operation symbols σ of basic arity.

Definition 3.2 Let $\mathcal{S}$ be a family of sorts, let Σ be an $\mathcal{S}$-sorted cosignature and let X be a set. For each arity $\alpha_1 + \ldots + \alpha_k \in \text{Arity}(\mathcal{S})$ we inductively define a corresponding set X_α by putting

$$X_{(S_1\ldots S_k,S)} := S^{S_1\times\ldots\times S_k} \qquad X_{(S_1\ldots S_k,\bullet)} := X^{S_1\times\ldots\times S_k} \qquad X_{\alpha_1+\alpha_2} := X_{\alpha_1} + X_{\alpha_2}$$

A *co-operation* of arity $a(\sigma) \in \text{Arity}(\mathcal{S})$ is a function $f : X \to X_{a(\sigma)}$.

Definition 3.3 Let $\mathcal{S}$ be a family of sorts and let Σ be an $\mathcal{S}$-sorted cosignature. A Σ-coalgebra $(X, \langle f_\sigma : \sigma \in \Sigma\rangle)$ consists of a set X and a collection of functions $\{f_\sigma : X \to X_{a(\sigma)}\}_{\sigma\in\Sigma}$. In other words, a Σ-coalgebra is a coalgebra for the functor

$$\begin{array}{lcl} G_\Sigma : \mathbf{Set} \to \mathbf{Set} & & \\ \qquad X & \mapsto & \prod_{\sigma\in\Sigma} X_{a(\sigma)} \\ X \xrightarrow{h} Y & \mapsto & \langle h_\sigma : \sigma \in \Sigma\rangle \end{array}$$

where $h_\sigma : X_{a(\sigma)} \to Y_{a(\sigma)}$ is defined in the obvious way. We call $g : Y_1 \to Y_2$ a Σ-coalgebra morphism from $(Y_1, \langle o^1_\sigma : \sigma \in \Sigma\rangle)$ to $(Y_2, \langle o^2_\sigma : \sigma \in \Sigma\rangle)$ if g is a G_Σ-coalgebra morphism.

Remark 3.4 The notion of a cosignature that we are using goes back to [3].

3.2 Complete sets of co-operations

If we instantiate Definition 2.1 of an observational coalgebra to the case of the more concrete Σ-coalgebras we obtain our notion of a complete set of co-operations.

Definition 3.5 Let X be a set, let $\mathcal{S}$ be a set of sorts and Σ an $\mathcal{S}$-sorted signature. A set of co-operations $\{f_\sigma : X \to X_{a(\sigma)}\}_{\sigma\in\Sigma}$ is called *complete* for X if the final map $\varphi : X \to \Omega_{G_\Sigma}$ from the corresponding Σ-coalgebra $(X, \langle f_\sigma : \sigma \in \Sigma\rangle)$ into the final Σ-coalgebra $(\Omega_{G_\Sigma}, \omega_\Sigma)$ is injective:

$$\begin{array}{ccc} X & \overset{\exists!\varphi}{\dashrightarrow} & \Omega_{G_\Sigma} \\ {\scriptstyle\langle f_\sigma:\sigma\in\Sigma\rangle}\downarrow & & \downarrow{\scriptstyle\omega_\Sigma} \\ G_\Sigma X & \underset{G_\Sigma\varphi}{\dashrightarrow} & G_\Sigma\Omega_{G_\Sigma} \end{array}$$

[4] Here $\alpha_1 + \ldots + \alpha_m$ denotes the word $\alpha_1 \ldots \alpha_m$ - the +'s have no formal meaning and are only there in order to make the structure of a given $\mathcal{S}$-arity more clear.

Examples of complete sets of co-operations can be found in Example 2.3 (ii)-(vi) above.

Example 3.6 (i) In Example 2.3(ii) the set $\mathcal{S}$ of sorts consists only of the one-element set 1. The co-operation P has arity $(\epsilon, 1) + (\epsilon, \bullet)$, where ϵ denotes the empty word, and $\{P\}$ ia a complete set of co-operations for $\mathbb{N}$.

(ii) Example 2.3(iii) gives not immediately rise to a complete set of co-operations. We first have to split the given function $P_p : \mathbb{N} \to 1 + \{0, \dots, p-1\}$ into two functions $P_p^1 : \mathbb{N} \to 1 + \{0, \dots, p-1\}$ and $P_p^2 : \mathbb{N} \to 1 + \mathbb{N}$ by letting $P_p^1(0) = P_p^2(0) = * \in 1$ and by putting for all $n \in \mathbb{N}$, $P_p^1(n) := n \bmod p$ and $P_p^2(n) := \lfloor \frac{n}{p} \rfloor$. It is now easy to see that $\{P_p^1, P_p^2\}$ is a complete set of co-operations for $\mathbb{N}$ with $a(P_p^1) = (\epsilon, 1) + (\epsilon, \{0, \dots, p-1\})$ and $a(P_p^2) = (\epsilon, 1) + (\epsilon, \bullet)$.

(iii) In Example 2.3(iv) the set $\mathcal{S}$ consists of the set A and the co-operations $h : A^{\mathbb{Z}} \to A$, $l : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ and $r : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ with arities (ϵ, A), $(\epsilon, \bullet)$ and $(\epsilon, \bullet)$, respectively, form a complete set of co-operations for $A^{\mathbb{Z}}$.

Remark 3.7 Equivalently, we could have defined complete sets of co-operations in the following way:

(1) $\{f_\sigma\}_{\sigma \in \Sigma}$ is complete for X if $(X, \{f_\sigma\}_{\sigma \in \Sigma})$ is strongly extensional.

That the completeness condition in (1) is implied by the one in Def. 3.5 is an immediate consequence of Prop. 2.5. The converse direction can be proven using the observation that for any functor of the form G_Σ for some cosignature Σ the relation $\leftrightarrows_{G_\Sigma}$ is transitive.

3.3 *Example: Completeness of* {head, even, odd}

In this section we show that

$$\{\mathrm{head} : A^\omega \to A, \mathrm{even} : A^\omega \to A^\omega, \mathrm{odd} : A^\omega \to A^\omega\}$$

is a complete set of co-operations for the set of A-streams, where for any infinite A-stream $\alpha = a_0 a_1 a_2 a_3 a_4 a_5 \ldots \in A^\omega$ we have

$$\mathrm{head}(\alpha) := a_0 \qquad \mathrm{even}(\alpha) := a_0 a_2 a_4 \ldots \qquad \mathrm{odd}(\alpha) := a_1 a_3 a_5 \ldots$$

Definition 3.8 Let $A^{2^*} := \{t \mid t : 2^* \to A\}$ be the set of infinite binary A-labelled trees. For a tree $t \in A^{2^*}$ and a word $w \in 2^*$ we denote by t_w the tree given by $t_w(v) := t(vw)$ for all $v \in 2^*$.

In the following we will work with the binary coding of natural numbers.

Remark 3.9 We follow the convention that the most significant digit of the binary coding of a natural number is the leftmost digit, e.g. the natural number 13 is encoded as the sequence 1101.

Definition 3.10 We denote by $\mathrm{bin} : \mathbb{N} \to 2^*$ the function that maps a natural number to its representation in binary coding. Furthermore we denote by nat :

$2^* \to \mathbb{N}$ the function that maps a binary code to the corresponding natural number. By convention we put $\mathrm{nat}(\epsilon) := 0$.

The following is a well-known fact from universal coalgebra.

