Explicit lower and upper bounds on the entangled value of multiplayer XOR games

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Abstract

The study of quantum-mechanical violations of Bell inequalities is motivated by the investigation, and the eventual demonstration, of the nonlocal properties of entanglement. In recent years, Bell inequalities have found a fruitful re-formulation using the language of multiplayer games originating from Computer Science. This paper studies the nonlocal properties of entanglement in the context of the simplest such games, called XOR games. When there are two players, it is well known that the maximum bias — the advantage over random play — of players using entanglement can be at most a constant times greater than that of classical players. Recently, Pérez-García et al. [Comm. Math. Phys. 279 (2), 2008] showed that no such bound holds when there are three or more players: the use of entanglement can provide an unbounded advantage, and scale with the number of questions in the game. Their proof relies on non-trivial results from operator space theory, and gives a non-explicit existence proof, leading to a game with a very large number of questions and only a loose control over the local dimension of the players' shared entanglement.

We give a new, simple and explicit (though still probabilistic) construction of a family of three-player XOR games which achieve a large quantum-classical gap (QC-gap). This QC-gap is exponentially larger than the one given by Pérez-García et. al. in terms of the size of the game, achieving a QC-gap of order $\sqrt{N}$ with N^2 questions per player. In terms of the dimension of the entangled state required, we achieve the same (optimal) QC-gap of $\sqrt{N}$ for a state of local dimension N per player. Moreover, the optimal entangled strategy is very simple, involving observables defined by tensor products of the Pauli matrices.

Additionally, we give the first upper bound on the maximal QC-gap in terms of the number of questions per player, showing that our construction is only quadratically off in that respect. Our results rely on probabilistic estimates on the norm of random matrices and higher-order tensors which may be of independent interest.

1   Introduction

Multiplayer games, already a very successful abstraction in theoretical computer science, were first proposed as an ideal framework in which to study the nonlocal properties of entanglement

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by Cleve et al. [CHTW04]. Known as *nonlocal*, or *entangled*, games, they can be thought of as an interactive re-framing of the familiar setting of Bell inequalities [Bel64]: a referee (the experimentalist) interacts with a number of players (the devices). The referee first sends a classical question (a setting) to each player. The players are all-powerful (there is no restriction on the shared state or the measurements applied) but not allowed to communicate: each of them must make a local measurement on his or her part of a shared entangled state, and provide a classical answer (the outcome) to the referee's question. The referee then decides whether to accept or reject the players' answers (he evaluates the Bell functional).

In their paper, Cleve et al. gave an in-depth study of the simplest class of multiplayer games, *two-player XOR games*. The XOR property refers to the fact that in such games each player answers with a single bit, and the referee's acceptance criterion only depends on the parity of the bits he receives as answers. One of the most fundamental Bell inequalities, the CHSH inequality [CHSH69], fits in this framework. In the corresponding XOR game the acceptance criterion dictates that the parity of the players' answers must equal the product of their questions, a uniform i.i.d. bit each. The laws of quantum mechanics predict that the CHSH game has the following striking property: there is a quantum strategy in which the players share a simple entangled state — a single EPR pair — and use it to achieve a strictly higher success probability than the best classical, unentangled strategy: roughly 85%, as compared to 75%. This example demonstrates that quantum mechanics is *nonlocal*: predictions made by the theory cannot be reproduced classically, or more generally by any local hidden variable model, a "paradox" most famously put forward by Einstein, Podolsky and Rosen [EPR35].

