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Relators and Metric Bisimulations

(Extended Abstract)

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Abstract

Coalgebras of set functors preserving weak pullbacks are particularly well-behaved. Invoking a result by Carboni, Kelly, and Wood (1990), we show that this can be explained by the fact that such functors can be uniquely extended to a *relator*. This insight next suggests a definition of metric bisimulation.

1 Preliminaries

Let $F : Set \rightarrow Set$ be a functor. An *F-coalgebra* is a pair (S, α_S) consisting of a set S and a function $\alpha_S : S \rightarrow F(S)$. Let (S, α_S) and (T, α_T) be two F-coalgebras. A function $f : S \rightarrow T$ is a homomorphism of F-coalgebras, or F-homomorphism, if $F(f) \circ \alpha_S = \alpha_T \circ f$. The collection of F-coalgebras and F-homomorphisms forms a category, denoted by Set_F. A relation $R \subseteq S \times T$ is called an *F-bisimulation* [AM89] between (S, α_S) and (T, α_T) if there exists an F-coalgebra structure $\alpha_R : R \rightarrow F(R)$ on R such that the projections $\pi_1 : R \rightarrow S$ and $\pi_2 : R \rightarrow T$ are F-homomorphisms. The following proposition is readily proved.

Proposition 1.1 *A function $f : S \rightarrow T$ is a homomorphism iff its graph $\phi(f) = \{\langle s, f(s)\rangle \mid s \in S\}$ is a (functional) F-bisimulation.*

As we shall see, the following two observations turn out to be closely related, and are best understood in the world of relators, to be introduced in Section 2.

Theorem 1.2 *A relation $R \subseteq S \times T$ is an F-bisimulation iff for all s in S and t in T,*

$$\langle s, t\rangle \in R \Rightarrow \langle \alpha_S(s), \alpha_T(t)\rangle \in F(\pi_2) \circ F(\pi_1)^{-1},$$

where the latter term denotes the relational composition of the inverse of $F(\pi_1)$ followed by $F(\pi_2)$.

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Theorem 1.3 *If a functor $F : Set \to Set$ preserves weak pullbacks then the category Set_F is particularly well-behaved [Rut96]:*

(i) *The composition of two F-bisimulations is again an F-bisimulation.*

(ii) *If F has a final coalgebra (P, π) then for any F-coalgebra (S, α_S), the by finality unique F-homomorphism $f : (S, \alpha_S) \to (P, \pi)$ identifies precisely those elements of S that are bisimilar: $f(s) = f(s')$ iff there exists an F-bisimulation R on (S, α_S) with $\langle s, s' \rangle \in R$.*

2 Relators

Let *Rel* be the category with sets as objects and relations as arrows. Identity arrows are given, for any set S, by $\Delta_S = \{\langle s, s \rangle \mid s \in S\}$. Composition is given by the usual relational composition, which we shall write in the same order as composition of functions; i.e., for relations $R \subseteq S \times T$ and $Q \subseteq T \times U$,

$$Q \circ R = \{\langle s, u \rangle \in S \times U \mid \exists t \in T,\ \langle s, t \rangle \in R \text{ and } \langle t, u \rangle \in Q\}.$$

A functor $\mathcal{F} : Rel \to Rel$ is called a *relator* [1]. Notably, relators preserve identities and composition: $\mathcal{F}(\Delta_S) = \Delta_{\mathcal{F}(S)}$, and $\mathcal{F}(Q \circ R) = \mathcal{F}(Q) \circ \mathcal{F}(R)$.

A well-known fact is that the category *Set* can be embedded into the category *Rel* by the functor $\phi : Set \to Rel$ which is the identity on sets and maps a function $f : S \to T$ to its graph $\phi(f)$. We say that a functor $\mathcal{F} : Rel \to Rel$ *extends* a functor $F : Set \to Set$ if the following two conditions are satisfied:

(i) For all sets S, $\mathcal{F}(S) = F(S)$.

(ii) For all functions $f : S \to T$, $\mathcal{F}(\phi(f)) = \phi(F(f))$.