Fact 3.11 *Define $h : A^{2^*} \to A$ by $h(t) := t(\epsilon)$, $l : A^{2^*} \to A^{2^*}$ by $l(t) := t_0$ and $r : A^{2^*} \to A^{2^*}$ by $r(t) := t_1$. The set A^{2^*} together with the map $\langle h, l, r\rangle : A^{2^*} \to A \times A^{2^*} \times A^{2^*}$ form a final coalgebra for the functor $A \times \mathrm{Id} \times \mathrm{Id}$.*

We now prove that $\{\mathrm{head}, \mathrm{even}, \mathrm{odd}\}$ is a complete set of co-operations.

Proposition 3.12 *Let $j : A^\omega \to A^{2^*}$ be the function that maps a stream τ to the binary tree $j(\tau)$ with*

$$j(\tau)(w) := \tau_{\mathrm{nat}(w)} \quad \textit{for all } w \in 2^*.$$

Then j is the unique coalgebra morphism from $(A^\omega, \langle \mathrm{head}, \mathrm{even}, \mathrm{odd}\rangle)$ into the final coalgebra $(A^{2^*}, \langle h, l, r\rangle)$.

Proof. We have to prove that the following diagram commutes:

$$\begin{array}{ccc} A^\omega & \xrightarrow{\quad j \quad} & A^{2^*} \\ {\scriptstyle \langle \mathrm{head},\mathrm{even},\mathrm{odd}\rangle}\downarrow & & \downarrow{\scriptstyle \langle h,l,r\rangle} \\ A \times A^\omega \times A^\omega & \xrightarrow[\mathrm{id}\times j\times j]{} & A \times A^{2^*} \times A^{2^*} \end{array}$$

Let $\tau \in A^\omega$ be a stream. Then $\mathrm{head}(\tau) = \tau_0 = \tau_{\mathrm{nat}(\epsilon)} = j(\tau)(\epsilon) = h(j(\tau))$. Furthermore for $w \in 2^*$ we get

$$\begin{aligned} l(j(\tau))(w) &= j(\tau)_0(w) = j(\tau)(w0) \\ &= \tau_{\mathrm{nat}(w0)} = \tau_{2*\mathrm{nat}(w)} = \mathrm{even}(\tau)_{\mathrm{nat}(w)} = j(\mathrm{even}(\tau))(w) \end{aligned}$$

and

$$\begin{aligned} r(j(\tau))(w) &= j(\tau)_1(w) = j(\tau)(w1) \\ &= \tau_{\mathrm{nat}(w1)} = \tau_{2*\mathrm{nat}(w)+1} = \mathrm{odd}(\tau)_{\mathrm{nat}(w)} = j(\mathrm{odd}(\tau))(w). \end{aligned}$$

Hence $l(j(\tau)) = j(\mathrm{even}(\tau))$ and $r(j(\tau)) = j(\mathrm{odd}(\tau))$ which finishes the proof that the above diagram commutes. Therefore j is the unique coalgebra morphism into the final coalgebra $(A^{2^*}, \langle h, l, r\rangle)$. □

Corollary 3.13 *The set $\{\mathrm{head}, \mathrm{even}, \mathrm{odd}\}$ is a complete set of co-operations for A^ω.*

Proof. This follows immediately from Prop. 3.12 and the fact that $j : A^\omega \to A^{2^*}$ is obviously injective. □

4 The proof principle

We are now going to discuss G_Σ-bisimulations and the resulting Σ-proof principle. It follows from Prop. 2.5 that a Σ-coalgebra $(\mathbf{X}, \langle f_\sigma : \sigma \in \Sigma\rangle)$ is strongly extensional w.r.t. G_Σ-bisimilarity, i.e., $\tau_1 \leftrightarrows_{G_\Sigma} \tau_2$ implies $\tau_1 = \tau_2$ for all $\tau_1, \tau_2 \in X$. Let us first spell out the definition of a Σ-bisimulation.

Fact 4.1 *Let* $(\mathbf{X}, \langle f_\sigma : \sigma \in \Sigma \rangle)$ *be a* Σ*-coalgebra. A relation* $R \subseteq \mathbf{X} \times \mathbf{X}$ *is a* Σ*-bisimulation if for all* $(\tau_1, \tau_2) \in R$ *and* $f_\sigma : \mathbf{X} \to \mathbf{X}_{\alpha_1} + \ldots + \mathbf{X}_{\alpha_n}$ *with basic arities* $\alpha_1, \ldots, \alpha_n$ *we have*

(i) $f_\sigma(\tau_1) \in \mathbf{X}_{\alpha_i}$ *iff* $f_\sigma(\tau_2) \in \mathbf{X}_{\alpha_i}$ *for* $1 \le i \le n$,

(ii) *if* $f_\sigma(\tau_1), f_\sigma(\tau_2) \in \mathbf{X}_{\alpha_i}$ *and* $\alpha_i = (S_1 \ldots S_m, S)$ *with* $S \in \mathcal{S}$ *we have*
$$f_\sigma(\tau_1)(s_1, \ldots, s_m) = f_\sigma(\tau_2)(s_1, \ldots, s_m) \qquad \text{for all } s_i \in S_i, 1 \le i \le m,$$

(iii) *if* $f_\sigma(\tau_1), f_\sigma(\tau_2) \in \mathbf{X}_{\alpha_i}$ *and* $\alpha_i = (S_1 \ldots S_m, \bullet)$ *we have*
$$(f_\sigma(\tau_1)(s_1, \ldots, s_m), f_\sigma(\tau_2)(s_1, \ldots, s_m)) \in R \qquad \text{for all } s_i \in S_i, 1 \le i \le m.$$

As observational coalgebras are strongly extensional we obtain the following Σ-coinduction proof principle for a set $\mathbf{X}$ that is equipped with a complete set of co-operations.

Proposition 4.2 *Let* Σ *be a cosignature and suppose* $\mathcal{O} = \{f_\sigma : \sigma \in \Sigma\}$ *is a complete set of co-operations for a set* $\mathbf{X}$*. For all* $\tau_1, \tau_2 \in \mathbf{X}$ *and all* Σ*-bisimulations* $R \subseteq \mathbf{X} \times \mathbf{X}$ *we have* $(\tau_1, \tau_2) \in R$ *implies* $\tau_1 = \tau_2$.

Proof. The claim follows from 2.5. □

The following proposition describes a special, slightly simpler case of the Σ-coinduction proof principle.

Proposition 4.3 *Let* Σ *be a cosignature, let* $\mathcal{O} = \{f_\sigma : \sigma \in \Sigma\}$ *be a complete set of co-operations for a set* $\mathbf{X}$ *and let* $\tau_1, \tau_2 \in \mathbf{X}$*. Suppose for all co-operations* $f_\sigma : \mathbf{X} \to \mathbf{X}_{\alpha_1} + \ldots + \mathbf{X}_{\alpha_n}$ *the following holds:*

(i) $f_\sigma(\tau_1) \in \mathbf{X}_{\alpha_i}$ *iff* $f_\sigma(\tau_2) \in \mathbf{X}_{\alpha_i}$ *for* $1 \le i \le n$

(ii) *if* $f_\sigma(\tau_1), f_\sigma(\tau_2) \in \mathbf{X}_{\alpha_i}$ *and* $\alpha_i = (S_1 \ldots S_m, T)$ *we have*
$$f_\sigma(\tau_1)(s_1, \ldots, s_m) = f_\sigma(\tau_2)(s_1, \ldots, s_m) \qquad \text{for all } s_i \in S_i, 1 \le i \le m.$$

Then we can conclude that $\tau_1 = \tau_2$.

Proof. Given the assumptions of the proposition it is straightforward to see that the relation $\Delta_{\mathbf{X}} \cup \{(\tau_1, \tau_2)\}$ is a Σ-bisimulation, where $\Delta_{\mathbf{X}} \subseteq \mathbf{X} \times \mathbf{X}$ denotes the identity relation (the $\mathbf{X}$-"diagonal"). Therefore the claim follows using Proposition 4.2. □

We now turn to an example that should demonstrate that a good choice of a complete set of co-operations for a given set $\mathbf{X}$ can lead to relatively simple proofs by Σ-coinduction. A further application of Σ-coinduction can be found in Section 5 (cf. Prop 5.15 below).