Any XOR game G can be won with probability $1/2$ by players who independently answer each question with the outcome of a random coin flip. It is therefore natural to measure the success of quantum (resp. classical) players through their maximum achievable *bias* $\beta^*(G)$ (resp. $\beta(G)$), defined as their maximum[1] winning probability in the game, *minus* the success probability that would be achieved by random play. As has become standard practice, we will measure the advantage of quantum over classical players through the ratio $\beta^*(G)/\beta(G)$, referred to as the quantum-classical gap, or *QC-gap* for short.[2] The CHSH example demonstrates the existence of a game for which $\beta^*(G) \geq \sqrt{2}\beta(G)$, and Tsirelson [Tsi87] proved that this gap was close to best possible. By making a connection to the celebrated *Grothendieck inequality* he showed that for any two-player XOR game G, we have $\beta^*(G)/\beta(G) \leq K_G^{\mathbb{R}}$, where $K_G^{\mathbb{R}}$ is the real Grothendieck constant.[3] The exact value of $K_G^{\mathbb{R}}$ is unknown, and the best upper bound currently known, $K_G^{\mathbb{R}} \lesssim 1.78$, appeared in recent work of Braverman et. al. [BMMN11]. Although experiments based on the CHSH game have been performed [AGR81, ADR82], the relatively small gap forces the use of state-of-the-art devices in terms of precision and timing in order to differentiate a truly nonlocal strategy from one that can be explained by local hidden-variable models. In order to observe larger quantum-

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[1]In this paper we only consider players who are allowed to use entanglement, and observables, of arbitrarily large but *finite* dimensions.

[2]See Section 5 for a brief discussion of other ways of measuring the quantum advantage, such as through the difference $\beta^*(G) - \beta(G)$.

[3]The subscript G in $K_G^{\mathbb{R}}$ stands for "Grothendieck", and is not related to the game G!

classical gaps, more general classes of games need to be considered, prompting a question that has driven much recent research in this area: *For a given QC-gap, what is the simplest game (in terms of the number of players, questions and answers) which demonstrates such a QC-gap (if one at all exists)?*

There are two main directions in which one can look for generalizations of two-player XOR games. The first is to increase the number of possible answers from each player. This option has so far been the preferred one, and has by now been relatively well explored [CHTW04, KRT10, JPPG+10, JP11, Reg11, BRSdW11]. In particular it is known that the largest possible quantum-classical gap is bounded by a constant times the minimum of the number of questions, the number of answers, and the local dimension of the players [JPPG+10], and there are explicit constructions of games (i.e., games whose existence is proved through a constructive proof) which come close to achieving these bounds [BRSdW11]. Unfortunately, these games require the players to perform complex measurements, involving large numbers of outcomes, making them ill-suited to experiment.

The second possible avenue for generalization consists of increasing the number of players, while remaining in the simple setting of binary answers and an XOR-based acceptance criterion. Our limited understanding of multipartite entanglement makes this setting more challenging, and for a long time little more than small, constant-size examples were known [Mer90, Zuk93]. However, recently, Pérez-García et al. [PGWP+08] discovered that adding even just one player allowed for a very different scaling of the QC-gap. They demonstrated the existence of an infinite family of three-player XOR games $(G_N)_{N \in \mathbb{N}}$ for which $\lim_{N \to \infty} \beta^*(G_N)/\beta(G_N) = +\infty$ — an unbounded gap! This exciting result demonstrated for the first time that very large violations could be observed even in the relatively simple context of three-player XOR games.

The results in [PGWP+08] were proved by establishing a surprising connection between XOR games and certain natural norms on the tensor product of operator spaces, enabling the authors to leverage powerful techniques from the latter area to establish their results on XOR games. Since their seminal paper, similar techniques have been successfully applied to other settings, such as general two-player games [JPPG+10] and games with quantum communication [CJPPG11].

For the games G_N from [PGWP+08], however, the above-mentioned techniques have a few somewhat unfortunate consequences. First of all, these techniques resulted in a highly non-explicit existence proof. While Pérez-García et al. show the *existence* of the games, it seems quite hard to even get the slightest idea of what the games would look like. Moreover, their use of the theory of operator spaces gives a very large game, with an exponential (in the QC-gap) number of questions per player. Finally, the strategies required for the players to achieve the promised QC-gap are not explicitly known, and may for instance require an entangled a state with unbounded dimension on two of the players; only the first player's dimension is controlled. We note that after the completion of our work, but independently from it, Pisier [Pis12a, Pis12b] showed that the construction in [PGWP+08] could be improved to require only a polynomial number of questions to each player, and that one could keep a control of the entanglement dimension on all three players. The resulting parameters, however, are still worse than the ones that we achieve here.