In other words, on the subcategory *Set* of *Rel*, the functor $\mathcal{F}$ essentially behaves like F. One can easily prove that extensions are unique, and are locally monotone: if $R \subseteq Q$ then $\mathcal{F}(R) \subseteq \mathcal{F}(Q)$. Often the extension $\mathcal{F}$ will be denoted again by F.

Many functors on *Set* have a (unique) extension to *Rel*. The following theorem, due to Carboni, Kelly, and Wood, makes precise which ones.

Theorem 2.1 *[CKW90]*
A functor $F : Set \to Set$ can be extended to a relator $\mathcal{F} : Rel \to Rel$ iff the functor F preserves weak pullbacks. In that case, the extension $\mathcal{F}$ is given, for any relation $R \subseteq S \times T$ with projections $\pi_1 : R \to S$ and $\pi_2 : R \to T$, by

$$\mathcal{F}(R) = F(\pi_2) \circ F(\pi_1)^{-1}.$$

(Equivalently, $\mathcal{F}(R)$ is the image of the function $\langle F(\pi_1), F(\pi_2) \rangle : F(R) \to F(S) \times F(T)$.)

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[1] The notion of relator has been defined by different authors in different ways. Relators also play a role in certain algebraic approaches to programming; see [BdM97] and the references therein.

In fact, [CKW90] contains the following more general version of the theorem above: Any endofunctor F on a regular category can be extended to the corresponding category of relations iff 1. F preserves strong epimorphisms, and 2. F nearly preserves pullbacks. The above theorem on *Set* is an immediate consequence of this more general statement, since condition 1 is trivially fulfilled for any endofunctor on *Set*, and condition 2 is equivalent to the preservation of weak pullbacks.

Example 2.2 *The definitions of the following relators, which are all stemming from weak-pullback-preserving functors, follow from Theorem 2.1. Let A be an arbitrary set and R a relation on sets S and T.*

(i) *The relator $A \times (-) : Rel \to Rel$ is defined on sets as usual and on relations by*

$$(\langle a, s\rangle, \langle b, t\rangle) \in A \times R \text{ iff } a = b \text{ and } \langle s, t\rangle \in R.$$

(ii) *For $A \to (-) : Rel \to Rel$, we have*

$$\langle f, g\rangle \in A \to R \text{ iff } \forall a \in A, \langle f(a), g(a)\rangle \in R.$$

(iii) *The powerset functor extends to a relator $\mathcal{P} : Rel \to Rel$ assigning to a set the collection of all its subsets, and to a relation R the following relation:*

$$\begin{aligned}\langle V, W\rangle \in \mathcal{P}(R) \text{ iff } & \forall v \in V\, \exists w \in W, \langle v, w\rangle \in R \quad \text{and} \\ & \forall w \in W\, \exists v \in V, \langle v, w\rangle \in R.\end{aligned}$$

3 Coalgebras of relators

In this section, let F be a functor that preserves weak pullbacks and hence extends to a relator, again denoted by F. Let the category Rel_F be defined as follows. Objects are, as before, coalgebras (S, α_S), consisting of a set S and a *function* $\alpha_S : S \to F(S)$. Arrows between coalgebras (S, α_S) and (T, α_T) are *relations* $R \subseteq S \times T$ such that

$$\langle s, t\rangle \in R \Rightarrow \langle \alpha_S(s), \alpha_T(t)\rangle \in F(R).$$

This definition generalizes an earlier definition of Hermida and Jacobs [HJ98] for polynomial functors to the case of arbitrary (weak-pullback-preserving) functors. (See also [Len98].)

The following is immediate by Theorem 1.2 and Theorem 2.1.