4.1 The proof principle: an example

Consider the set $\mathbb{R}^\omega$ of streams of real numbers together with the complete set of observers $\{h, \Delta\}$ from Example 2.3(v). We will recall a bit of so-called stream calculus; see [11] for all details. Let $X = (0, 1, 0, 0, 0, \ldots)$. We denote the convolution product of two streams σ and τ in $\mathbb{R}^\omega$ by $\sigma \times \tau$. The multiplicative inverse of τ is denoted by $1/\tau$ (which exists whenever $\tau(0) \neq 0$). As usual, σ/τ denotes $\sigma \times (1/\tau)$. We define the following so-called *falling powers* of X, for all $n \geq 0$, by

$$X^{\underline{n}} = X^n/(1-X)^{n+1}$$

As usual, we include the set of reals $\mathbb{R}$ into the set of streams $\mathbb{R}^\omega$ by the notational convention

$$r = (r, 0, 0, 0, \ldots)$$

Note that $\Delta X^{\underline{0}} = \Delta 1/(1-X) = 0$ and

$$\Delta X^{\underline{n+1}} = X^{\underline{n}}$$

For $\sigma \in \mathbb{R}^\omega$ we define

$$r_n^\sigma = \left(\Delta^{(n)} \sigma \right)(0)$$

Now let

$$sum(\sigma) = r_0^\sigma \times X^{\underline{0}} + r_1^\sigma \times X^{\underline{1}} + r_2^\sigma \times X^{\underline{2}} + \cdots$$

Theorem 4.4 *For all $\sigma \in \mathbb{R}^\omega$,*

$$\sigma = sum(\sigma)$$

Proof. We show that

$$R = \{ (\sigma,\ sum(\sigma)) \mid \sigma \in \mathbb{R}^\omega \}$$

is an $\{h, \Delta\}$-bisimulation. Clearly,

$$h(\sigma) = \sigma(0) = h(sum(\sigma))$$

Furthermore we have

$$\begin{aligned}
&\Delta\, sum(\sigma) \\
&= \Delta\ \left(r_0^\sigma \times X^{\underline{0}} + r_1^\sigma \times X^{\underline{1}} + r_2^\sigma \times X^{\underline{2}} + \cdots \right) \\
&= r_0^\sigma \times \Delta\, X^{\underline{0}} + r_1^\sigma \times \Delta\, X^{\underline{1}} + r_2^\sigma \times \Delta\, X^{\underline{2}} + \cdots \\
&= r_1^\sigma \times X^{\underline{0}} + r_2^\sigma \times X^{\underline{1}} + r_3^\sigma \times X^{\underline{2}} + \cdots \\
&= sum(\Delta\sigma)
\end{aligned}$$

where for the latter equality we use

$$r_{n+1}^\sigma = r_n^{\Delta\sigma}$$

As a consequence, we have

$$(\Delta\sigma,\ \Delta\, sum(\sigma)) = (\Delta\sigma,\ sum(\Delta\sigma)) \in R$$

This proves that R is an $\{h, \Delta\}$-bisimulation. □

The theorem above is already present in [11, Thm 11.1]. The reader is invited to compare the proof there with the present one. (Giving away the clue, the present one is quite a bit simpler ;-))

5 The definition scheme

5.1 The general case

Remark 5.1 We restrict our attention to basic cosignatures, i.e., the arities of the occurring co-operations do not involve the coproduct. The case in which co-operations involve the coproduct is considerably more complicated and is left as future work.

In this subsection we are considering the situation in which we are given:

- a collection $\mathcal{S}$ of sets ("visible sorts") and a hidden sort $\bullet$
- a set $\mathbf{X}$, a basic cosignature Σ and a complete set of co-operations $\mathcal{O} = \{f_\sigma : \sigma \in \Sigma\}_{\sigma\in\Sigma}$ for $\mathbf{X}$,
- a set Δ of function symbols for the functions that we want to define; we write $\Delta_i \subseteq \Delta$ for the set of function symbols in Δ with $i \in \mathbb{N}$ arguments,

We are going to define a generic definition scheme for $\mathbf{X}$-constants and $\mathbf{X}$-functions, generalising the scheme that has been presented in [12] for the case that $\mathbf{X}$ is the set of infinite binary A-labelled trees. In order to be able to formulate what a well-formed definition of $\mathbf{X}$-constants and functions is, we have to introduce some syntax.

5.1.1 The terms

We first define the set $\mathcal{FE}$ of flat equation terms, the set $\mathcal{E}$ of equation terms and the set of $\mathcal{E}_r$ of restricted equation terms. These terms are sorted, i.e. we write $t : S$ to indicate that t is a term of sort $S \in \mathcal{S} \cup \{\bullet\}$. In our scheme we are allowed to freely use "help functions" of visible sort.

Definition 5.2 For a set $\mathcal{S}$ of visible sorts we define the set of admissible help functions by putting $\mathtt{Help}_{\mathcal{S}} := \{h \mid h \text{ is a function of type } S_1 \times \ldots \times S_j \to T \text{ for some } j \in \mathbb{N} \text{ and some } S_1, \ldots S_j, T \in \mathcal{S}\}$.

Definition 5.3 Given a cosignature Σ, a set of visible sorts $\mathcal{S}$, a set Δ of constants and function symbols and a sorted set $\mathcal{X} = (X_S)_{S\in\mathcal{S}\cup\{\bullet\}}$ of variables. We define the sets $\mathcal{FE}$ of flat equation terms, $\mathcal{E}$ of equation terms and $\mathcal{E}_r$ of restricted equation terms by putting:

$$
\begin{aligned}
\mathcal{FE} \ni t ::= {} & x : S, x \in X_S, S \in \mathcal{S} \cup \{\bullet\} \mid \underline{s}, s \in S, S \in \mathcal{S} \\
& \mid F_\sigma(t) : S_1 \times \ldots \times S_n \to T,\ \sigma \in \Sigma, a(\sigma) = (S_1 \ldots S_n, T), t : \bullet \\
& \mid t(t_1, \ldots, t_l) : S,\ t : S_1 \times \ldots \times S_l \to S, t_i : S_i \text{ for } 1 \le i \le l
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E} \ni s ::= {} & x : S, x \in X_S, S \in \mathcal{S} \cup \{\bullet\} \mid \underline{s}, s \in S, S \in \mathcal{S} \mid \underline{\tau} : \bullet, \tau \in \mathbf{X} \\
& \mid F_\sigma(t) : S_1 \times \ldots \times S_n \to T,\ \sigma \in \Sigma, a(\sigma) = (S_1 \ldots S_n, T), t : \bullet \\
& \mid h : S_1 \times \ldots \times S_l \to S \in \mathtt{Help} \mid g : (\bullet)^n \to \bullet,\ g \in \Delta_n, n \in \mathbb{N} \\
& \mid t(t_1, \ldots, t_l) : S,\ t : S_1 \times \ldots \times S_l \to S, t_i : S_i \text{ for } 1 \le i \le l
\end{aligned}
$$

and by defining $\mathcal{E}_r \subseteq \mathcal{E}$ to consist exactly of those terms in $\mathcal{E}$ in which for every $\sigma \in \Sigma$ the symbol F_σ is applied to variables only. Finally we put

$$\mathcal{T} \ni t ::= \underline{\tau}, \tau \in \mathbf{X} \mid g(t_1, \ldots, t_n), g \in \Delta_n, n \in \mathbb{N}.$$

We write $t(x_1 : S_1, \ldots, x_n : S_n)$ in order to indicate that t is a term with variables contained in $\{x_1 : S_1, \ldots, x_n : S_n\}$.