1.1 Our results

In this paper we give a new and improved proof of the existence of a family of three-player XOR games for which the QC-gap is unbounded. In turn, this implies the existence of tripartite Bell correlation inequalities that exhibit arbitrarily large (as the number of settings per site is allowed to grow) violations in quantum mechanics. Our proof technique uses the probabilistic method: we describe a simple *probabilistic* procedure that outputs a game with the desired properties *with high probability*. As such it is much more explicit than previous results [PGWP+08], albeit not fully constructive. Our construction is outlined in Section 1.2 below. For a desired ratio $\sqrt{N}$, our game has order N^2 questions per player, which, as we show, is within a factor $\tilde{O}(N)$ of the smallest number possible.[4] Moreover, to achieve such a gap entangled players only need to use Pauli observables and an entangled state of local dimension N per player. The simplicity of our construction enables us to give concrete values for most of the parameters, leading to a rigorous control of the constants involved. We prove the following:

Theorem 1. *For any integer n and $N = 2^n$ there exists a three-player XOR game G_N, with N^2 questions per player, such that $\beta^*(G_N) \geq \Omega(\sqrt{N} \log^{-5/2} N) \beta(G_N)$. Moreover, there is an entangled strategy which achieves a bias of $\Omega(\sqrt{N} \log^{-5/2} N) \beta(G_N)$, uses an entangled state of local dimension N per player, and in which the players' observables are tensor products of n Pauli matrices.*

Additionally, we prove that the dependence of the QC-gap on the number of questions obtained in Theorem 1 is close to optimal.[5] This improves upon an independent previous result by Loubenets [Lou12], who showed that $\beta^*(G) \leq (2Q-1)^2 \beta(G)$.

Theorem 2. *For any 3-player XOR game G in which there are at most Q possible questions to the third player,*

$$\beta^*(G) \leq \sqrt{Q} K_G^{\mathbb{R}} \beta(G),$$

where $K_G^{\mathbb{R}} < 1.783$ is the real Grothendieck constant.

Finally, we also show that the dependence on the local dimension of the entangled state is optimal, re-proving in a simpler language a result first proved in [PGWP+08].

Theorem 3. *Let G be a 3-player XOR game in which the maximal entangled bias $\beta^*(G)$ is achieved by a strategy in which the third player's local dimension is d. Then*

$$\beta^*(G) \leq 3\sqrt{2d} \left(K_G^{\mathbb{C}}\right)^{3/2} \beta(G),$$

where $K_G^{\mathbb{C}} < 1.405$ is the complex Grothendieck constant.

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[4]As has become customary, we use the $\tilde{O}$, $\tilde{\Omega}$ notation to designate bounds that ignore possible poly-logarithmic factors, e.g. $\tilde{O}(N) = O(N \log^c N)$ for some constant c independent of N.

[5]A similar result was recently communicated to us by Carlos Palazuelos [Pal11].

Generalizations   While we present our results in the case of three-player XOR games, they have straightforward extensions to an arbitrary number of players. In particular, one can show that the following holds, for any $r \geq 3$:

1. For any integer N that is a power of 2, there exists a r-player XOR game G, with N^2 questions per player, such that $\beta^*(G) \geq \Omega\big((N \log^{-5} N)^{(r-2)/2}\big)\beta(G)$, and there is a entangled strategy achieving this gap that involves only N-dimensional Pauli observables.

2. If G is a r-player XOR game in which at least $r-2$ of the players have at most Q possible questions each, then $\beta^*(G) \leq O(Q^{(r-2)/2})\beta(G)$.

3. If G is a r-player XOR game in which the shared state of the players is restricted to have local dimension d on at least $r-2$ of the players, then $\beta^*(G) \leq O\big(d^{(r-2)/2}\big)\beta(G)$.