Corollary 3.1 *Let (S, α_S) and (T, α_T) be two F-coalgebras. A relation $R \subseteq S \times T$ is an F-bisimulation iff $R : (S, \alpha_S) \to (T, \alpha_T)$ is an arrow in Rel_F.*

The above corollary establishes an equivalence between the definition of F-bisimulation in the style of Aczel and Mendler, on the one hand, and a definition of bisimulation in the style of Milner and Park [Mil80,Par81], on the other. For a prototypical example, consider the relator $\mathcal{P}(A \times -) : Rel \to Rel$, the coalgebras of which are (A-)labelled transition systems. Writing $s \xrightarrow{a} s'$ for $\langle a, s'\rangle \in \alpha_S(s)$, for a coalgebra (S, α_S) of $\mathcal{P}(A \times -)$, we have:

R is a $\mathcal{P}(A \times -)$-bisimulation
iff R is an arrow in the category $Rel_{\mathcal{P}(A\times -)}$
iff $\langle s,t\rangle \in R \Rightarrow \langle \alpha_S(s),\, \alpha_T(t)\rangle \in \mathcal{P}(A \times R)$
iff $\langle s,t\rangle \in R \Rightarrow \forall s \xrightarrow{a} s'\ \exists t' \xleftarrow{a} t,\ \langle s',t'\rangle \in R$
and $\forall t \xrightarrow{a} t'\ \exists s' \xleftarrow{a} s,\ \langle s',t'\rangle \in R.$

There is also the following corollary of Corollary 3.1 and Proposition 1.1.

Corollary 3.2 *The embedding $\phi : Set \to Rel$ can be extended to an embedding $\phi : Set_F \to Rel_F$.*

We briefly return to the second observation (Theorem 1.3) of Section 1. The fact that Set_F is well-behaved for functors F that preserve weak pullbacks is best explained by the fact that for such functors, Set_F is in essence a subcategory of Rel_F. For instance, the fact that the composition of bisimulations is again a bisimulation is equivalent to the observation that Rel_F indeed is a category, that is, Rel_F is closed under composition of arrows. Working in the 'larger' category Rel_F has some additional advantages. For instance, the greatest bisimulation relation (bisimilarity) on an F-coalgebra (S, α_S) can be characterized, as in the original definitions of Milner and Park, as the largest (post-)fixpoint of the following monotone operator $\Lambda : \mathcal{P}(S \times S) \to \mathcal{P}(S \times S)$, defined, for $R \subseteq S \times S$, by

$$\langle s, s'\rangle \in \Lambda(R) \text{ iff } \langle \alpha_S(s),\, \alpha_S(s')\rangle \in F(R).$$

Using Λ, the *coinduction* proof principle for a final F-coalgebra (P, π), looks like

$$\Delta_P = \bigcup \{R \mid R \subseteq \Lambda(R)\}.$$

4 Metric bisimulations

In this section, we sketch how to extend the framework of sets and relations to metric spaces, which are, in some sense, more expressive. We give a few examples of dynamical systems on metric spaces X:

(i) One-dimensional discrete-time dynamical systems, which are continuous functions $f : X \to X$.

(ii) Contractions $f : \mathcal{P}_c(X) \to \mathcal{P}_c(X)$ on the space of compact subsets of X are of particular interest, since their unique fixed points can be used to model fractals.

(iii) Probabilistic transition systems can be modelled by continuous functions $f : X \to \mathcal{M}(X)$, which map a state x in X to a Borel measure on X.

Taking Lawvere's [Law73] enriched-categorical view on metric spaces, the categories of interest are *Met* and *MRel*. The objects in *Met* are (generalized) metric spaces X, which consist of a set X and a distance function d_X taking values in $[0, \infty]$; arrows are non-expansive functions. (See [BBR98] for a detailed account of generalized metric spaces.) The category *MRel* has the same objects, and arrows are what we shall call *metric relations*, also called bimodules or distributors. A metric relation $\Phi : X \Rightarrow Y$ is a non-expansive function $\Phi : Y^{op} \times X \to [0,\infty]$ [2]. A metric relation Φ can be thought of as a fuzzy relation; it measures for any y and x the amount $\Phi(y,x)$ to which these elements are related by Φ ('the smaller this number the better'). The composition $\Psi \circ \Phi : X \Rightarrow Z$ of metric relations $\Phi : X \Rightarrow Y$ and $\Psi : Y \Rightarrow Z$ is given by the following 'least cost' composition:

$$\Psi \circ \Phi(z,x) = \inf_{y \in Y} \{\Phi(y,x) + \Psi(z,y)\}.$$

The distance function $d_X : X^{op} \times X \to [0,\infty]$ on a metric space X is an identity for this composition.