Definition 5.4 Let $\mathbb{Y} = (Y, \langle o_\sigma : \sigma \in \Sigma\rangle)$ be a Σ-coalgebra and let V be a set of sorted variables from $\mathcal{X}$. A *variable assignment on* V is a function α that assigns to each variable $x \in V$ of sort $T \in \mathcal{S} \cup \{\bullet\}$ an element $s \in S$ if $T = S \in \mathcal{S}$ or a state $y \in Y$ if $T = \bullet$.

Definition 5.5 Let $\mathbb{Y} = (Y, \langle o_\sigma : \sigma \in \Sigma\rangle)$ be a Σ-coalgebra and suppose that for every $g \in \Delta_m$ there is an operation $g^{\mathbb{Y}} : Y^m \to Y$. For every term $t(x_1, \ldots, x_n) \in \mathcal{E}$ and every variable assignment α on $\{x_1, \ldots, x_n\}$ we define by induction on t its interpretation $(t[\alpha])^{\mathbb{Y}}$ as follows:

$$\begin{array}{rlll} (x[\alpha])^{\mathbb{Y}} := \alpha(x) & (\underline{s}[\alpha])^{\mathbb{Y}} := s \in S & & (\underline{\tau}[\alpha])^{\mathbb{Y}} := \tau \in \mathbf{X} \\ (h[\alpha])^{\mathbb{Y}} := h & (g[\alpha])^{\mathbb{Y}} := g^{\mathbb{Y}} & & \\ (F_\sigma(t)[\alpha])^{\mathbb{Y}} := o_\sigma((t[\alpha])^{\mathbb{Y}}) & & & \\ (t(t_1, \ldots, t_n)[\alpha])^{\mathbb{Y}} := (t[\alpha])^{\mathbb{Y}}((t_1[\alpha])^{\mathbb{Y}}, \ldots, t_n[\alpha])^{\mathbb{Y}}). & & & \end{array}$$

Where necessary, we explicitly mention the interpretations of the function symbols in Δ and write $(t[\alpha])^{(\mathbb{Y}, \{g^{\mathbb{Y}}\}_{g \in \Delta})}$ for $(t[\alpha])^{\mathbb{Y}}$. Similarly we define the interpretation $(t[\alpha])^{\mathbb{Y}}$ of a term $t(x_1, \ldots, x_n) \in \mathcal{FE}$ on an arbitrary Σ-coalgebra. An *equation* is a pair of terms $e_1(x_1, \ldots, x_n), e_2(x_1, \ldots, x_n) \in \mathcal{E}$. We say (e_1, e_2) is satisfied in $\mathbb{Y}$ by an assignment α if $(e_1[\alpha])^{\mathbb{Y}} = (e_2[\alpha])^{\mathbb{Y}}$. We write $\mathbb{Y}, \alpha \models (e_1, e_2)$ if (e_1, e_2) is satisfied by α. Furthermore we write $x \mapsto t$ for the variable assignment that maps the variable x to the term t.

Definition 5.6 A *state equation* is a pair $e = (e_1, e_2)$ of terms $e_1(x : \bullet), e_2(x : \bullet) \in \mathcal{FE}$. Given a Σ-coalgebra $\mathbb{Y} = (Y, \langle o_\sigma : \sigma \in \Sigma\rangle)$ we say that e is satisfied at a state y if $\mathbb{Y}, (x \mapsto y) \models (e_1, e_2)$, ie., if $(e_1[x \mapsto y])^{\mathbb{Y}} = (e_2[x \mapsto y])^{\mathbb{Y}}$. We write $y \models e$ if e is satisfied at y and we write $\mathbb{Y} \models e$ if $y \models e$ for all $y \in Y$.

We will use the fact that Σ-coalgebra morphisms preserve state equations: if f is a coalgebra morphism and e is some state equation satisfied at a state x then e is also satisfied at $f(x)$. This is the content of the following two lemmas.

Lemma 5.7 *Let* $\mathbb{Y}_1 = (Y_1, \langle o^1_\sigma : \sigma \in \Sigma\rangle)$ *and* $\mathbb{Y}_2 = (Y_2, \langle o^2_\sigma : \sigma \in \Sigma\rangle)$ *be* Σ-*coalgebras and let* $f : \mathbb{Y}_1 \to \mathbb{Y}_2$ *be a* Σ-*coalgebra morphism. For all states* $y \in Y_1$, *all* $\sigma \in \Sigma$ *and all* $(s_1, \ldots, s_n) \in S_1 \times \ldots S_n$ *we have*

$$\begin{array}{rl} o^1_\sigma(y)(s_1, \ldots, s_n) = o^2_\sigma(f(y))(s_1, \ldots, s_n) & \textit{for all } (s_1, \ldots, s_n) \in S_1 \times \ldots S_n, \\ & \textit{if } \mathrm{Arity}(\sigma) = (S_1 \ldots S_n, S),\ S \neq \bullet \\ f(o^1_\sigma(y)(s_1, \ldots, s_n)) = o^2_\sigma(f(y))(s_1, \ldots, s_n) & \textit{for all } (s_1, \ldots, s_n) \in S_1 \times \ldots S_n \\ & \textit{if } \mathrm{Arity}(\sigma) = (S_1 \ldots S_n, \bullet) \end{array}$$

Proof. The claim can be easily proven by spelling out the definitions of a Σ-coalgebra morphism. $\Box$

As a consequence we get that state equations are "preserved" under coalgebra morphisms.

Lemma 5.8 *Let $\mathbb{Y}_1 = (Y_1, \langle o^1_\sigma : \sigma \in \Sigma\rangle)$ and $\mathbb{Y}_2$ be Σ-coalgebras, let $f : \mathbb{Y}_1 \to \mathbb{Y}_2$ be a Σ-coalgebra morphism and let $e = (e_1, e_2)$ be a state equation. Then for all $y \in Y_1$ we have*

$$y \models e \quad \textit{implies} \quad f(y) \models e.$$

Proof. Using the previous lemma one can show by a straightforward induction on the term structure that for all terms $t(x : \bullet) \in \mathcal{FE}$ the following holds

$$\begin{aligned} f\left((t[x \mapsto y])^{\mathbb{Y}_1}\right) &= \left(t[x \mapsto f(y)]^{\mathbb{Y}_2}\right) \text{ if } t : \bullet \\ (t[x \mapsto y])^{\mathbb{Y}_1} &= (t[x \mapsto f(y)])^{\mathbb{Y}_2} \text{ if } t \text{ is of visible sort.} \end{aligned}$$

This clearly implies the claim. $\Box$

In order to be able to use the fact that $\mathbf{X}$ together with the set of co-operations $\{f_\sigma : \sigma \in \Sigma\}$ is (isomorphic to) a subcoalgebra (U, γ_U) of the final Σ-coalgebra, we have to concretely describe (U, γ_U) using state equations: if we characterise (U, γ_U) by a set of state equations E, we know that (U, γ_U) and consequently also $(\mathbf{X}, \langle f_\sigma : \sigma \in \Sigma\rangle)$ is relatively final among all Σ-coalgebras that validate the state equations in E.

Definition 5.9 Let $\Omega = (\Omega_\Sigma, \langle \omega_\sigma : \sigma \in \Sigma\rangle)$ be the final Σ-coalgebra and let $P \subseteq \Omega_\Sigma$ be a subset of Ω_Σ. We denote by $\Box\mathbb{P} = (\Box P, \langle \omega^{\Box P}_\sigma : \sigma \in \Sigma\rangle)$ the largest subcoalgebra of Ω that is contained in P.