Applications to operator space theory   The paper [PGWP$^+$08] was, to the best of our knowledge, the first to introduce methods originating in functional analysis (more specifically, operator space theory) to the analysis of Bell inequalities. The results in that paper also had consequences in functional analysis itself: the existence of very large violations of tripartite Bell inequalities by Quantum Mechanics implies in turn that a certain trilinear extension of Grothendieck's inequality does not hold. Our construction leads to an improved obstruction: the three-player XOR game constructed in Theorem 1 can be used to show that two different norms are not equivalent on the space of trilinear forms on $\ell_\infty \times \ell_\infty \times \ell_\infty$. More precisely, Theorem 1 implies that for any $N = 2^n$ there is a trilinear form $T : \ell_\infty^{N^2} \times \ell_\infty^{N^2} \times \ell_\infty^{N^2} \to \mathbb{C}$ such that

$$\|T\|_{\mathrm{cb}} \ \geq \ \Omega\big(\sqrt{N} \log^{-5/2} N\big) \|T\|, \tag{1}$$

and the Nth amplifications in the completely bounded norm suffice. Put differently, the injective and minimal tensor norms are inequivalent on $\ell_1 \otimes \ell_1 \otimes \ell_1$. This improves on the estimate from [PGWP$^+$08], in which the bound was logarithmic in N. For more details and background on relevant aspects of Grothendieck's inequality we refer to the excellent survey [Pis12a], and to Section 20 in particular for the connection with XOR games.

In addition, Pisier [Pis12b] recently applied our result to prove an almost-tight estimate on the norm of the re-ordering map

$$J : \ (H_1 \otimes_2 K_1) \otimes_\epsilon \cdots \otimes_\epsilon (H_r \otimes_2 K_r) \ \to \ (H_1 \otimes_2 \cdots \otimes_2 H_r) \otimes_\epsilon (K_1 \otimes_2 \cdots \otimes_2 K_r), \tag{2}$$

where H_i, K_i are N-dimensional Hilbert spaces, proving that $\|J\| = \tilde{\Omega}\big(N^{r-1}\big)$.

1.2   Proof overview and techniques

Lower bound   Our construction of a three-player XOR game G_N proceeds through two independent steps. In the first step we assume given a 3-tensor $T = T_{(i,i'),(j,j'),(k,k')}$ of dimension $N^2 \times N^2 \times N^2$, where N is a power of 2. Based on T, we define a three-player XOR game $G_N =$

$G(T)$. Questions in this game are N-dimensional Pauli matrices P, Q, R, and the corresponding game coefficient[6] is defined as

$$G(P, Q, R) = \langle T, P \otimes Q \otimes R \rangle := \sum_{(i,i'),(j,j'),(k,k')} T_{(i,i'),(j,j'),(k,k')} \overline{P_{i,i'} Q_{j,j'} R_{k,k'}}.$$

This definition results in a game whose entangled and classical biases can be directly related to *spectral properties* of the tensor T. On the one hand we show that the classical bias $\beta(G_N)$ reflects the tripartite structure of T, and is upper-bounded by the norm of T as a trilinear operator. On the other hand we show that the entangled bias $\beta^*(G_N)$ is *lower-bounded* by the norm of T *as a matrix* — a bilinear operator on N^3-dimensional vectors, obtained by pairing up the indices (i, j, k) and (i', j', k'). This new connection reduces the problem of constructing a game with large QC-gap to constructing a tensor T with appropriate spectral properties.