As with sets and relations, there exists an embedding functor $G : Met \to MRel$, which is the identity on metric spaces, and maps a non-expansive function $f : X \to Y$ to its fuzzy graph, defined by $G(f)(y,x) = d_Y(y, f(x))$.

A *metric relator* is now any endofunctor on the category *MRel*. As before, we say that a functor $\mathcal{F} : MRel \to MRel$ extends a functor $F : Met \to Met$ if it commutes with the functor $G : Met \to MRel$: $\mathcal{F} \circ G = G \circ F$. The guiding principle in extending a functor F on *Met* to a functor $\mathcal{F}$ on *MRel* is the fact that F is already defined on one particular kind of metric relation: F assigns to any set and distance function again a set and a distance function. Distance functions are identity metric relations, so we already know what $\mathcal{F}$ should look like on those. And this can often be easily generalized to arbitrary metric relations.

This may be illustrated by the following example. Let $\mathcal{P} : Met \to Met$ be defined as the powerset functor on metric spaces, assigning to a metric space X the collection of subsets of X, with the familiar Hausdorff distance:

$$d_{\mathcal{P}(X)}(V,W) = \max\{\sup_{v \in V} \inf_{w \in W} d_X(v,w),\ \sup_{w \in W} \inf_{v \in V} d_X(v,w)\}.$$

Now $\mathcal{P}$ can be extended to *MRel* by defining, for any metric relation $\Phi : X \Rightarrow Y$, a relation $\mathcal{P}(\Phi) : \mathcal{P}(X) \Rightarrow \mathcal{P}(Y)$, given by

$$\mathcal{P}(\Phi)(V,W) = \max\{\sup_{v \in V} \inf_{w \in W} \Phi(v,w),\ \sup_{w \in W} \inf_{v \in V} \Phi(v,w)\}.$$

This definition generalizes in a very precise sense the definition of the powerset relator (read 'max' as 'and', 'sup' as 'for all', and 'inf' as 'there exists').

Let $F : Met \to Met$ be a functor with a (unique) extension, denoted by the same symbol, $F : MRel \to MRel$. The category $MRel_F$ is defined as follows: objects are F-coalgebras (X, α_X), consisting of a metric space X and a non-expansive function $\alpha_X : X \to F(X)$. Arrows in $MRel_F$ are *metric bisimulations*: given two metric F-coalgebras (X, α_X) and (Y, α_Y), a metric F-bisimulation is a metric relation $\Phi : X \Rightarrow Y$ satisfying

$$F(\Phi)(\alpha_Y(y),\ \alpha_X(x)) \le \Phi(y,x).$$

[2] The space Y^{op} is like Y but with distance $d_{Y^{op}}(y,y') = d_Y(y',y)$. Note that generalized distances are generally non-symmetric.

Examples of metric bisimulations are bismulations 'up-to-depth-n'. An early example can be found in [TR98].

The goal will now be to develop a theory of metric coalgebras and bisimulations along the lines of the theory of (universal) coalgebra for sets. As an example of a theorem in such a theory, there is the following principle of *metric coinduction*.

Theorem 4.1 *Under reasonable conditions*[3] *on the functor F, the distance function d_P of a final F-coalgebra (P, π) satisfies:*

$$d_P = \inf \{\Phi \mid \Phi \text{ is a metric bisimulation on } (P, \pi)\}.$$

Again, this generalizes in a precise sense the coinduction principle for final coalgebras of endofunctors on *Set* (formulated at the very end of Section 3).

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[3] The functor F should preserve (forward) completeness, and be locally non-expansive and continuous.

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