The well-definedness of $\Box\mathbb{P}$ follows from the fact that for any $P \subseteq \Omega_\Sigma$ the largest subcoalgebra of Ω contained in P exists (cf. e.g. [5, Thm. 4.7]).

Definition 5.10 Let Σ be a cosignature and let $\mathcal{O} = \{f_\sigma \mid \sigma \in \Sigma\}$ be a complete set of Σ-co-operations for $\mathbf{X}$. We say that a set of equations E *completely specifies* $(\mathbf{X}, \langle f_\sigma : \sigma \in \Sigma\rangle)$ if

$$\Box\mathbb{P}_E \cong (\mathbf{X}, \langle f_\sigma : \sigma \in \Sigma\rangle)$$

where $P_E := \{y \in \Omega_\Sigma \mid \forall e \in E.\ y \models e\}$. In this case we call $(\mathcal{O}, E)$ a *complete (Σ-)specification of* $\mathbf{X}$.

Lemma 5.11 *Let $(\mathcal{O}, E)$ be a complete Σ-specification of $\mathbf{X}$. For all Σ-coalgebras $\mathbb{Y} = (Y, \langle o_\sigma : \sigma \in \Sigma\rangle)$ such that $\mathbb{Y} \models e$ for all $e \in E$ there exists a unique Σ-coalgebra morphism $\iota_\mathbb{Y} : Y \to \mathbf{X}$.*

Proof. Let $\varphi : Y \to \Omega_\Sigma$ be the unique Σ-coalgebra morphism into the final Σ-coalgebra Ω. It follows from Lemma 5.8 that $\mathbb{Y} \models e$ for all $e \in E$ implies range$(\varphi) \subseteq P_E$. As range(φ) is a subcoalgebra of Ω we get range$(\varphi) \subseteq \Box\mathbb{P}_E$. The existence of ι follows now from Proposition 2.7. $\Box$

5.1.2 The differential equations

We now have introduced the necessary terminology in order to be able to state the main definition of this section. This definition involves the notion of an equation being provable in conditional equational logic. We do not want to spell out this notion, instead, the reader is referred to the brief overview in [13, Sec. 7.3] and references therein.

Definition 5.12 Let $(\mathcal{O}, E)$ be a complete specification of $\mathbf{X}$. A *well-formed* system of behavioural differential equations for $(\mathcal{O}, E)$ and Δ is a set $\mathsf{Equ} \subseteq \mathcal{E} \times \mathcal{E}_r$ of equations which contains for every $g \in \Delta$, and for any $\sigma \in \Sigma$ with $a(\sigma) = (S_1 \dots S_n, T)$ an equation

$$(F_\sigma(g(x_1 : \bullet, \dots, x_r : \bullet)))\,(y_1 : S_1, \dots, y_n : S_n) = \\ t^g_\sigma(x_1 : \bullet, \dots, x_r : \bullet, y_1 : S_1, \dots, y_n : S_n) : T,$$

where the t^g_σ's are terms in $\mathcal{E}_r$ and with the property that for all $(e_1, e_2) \in E$ and all $g \in \Delta$ the following conditional equation is provable in conditional equation logic

$$(2) \qquad (e_1[x := g(x_1, \dots, x_n)] = e_2[x := g(x_1, \dots, x_n)]) \Leftarrow E(x_1, \dots, x_n) \cup \mathsf{Equ},$$

where $E(x_1, \dots, x_n) := \{(e_1[x := x_i], e_2[x := x_i]) \mid (e_1, e_2) \in E, 1 \leq i \leq n\}$.

Now that we know what a well-behaved system of equations is, we also want to see what a solution of these equations looks like.

Definition 5.13 A solution of Equ is a family of functions $\{\hat{g}\}_{g\in\Delta}$ that contains for all $g \in \Delta$ a function $\hat{g} : \mathbf{X}^n \to \mathbf{X}$ such that for all $\sigma \in \Sigma$:

$$f_\sigma(\hat{g}(\tau_1, \dots, \tau_n))(s_1, \dots, s_l) = (t^g_\sigma[y_j := \underline{s}_j][x_i \mapsto \underline{\tau}_i])^{\mathbb{X}},$$

where n is the arity of g and l is the arity of σ.

Before we demonstrate that such a solution exists for any well-formed system of behavioural differential equations we demonstrate that a solution has to be necessarily unique. The following technical lemma will be useful.

Lemma 5.14 *Let $\{\hat{g}\}$ be a solution of Equ. For all terms $(t : \bullet) \in \mathcal{E}_r$, all $f_\sigma \in \mathcal{O}$ and all $s_i \in S_i$ for $1 \leq i \leq m$ there exists $t^* \in \mathcal{E}_r$ such that*

$$(3) \qquad f_\sigma\left((t[\alpha])^{(\mathbb{X},\{\hat{g}\}_{g\in\Delta})}\right)(s_1, \dots, s_m) = (t^*[\alpha])^{(\mathbb{X},\{\hat{g}\}_{g\in\Delta})}$$

where $a(\sigma) = (S_1 \dots S_m, T)$.

Proof. The claim can be proven by induction on the structure of t. □

We are ready to prove that any solution of a well-formed system of behavioural differential equations has to be unique.

Proposition 5.15 *If $\{\hat{g}\}_{g\in\Delta}$ and $\{g'\}_{g\in\Delta}$ are solutions of Equ, then for all $g \in \Delta$ we have $\hat{g} = g'$.*

Proof. In order to prove the proposition, we first observe that for any term $t(x_1, \dots, x_m) : S \in \mathcal{E}_r$ of observable sort $S \in \mathcal{S}$ and all variable assignments α, we have

(4) $$(t[\alpha])^{(\mathbb{X},\{\hat{g}\}_{g\in\Delta})} = (t[\alpha])^{(\mathbb{X},\{g'\}_{g\in\Delta})}$$

We leave the easy inductive proof to the reader. The key observation is that t cannot contain any co-operation symbol $g \in \Delta$, because t is of observable sort and the operation symbols F_σ - the only operations that can transform a term of sort $\bullet$ into a term of observable sort - are exclusively applied to variables because t is a term in $\mathcal{E}_r$.

We now prove the proposition using the principle of Σ-coinduction from Prop. 4.2. We put

$$R := \{(\hat{t}, t') \mid (t : \bullet) \in \mathcal{E}_r,\ \hat{t} = (t[\alpha])^{(\mathbb{X},\{\hat{g}\})},\ t' = (t[\alpha])^{(\mathbb{X},\{g'\})} \text{ for some variable assignment } \alpha : \mathcal{X} \to \mathbf{X}\}.$$

Let $g \in \Delta$ be some n-ary function symbol. By definition of R we have $(\hat{g}(\tau_1,\ldots,\tau_n), g'(\tau_1,\ldots,\tau_n)) \in R$ for arbitrary $\tau_1,\ldots,\tau_n \in \mathbf{X}$. Therefore, in order to prove that $\hat{g}(\tau_1,\ldots,\tau_n) = g'(\tau_1,\ldots,\tau_n)$, it suffices to show that R is a Σ-bisimulation.