The second step of the proof is our main technical contribution. We give a probabilistic construction of a 3-tensor T having large norm when seen as a bilinear operator (giving a large entangled bias), but low norm when seen as a trilinear operator (giving a low classical bias). To this end, we simply take T to correspond to an (almost) rank-1 matrix: letting (g_{ijk}) be a random N^3-dimensional vector with i.i.d. entries distributed as standard Gaussians,[7] the $(i, i'), (j, j'), (k, k')$-th entry of T is $g_{ijk} g_{i'j'k'}$ if $i \neq i'$, $j \neq j'$ and $k \neq k'$, and 0 otherwise. The fact that T, when seen as a matrix, is close to having rank 1 makes it easy to lower bound its spectral norm. An upper bound on the norm of T as a trilinear operator is proved in two steps. In the first step we apply a concentration bound due to Latała to show that for any fixed Hermitian X, Y, Z with Frobenius norm at most 1, the product $|\langle T, X \otimes Y \otimes Z \rangle|$ is highly concentrated around its expected value, where the concentration is over the random choice of T. We then conclude by a union bound, using a delicate ε-net construction based on a decomposition of Hermitian matrices with Frobenius norm at most 1 as linear combinations of (normalized, signed) projectors.

Upper bounds   We prove upper bounds on the largest possible QC-gap achievable by any three-player XOR game, both as a function of the local dimension of an optimal strategy, and of the number of questions per player in the game. Both bounds follow the same overall proof strategy: using a decoupling argument, we show that the third player can be restricted to applying a classical strategy while incurring only a bounded factor loss in the bias. We conclude by applying (the easy direction of) Tsirelson's Theorem and Grothendieck's inequality (see Section 2.6) to show that the first two players can be made classical at a further loss of a constant factor only.

Organization of the paper   We start with some preliminaries in Section 2. We describe our construction of a game with unbounded QC-gap in Section 3. Our upper bounds on the QC-gap as a function of the number of questions and the local dimension are proved in Section 4. We conclude with some open questions in Section 5.

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[6]The equation below defines a complex number. Taking its real or imaginary part would result in a Bell functional, which can in turn easily be transformed into an XOR game through a proper normalization.

[7]Our results also hold with the Gaussians replaced by i.i.d. standard Bernoulli random variables.

2 Preliminaries

This section is devoted to some preliminary definitions and results that will be useful in order to prove our main theorems. We start by setting some notation in Section 2.1. In Section 2.2 we introduce 3-dimensional tensors, and two norms on such tensors that will be central to our results, the $\|\cdot\|_{3,3}$ and the $\|\cdot\|_{2,2,2}$ norms. In Section 2.3 we recall the standard definitions of XOR games and the associated biases. In Section 2.4 we introduce a (previously known) construction of an ε-net over Hermitian matrices (with the Frobenius norm), and in Section 2.5 we recall some strong concentration bounds. Both will be used in combination in the proof of our main theorem, Theorem 1. Finally, in Section 2.6 we state Grothendieck's inequality, which is used in the proofs of both Theorem 2 and Theorem 3.

2.1 Notation

For a positive integer N we define $[N] := \{1, \ldots, N\}$. For a positive integer K we denote by $[N]^K$ the Cartesian product of the set $[N]$ with itself K times (i.e., $[N] \times \cdots \times [N]$).

For a subspace $\mathcal{W} \subseteq \mathcal{V}$ of a normed vector space $(\mathcal{V}, \|\cdot\|)$ we let $S(\mathcal{W}) := \{X \in \mathcal{W} : \|X\| = 1\}$ be the unit sphere, $B(\mathcal{W}, \tau) := \{X \in \mathcal{W} : \|X\| \leq \tau\}$ the ball of radius τ and $B(\mathcal{W}) := B(\mathcal{W}, 1)$ the unit ball. We let $\|\cdot\|_2$ denote the usual Euclidean norm. Throughout we endow $\mathbb{C}^N$ with this norm.

We will usually use $g \sim \mathcal{N}(0,1)$ to denote a real-valued random variable distributed according to a standard normal (Gaussian) distribution (i.e., a variable with mean 0 and variance 1), and $|g\rangle \sim \mathcal{N}(0,1)^N$ for an N-dimensional vector whose entries are i.i.d. standard normal random variables.