Let $(\hat{t}, t') \in R$, i.e., $\hat{t} = (t[\alpha])^{(\mathbb{X},\{\hat{g}\})}$ and $t' = (t[\alpha])^{(\mathbb{X},\{g'\})}$ for some $t \in \mathcal{E}_r$ and some variable assignment α, and let $\sigma \in \Sigma$ with $a(\sigma) = (S_1 \ldots S_n, T)$. Then by (3) in Lemma 5.14 there exists for all $s_i \in S_i$, $1 \le i \le n$ a term $t^* \in \mathcal{E}_r$ such that

(5) $$f_\sigma(\hat{t})(s_1,\ldots,s_n) = (t^*[\alpha])^{(\mathbb{X},\{\hat{g}\})} \qquad f_\sigma(t')(s_1,\ldots,s_n) = (t^*[\alpha])^{(\mathbb{X},\{g'\})}$$

In case $T \in \mathcal{S}$, i.e., in case T is an observable sort, (4) and (5) imply that $f_\sigma(\hat{t})(s_1,\ldots,s_n) = f_\sigma(t')(s_1,\ldots,s_n)$. In the case that $T = \bullet$ the equations (5) yield $(f_\sigma(\hat{t})(s_1,\ldots,s_n), f_\sigma(t')(s_1,\ldots,s_n)) \in R$. As σ and the s_i's where arbitrary this means that R is indeed a Σ-bisimulation and hence for any $g \in \Delta$ we have $\hat{g}(\tau_1,\ldots,\tau_n) = g'(\tau_1,\ldots,\tau_n)$ for all $\tau_i \in \mathbf{X}$. □

5.1.3 The solution

Throughout this section we fix a cosignature Σ, a complete Σ-specification $(\mathcal{O}, E)$ of $\mathbf{X}$ and a well-formed system Equ of behavioural differential equations for $(\mathcal{O}, E)$.

For all $\sigma \in \Sigma$ with $a(\sigma) = (S_1 \ldots S_n, S)$, $S \in \mathcal{S}$ and for all $\tau \in \Sigma$ with $a(\tau) = (S_1 \ldots S_n, \bullet)$ we define functions

$$F_\sigma : \mathcal{T} \times S_1 \times \ldots \times S_n \to S \quad \text{and}$$
$$F_\tau : \mathcal{T} \times S_1 \times \ldots \times S_n \to \mathcal{T},$$

respectively. The F_σ's are defined by induction on the structure of the terms in $\mathcal{T}$.

Definition 5.16 The *term coalgebra* $\mathbb{T} = (\mathcal{T}, \langle F_\sigma : \sigma \in \Sigma\rangle)$ is defined inductively by putting for all $\sigma \in \Sigma$:

$$F_\sigma(\underline{\tau}) := \underline{f_\sigma(\tau)}$$

$$\text{where } \underline{f_\sigma(\tau)}(s_1,\ldots,s_n) := \begin{cases} \underline{f_\sigma(\tau)(s_1,\ldots,s_n)} \text{ if } f_\sigma(\tau)(s_1,\ldots,s_n) \in \mathbf{X} \\ f_\sigma(\tau)(s_1,\ldots,s_n) \text{ otherwise} \end{cases}$$

$$F_\sigma(g(t_1,\ldots,t_n)) := \lambda\vec{s}.(t^g_\sigma[y_j := \underline{s}_j][x_i \mapsto t_i])^{\mathbb{T}}.$$

Lemma 5.17 *Let $(\mathcal{O}, E)$ be a complete specification of $\mathbf{X}$ and let Equ be a well-formed system of behavioural differential equations for $(\mathcal{O}, E)$. Furthermore let $\mathbb{T} = (\mathcal{T}, \langle F_\sigma : \sigma \in \Sigma \rangle)$ be the term Σ-coalgebra defined above. For all $t \in \mathcal{T}$ and all $e \in E$ we get that $t \models e$.*

Proof. Let t be an element of $\mathcal{T}$ and let $(e_1, e_2) \in E$. We have to show that

$$(e_1[x \mapsto t])^{\mathbb{T}} = (e_2[x \mapsto t])^{\mathbb{T}}. \tag{6}$$

We prove the claim by induction on the structure of t.

Case $t = \underline{\tau}$ for some $\tau \in \mathbf{X}$. In order to show that (6) holds one first has to prove that the function $(\underline{\ }) : \mathbf{X} \to \mathcal{T}$ that maps an element $\tau \in \mathbf{X}$ to the corresponding constant $\underline{\tau} \in \mathcal{T}$ is a Σ-coalgebra morphism from $\langle \mathbf{X}, \langle f_\sigma : \sigma \in \Sigma \rangle \rangle$ to $\mathbb{T}$. This a matter of routine checking. By Lemma 5.8 and the fact that $\tau \models (e_1, e_2)$ it now follows that also $\underline{\tau} \models (e_1, e_2)$.

Case $t = g(t_1, \ldots, t_n)$ for some $g \in \Delta$. Let α be a variable assignment that maps for all $1 \leq i \leq n$ the variable x_i to the term t_i. Then by I.H. we have $(e_1[\alpha])^{\mathbb{T}} = (e_2[\alpha])^{\mathbb{T}}$, ie., $\mathbb{T}, \alpha \models (e_1, e_2)$, for all $e = (e_1, e_2) \in E(x_1, \ldots, x_n)$. Furthermore for all $\sigma \in \Sigma$ with $a(\sigma) = (S_1 \ldots S_l, T)$ and for all $(s_1, \ldots, s_l) \in S_1 \times \ldots \times S_l$ by definition we have $(F_\sigma(g(x_1, \ldots, x_n))(y_1, \ldots, y_l)[y_j := \underline{s}_j][\alpha])^{\mathbb{T}} = (t^g_\sigma[y_j := \underline{s}_j][\alpha])^{\mathbb{T}}$ and thus $\mathbb{T}, \alpha \models (e_1, e_2)$ for all equations (e_1, e_2) in Equ. By (2) it follows that for an arbitrary $e = (e_1, e_2) \in E$ we have $\mathbb{T}, \alpha \models (e_1[x := g(x_1, \ldots, x_n)], e_1[x := g(x_1, \ldots, x_n)])$, ie.,

$$(e_1[x := g(x_1, \ldots, x_n)][\alpha])^{\mathbb{T}} = (e_2[x := g(x_1, \ldots, x_n)][\alpha])^{\mathbb{T}},$$

which is equivalent to $(e_1[x \mapsto t])^{\mathbb{T}} = (e_2[x \mapsto t])^{\mathbb{T}}$. The latter shows that $t \models e$ as required.

□

The following is an immediate corollary.

Corollary 5.18 *There exists a unique Σ-coalgebra morphism*

$$\iota : (\mathcal{T}, \langle F_\sigma : \sigma \in \Sigma \rangle) \to (\mathbf{X}, \langle f_\sigma : \sigma \in \Sigma \rangle).$$

Proof. The claim follows from the fact that $(\mathbf{X}, \langle f_\sigma : \sigma \in \Sigma \rangle)$ is relatively final among all Σ-coalgebras that satisfy the equations in E (Lemma 5.11) and from the fact that the term coalgebra satisfies the equations in E (Lemma 5.17). □

The final map ι can be used in order to obtain the solution of the given system Equ of behavioural differential equations.

Definition 5.19 Let $(\mathcal{O}, E)$ be a complete specification of $\mathbf{X}$, let Equ be a well-formed system of behavioural differential equations for $(\mathcal{O}, E)$ and let ι be the Σ-coalgebra-morphism from Corollary 5.18. For every $g \in \Delta$ we define a function $\hat{g} : \mathbf{X}^n \to \mathbf{X}$ by putting $\hat{g}(\tau_1, \ldots, \tau_n) := \iota(g(\underline{\tau}_1, \ldots, \underline{\tau}_n))$.

The above definition yields the unique solution of a given well-formed system of behavioural differential equations.

Proposition 5.20 *Let $(\mathcal{O}, E)$ be a complete specification of* $\mathbf{X}$ *and let* Equ *be a well-formed system of behavioural differential equations for $(\mathcal{O}, E)$ and a given set of function symbols Δ. The family $\{\hat{g}\}_{g\in\Delta}$ from Definition 5.19 is the unique solution of* Equ.