Matrices Throughout $\mathcal{H}$ will denote a N-dimensional complex Hilbert space. We identify the set of linear operators $L(\mathcal{H})$ on $\mathcal{H}$ with the set $\mathrm{Mat}(N)$ of complex N-by-N matrices. Let $\mathrm{Herm}(\mathcal{H}) = \{X \in L(\mathcal{H}) : X^\dagger = X\}$ be the subset of Hermitian operators and $\mathrm{Obs}(\mathcal{H}) \subseteq \mathrm{Herm}(\mathcal{H})$ be the Hermitian operators with all eigenvalues in $\{-1, 1\}$. In other words, $\mathrm{Obs}(\mathcal{H})$ is the set of $\{-1, 1\}$-valued observables on $\mathcal{H}$. Note that operators in $\mathrm{Obs}(\mathcal{H})$ are unitary and square to the identity. We will use the notation $\mathrm{Herm}(N)$ and $\mathrm{Obs}(N)$ when we think of the operators' matrix representation. The space of matrices $\mathrm{Mat}(N)$ is a Hilbert space for the inner product $(A, B) \mapsto \langle A, B \rangle := \mathrm{Tr}(AB^\dagger)$. The resulting norm is the *Frobenius norm* $A \mapsto \|A\|_F := \sqrt{\mathrm{Tr}(AA^\dagger)}$. Throughout we tacitly endow $\mathrm{Mat}(N)$ with the Frobenius norm, so the balls and sphere are always defined with respect to this norm. Note that if we let the singular values of a matrix $X \in \mathrm{Mat}(N)$ be $\sigma_1(X) \geq \cdots \geq \sigma_N(X)$, then $\|X\|_F^2 = \sigma_1(X)^2 + \cdots + \sigma_N(X)^2$. We recall that for each eigenvalue λ of a Hermitian matrix X there is a corresponding singular value $\sigma = |\lambda|$. We denote by $\|\cdot\|_\infty = \sigma_1(X)$ the operator norm on $\mathrm{Mat}(N)$. Let $\mathrm{Proj}(N)_k \subseteq \mathrm{Herm}(N)$ be the set of rank-k (orthogonal) projectors on $\mathbb{C}^N$ and let $\overline{\mathrm{Proj}}(N)_k = \mathrm{Proj}(N)_k / \sqrt{k}$ be the set of rank-k projectors that are normalized with respect to the Frobenius norm. Define the set of all N-dimensional normalized projectors by $\overline{\mathrm{Proj}}(N) = \bigcup_{k=1}^N \overline{\mathrm{Proj}}(N)_k$.

If $N = 2^n$ for some positive integer n, we let $\mathcal{P}_N := \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \right\}^{\otimes n}$ be the set of n-fold tensor products of Pauli matrices. The letters P, Q, R will usually denote elements of $\mathcal{P}_N$. We have $|\mathcal{P}_N| = N^2$, and for $P, Q \in \mathcal{P}_N$ we have $\langle P, Q \rangle = N \delta_{P,Q}$: the set $\mathcal{P}_N$ forms an orthogonal basis of observables for $\mathrm{Mat}(N)$.

2.2 Tensors

Given positive integers $r, N_1, \ldots, N_r$, an *r-tensor of dimensions* $N_1 \times \cdots \times N_r$ is a map of the form $T : [N_1] \times \cdots \times [N_r] \to \mathbb{C}$. Every element $T(i_1, \ldots, i_r)$ of such a tensor is specified by an r-tuple of indices $(i_1, \ldots, i_r) \in [N_1] \times \cdots \times [N_r]$. We will mostly deal with 3-tensors of dimensions $N^2 \times N^2 \times N^2$ for some $N \in \mathbb{N}$. In this case we index the elements by three pairs of indices $(i, i'), (j, j')$ and $(k, k') \in [N]^2$. We will think of such a tensor in two different ways: as a *bilinear* functional acting on N^3-dimensional complex vectors, and as a *trilinear* functional acting on Hermitian $N \times N$ matrices. For the sake of concreteness we now describe in detail how these two perspectives relate to each other.