Proof. The fact that $\{\hat{g}\}_{g\in\Delta}$ is a solution of Equ can be easily checked. That the solution of a well-formed system of behavioural differential equations is unique has been proven in Proposition 5.15. □

5.2 Definition scheme: short examples

We now give a short list of examples that are instances of our definition scheme. An example that has been worked out in more detail can be found in Section 5.3 below.

(i) Consider the set of bi-infinite streams $\mathbb{Z}^{\mathbb{Z}}$ of integers together with the set of co-operations $\{h : \mathbb{Z}^{\mathbb{Z}} \to \mathbb{Z}, l : \mathbb{Z}^{\mathbb{Z}} \to \mathbb{Z}^{\mathbb{Z}}, r : \mathbb{Z}^{\mathbb{Z}} \to \mathbb{Z}^{\mathbb{Z}}\}$ (cf. Example 2.3(iv)). The equations $(F_l(F_r(x)), x)$ and $(F_r(F_l(x)), x)$ can be seen to completely specify $(\mathbb{Z}^{\mathbb{Z}}, \langle h, l, r\rangle)$. The following is a well-formed system of differential equations for $\Delta = \{\sigma\} \cup \{+(\_, z) \mid z \in \mathbb{Z}\}$:

$$\begin{aligned} F_h(\sigma) &= 0 & F_h(+(x, z)) &= F_h(x) + z \\ F_l(\sigma) &= +(\sigma, 1) & F_l(+(x, z)) &= +(F_l(x), a) \\ F_r(\sigma) &= +(\sigma, -1) & F_r(+(x, z)) &= +(F_r(x), z) \end{aligned}$$

where $z \in \mathbb{Z}$. Then the functions $+(\_, z) : \mathbb{Z}^{\mathbb{Z}} \to \mathbb{Z}^{\mathbb{Z}}$ for all $z \in \mathbb{Z}$ that add to a given bi-infinite stream the integer z and the constant

$$\sigma = (\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots),$$

form the unique solution.

(ii) Here is an example of an $\{h, \Delta\}$-differential equation (cf. Example 2.3(v)):

$$\Delta\sigma = \sigma\,, \quad \sigma(0) = 1$$

It has a unique solution:

$$\sigma = (2^0, 2^1, 2^2, \ldots).$$

A closed expression for his solution can be computed using the following identity, which can be viewed as the fundamental theorem of the difference calculus: for all $\tau \in \mathbb{R}^{\omega}$,

$$\tau = \frac{1}{1 - X} \times (\,\tau_0 + X \times \Delta\tau\,)$$

Using this and the differential equation above, one obtains

$$\sigma = \frac{1}{1 - 2X} = (2^0, 2^1, 2^2, \ldots)$$

(iii) The following is an example of an $\{h, d/dX\}$-differential equation (cf. Example 2.3(vi)):

$$\frac{d\sigma}{dX} = \sigma\,, \quad \sigma(0) = 1$$

Again, it has a unique solution, which is now given by

$$\sigma(n) = \frac{1}{n!}$$

(It is not obvious how to find a closed expression for σ.)

5.3 *Definition scheme for* {head, even, odd}

We now want to look at one instance of our definition scheme in somewhat more detail. We consider the set $\mathbf{X} = A^\omega$ of infinite A-streams for a given non-empty set A. For our scheme we first need a complete specification $(\mathcal{O}, E)$ of A^ω. In Section 3.3 we saw that the co-operations $\{\text{head}, \text{even}, \text{odd}\}$ are complete with respect to A^ω. Thus we put $\mathcal{O} = \{\text{head}, \text{even}, \text{odd}\}$, i.e., our cosignature consists of one constant head with $a(\text{head}) = (\epsilon, A)$, and two operation symbols even and odd with $a(\text{even}) = a(\text{odd}) = (\epsilon, \bullet)$ [5].

For a complete specification of A^ω, however, we also need some equations that characterise the subcoalgebra of the final $A \times \text{Id} \times \text{Id}$-coalgebra that is isomorphic to $(A^\omega, \langle \text{head}, \text{even}, \text{odd} \rangle)$: intuitively speaking this subcoalgebra consists of those binary A-labelled trees that do not change the label on paths that go to the left only - corresponding to the fact that the first element of a stream σ and the first element of $\text{even}(\sigma)$ are equal. This property can be expressed by the following state equation:

$$F_{\text{head}}(F_{\text{even}}(x)) = F_{\text{head}}(x) \text{ with some variable } x \in X_\bullet,$$

i.e., we put $E := \{(F_{\text{head}}(F_{\text{even}}(x)), F_{\text{head}}(x))\}$.

Recall the representation of the final $A \times \text{Id} \times \text{Id}$-coalgebra $(A^{2^*}, \langle h, l, r \rangle)$ from Fact 3.11 and let $j : A^\omega \to A^{2^*}$ be the (injective) coalgebra morphism from $(A^\omega, \langle \text{head}, \text{even}, \text{odd} \rangle)$ into the final coalgebra.

Lemma 5.21 *Using the terminology of Definition 5.10 we have* $P_E = \{t \in A^{2^*} \mid h(t) = h(l(t))\} \subseteq A^{2^*}$ *and* $j[A^\omega] = \Box P_E$*, i.e. ,* $A^\omega \cong \Box \mathbb{P}_E$*. Therefore* $(\mathcal{O}, E)$ *is a complete specification of* A^ω*.*

Proof. The first claim about P_E can be seen to be true by spelling out the definition of P_E. In order to show that $j[A^\omega] = \Box P_E$ we first prove $\Box P_E \subseteq j[A^\omega]$. Let $t \in \Box P_E$. Then it is easy to see that

$$\text{for all} \quad w \in 2^* \quad \text{we have } t_w \in P_E, \text{ i.e., } h(l(t_w)) = h(t_w). \tag{7}$$

We define a stream $\tau \in A^\omega$ by putting $\tau_n := h(t_{\text{bin}(n)})$ for all $n \in \omega$. Our claim is that $j(\tau) = t$. We prove $j(\tau)(w) = t(w)$ for all $w \in 2^*$ by induction on w.

Base case $w = \epsilon$. Then $j(\tau)(\epsilon) = \tau_0 = h(t) = t(\epsilon)$.

Case $w = 0v$. Then

$$\begin{aligned} j(\tau)(0w) = j(\tau)(w) &\overset{\text{I.H.}}{=} t(w) = h(t_w) \\ &\overset{(7)}{=} h(l(t_w)) = h(t_{0w}) = t(0w) \end{aligned}$$

[5] Note that we simply write head, even and odd instead of f_{head}, f_{even} and f_{odd}.

Case $w = 1v$. Then

$$j(\tau)(1w) = \tau_{\mathrm{nat}(1w)} \stackrel{\mathrm{Def.}}{=} h(t_{\mathrm{bin}(\mathrm{nat}(1w))}) = h(t_{1w}) = t(1w)$$

This concludes the proof of $\Box P_E \subseteq j[A^\omega]$. For the converse direction note that obviously $j[A^\omega] \subseteq P_E$. Therefore it suffices to show that $j[A^\omega]$ is an invariant - but this follows from the fact that j is a homomorphism and hence $j[A^\omega]$ is a subcoalgebra of $(A^{2^*}, \langle h, l, r\rangle)$. $\Box$

Now we are ready to concretely describe the stream definition scheme. Given a set of functions symbols Δ, each $g \in \Delta$ with an arity $a(g) \in \mathbb{N}$, the syntax for the definition scheme is defined as above - but now for the special case that $\mathbf{X} = A^\omega$, $\Sigma = \{\mathrm{head}, \mathrm{even}, \mathrm{odd}\}$ and $\mathcal{S} = \{A\}$. Then a *well-formed* system of behavioural differential equations for $(\mathcal{O}, E)$ and Δ is a set Equ of equations which contains for every $g \in \Delta_n$ three equations

$$\begin{aligned} F_{\mathrm{head}}(g(x_1,\ldots,x_n)) &:= c^g(F_{\mathrm{head}}(x_1),\ldots,F_{\mathrm{head}}(x_n)) \\ &\qquad \text{for some function } c : A^n \to A \\ F_{\mathrm{even}}(g(x_1,\ldots,x_n)) &:= t^g_{\mathrm{even}}(x_1,\ldots,x_n) \\ F_{\mathrm{odd}}(g(x_1,\ldots,x_n)) &:= t^g_{\mathrm{odd}}(x_1,\ldots,x_n) \end{aligned}$$

where t^g_{even} and t^f_{odd} are terms in $\mathcal{E}_r$ with variables contained in $\{x_1,\ldots,x_n\}$. Furthermore we require that we can prove for all $g \in \Delta$ the following conditional equation

$$F_{\mathrm{head}}(F_{\mathrm{even}}(g(\vec{x}))) = F_{\mathrm{head}}(g(\vec{x})) \;\Leftarrow\; \mathsf{Equ} \cup \{F_{\mathrm{head}}(F_{\mathrm{even}}(x_i)) = F_{\mathrm{head}}(x_i) \mid x_i \in X\}.$$

By Corollary 5.18 there exists a unique coalgebra morphism

$$\iota : (\mathcal{T}, \langle F_{\mathrm{head}}, F_{\mathrm{even}}, F_{\mathrm{odd}}\rangle) \to (A^\omega, \langle \mathrm{head}, \mathrm{even}, \mathrm{odd}\rangle),$$

i.e., ι makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{T} & \stackrel{\iota}{\dashrightarrow} & A^\omega \\ {\scriptstyle \langle F_{\mathrm{head}}, F_{\mathrm{even}}, F_{\mathrm{odd}}\rangle}\downarrow & & \downarrow{\scriptstyle \langle \mathrm{head},\mathrm{even},\mathrm{odd}\rangle} \\ A \times \mathcal{T} \times \mathcal{T} & \underset{\mathrm{id}\times\iota\times\iota}{\dashrightarrow} & A \times A^\omega \times A^\omega \end{array}$$

Furthermore the function ι can be used in order to compute the unique *solution* for the given set Equ of behavioural differential equations:

Proposition 5.22 *Let* Equ *be a well-formed system of behavioural differential equations for a given set* Δ *of function symbols and let* $\iota : \mathcal{T} \to A^\omega$ *be the coalgebra map that interprets terms* $t \in \mathcal{T}$ *as A-streams. Furthermore we define for every* $a(g)$*-ary function symbol* $g \in \Delta$ *a function* $\hat{g} : (A^\omega)^{a(g)} \to A^\omega$ *by putting* $\hat{g}(\tau_1,\ldots,\tau_{a(g)}) := \iota(g(\underline{\tau_1},\ldots,\underline{\tau_{a(g)}}))$*. Then the family* $\{\hat{g}\}_{g\in\Delta}$ *is the (unique) solution of* Equ*.*

Proof. This is just a special case of Proposition 5.20 above. $\Box$

As an example consider the following definition of the Thue-Morse sequence.

Example 5.23 Let $A = 2$ and $\Delta = \{\mathrm{inv}, \mathrm{TM}\}$. We define Equ to be the following set of equations

$$F_{\text{head}}(\text{inv}(x)) := 1 - F_{\text{head}}(x) \qquad F_{\text{head}}(\text{TM}) := 0$$
$$F_{\text{even}}(\text{inv}(x)) := \text{inv}(F_{\text{even}}(x)) \qquad F_{\text{even}}(\text{TM}) := \text{TM}$$
$$F_{\text{odd}}(\text{inv}(x)) := \text{inv}(F_{\text{odd}}(x)) \qquad F_{\text{odd}}(\text{TM}) := \text{inv}(\text{TM})$$

In order to see that this system of equations is well-formed one can easily check that the following conditional equations are theorems of conditional equational logic

$$F_{\text{head}}(F_{\text{even}}(TM)) = F_{\text{head}}(TM) \quad \Leftarrow \quad F_{\text{even}}(TM) = TM$$
$$F_{\text{head}}(F_{\text{even}}(\text{inv}(x))) = F_{\text{head}}(\text{inv}(x)) \quad \Leftarrow \quad \{F_{\text{head}}(F_{\text{even}}(x)) = F_{\text{head}}(x)\} \cup \mathsf{Equ}$$

The unique solution of this system of equations consists of the function $\text{inv} : 2^\omega \to 2^\omega$ that inverts a given bitstream and of the constant $\text{TM} : 1 \to 2^\omega$ which is the so-called Thue-Morse sequence, i.e., $\text{TM} = t_0t_1t_2\ldots$ with $t_n = s_2(n) \bmod 2$, where $s_2(n)$ denotes the sum of the digits of the binary representation of n.

6 Related and future work

6.1 Connection with Hidden Algebra

One source of inspiration for this paper was the work on hidden algebra (cf. e.g. [9]) and its close connection to coalgebra which has been described in the papers by Cîrstea (cf. [2,3]). A hidden specification consists of a many-sorted algebraic signature Σ, involving *visible* and *hidden* sorts, together with a set of equations that specify certain constraints on the given operations. The notion of a *cobasis* from hidden algebra defines when a given set of operations is "complete". If we think of the operations as ways for obtaining information about elements of hidden sort, completeness means that we can either distinguish two given elements of some hidden sort using the operations of the cobasis, or these elements should be considered to be equal.

Example 6.1 (Sketch) A possible hidden specification for streams over A contains the operations $\{\text{head}, \text{cons}, \text{even}, \text{odd}, \text{tail}, \text{zip}\}$ together with the equations that are to be expected (cf. e.g. [8]). The sorts in this example are Stream and A where Stream is the hidden sort. Possible cobases would be: the set of all operations, the set $\{\text{head}, \text{tail}\}$ and the set $\{\text{head}, \text{even}, \text{odd}\}$. But for example $\{\text{head}, \text{even}\}$ would not be a cobasis.

It follows from the results in [2] that the Σ-coalgebras for a basic cosignature Σ can be seen as hidden algebras. Cobases are closely related to complete sets of co-operations, but these two notions do not coincide: A cobasis is defined for a given specification and hence for *all* hidden algebras (or Σ-coalgebras) that are a model for this specification. In the above example the set of A-streams can be seen as one model of the specification. Thus any cobasis for the specification will give rise to a complete set of co-operations on the set A^ω. Complete sets of co-operations are defined relative to one given set only. In the above example one can easily construct a complete set of co-operations on A^ω that cannot be extended to a cobasis of the above stream specification. Summarising one could say that our definition of a complete set of co-operations is more basic then the notion of a cobasis. Nevertheless it gives rise to interesting coinductive definition and proof principles as we hope to have demonstrated.

6.2 Future Work

We believe that the value of our definition scheme lies in the fact that it is parametric in the type of objects under consideration and in the (complete) given set of co-operations. The generality of our approach, however, has the drawback that for concrete cases, approaches which have been designed explicitly for these cases put less restrictions on the format of a "correct" definition. We are thinking, for example, of the recent work on defining streams and stream functions in [4] where techniques from (infinite) term rewriting are employed. At the moment we are working on making our definition scheme more liberal, mainly by using a refined induction argument for defining the term coalgebra (cf. Def. 5.16). Furthermore we want our definition scheme to be applicable to arbitrary cosignatures, i.e., we want to incorporate arities of co-operations that involve the coproduct. Finally we want to explore in more detail possible differences between different sets of co-operations on a given set of objects. One question is, for example, whether one complete set of co-operations allows to define more or different functions on streams than another one.

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