The bilinear view Let T be a 3-tensor of dimension $N^2 \times N^2 \times N^2$. The dimensions of the tensor T allow us to view it as an N^3-by-N^3 complex matrix. Correspondingly, we define the spectral norm of T by

$$\|T\|_{3,3} := \max_{x,y \in S(\mathbb{C}^{N^3})} \left| \sum_{(i,j,k),(i',j',k') \in [N]^3} T_{(i,i'),(j,j'),(k,k')} x_{i,j,k} y_{i',j',k'} \right|.$$

Suppose that for some $n \in \mathbb{N}$, we have $N = 2^n$. Since the set $\mathcal{P}_{N^3} = \{X \otimes Y \otimes Z : X, Y, Z \in \mathcal{P}_N\}$ is an orthogonal basis for $\mathrm{Mat}(N^3)$, we can define the "Fourier coefficient" of T at (P, Q, R) as

$$\widehat{T}(P, Q, R) := \langle T, P \otimes Q \otimes R \rangle = \sum_{(i,i'),(j,j'),(k,k') \in [N]^2} T_{(i,i'),(j,j'),(k,k')} \overline{P_{i,i'} Q_{j,j'} R_{k,k'}}.$$

With this definition, T can be written as

$$T = N^{-3} \sum_{P,Q,R \in \mathcal{P}_N} \widehat{T}(P, Q, R) P \otimes Q \otimes R.$$

The trilinear view Let T be a 3-tensor of dimensions $N^2 \times N^2 \times N^2$. We can associate with T a trilinear functional $L_T : \mathrm{Herm}\,(N) \times \mathrm{Herm}\,(N) \times \mathrm{Herm}\,(N) \to \mathbb{C}$ defined by

$$L_T(X, Y, Z) = \langle T, X \otimes Y \otimes Z \rangle = \sum_{(i,j,k),(i',j',k') \in [N]^3} T_{(i,i'),(j,j'),(k,k')} \overline{X_{i,i'} Y_{j,j'} Z_{k,k'}},$$

where $X, Y, Z \in \mathrm{Herm}\,(N)$. The operator norm of L_T induces the following norm on T

$$\|T\|_{2,2,2} := \max_{X,Y,Z \in B(\mathrm{Herm}(N))} \left| L_T(X, Y, Z) \right| = \max_{X,Y,Z \in B(\mathrm{Herm}(N))} |\langle T, X \otimes Y \otimes Z \rangle|.^8$$

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[8] The restriction to Hermitian matrices in this definition is not essential, but it will be convenient later on.

2.3 XOR games

An r-player XOR game with N questions per player is fully specified by a joint probability distribution π on $[N]^r$ and an r-tensor $M : [N]^r \to \{-1, 1\}$. The *classical bias* of an XOR game $G = (\pi, M)$ is defined by

$$\beta(G) := \max_{\chi_1, \dots, \chi_r : [N] \to \{-1,1\}} \mathbb{E}_{(q_1, \dots, q_r) \sim \pi} \Big[M(q_1, \dots, q_r) \chi_1(q_1) \cdots \chi_r(q_r) \Big].$$

The maps $\chi_1, \dots, \chi_r$ in the above maximum are referred to as *strategies*: they should be interpreted as giving the players' answers to the questions $q_1, \dots, q_r$, respectively. The *entangled bias* of G is defined by

$$\beta^*(G) := \sup_{\substack{d \in \mathbb{N}, |\Psi\rangle \in S(\mathbb{C}^{d^r}) \\ A_1, \dots, A_r : [N] \to \mathrm{Obs}(d)}} \mathbb{E}_{(q_1, \dots, q_r) \sim \pi} \Big[M(q_1, \dots, q_r) \langle \psi | A_1(q_1) \otimes \cdots \otimes A_r(q_r) | \psi \rangle \Big].$$

We note that this definition restricts the players to the use of arbitrarily large but finite-dimensional strategies. In general it is an open problem, known as *Tsirelson's problem* [Tsi06], whether there exists games for which it is possible to achieve a strictly higher bias through the use of infinite-dimensional strategies.

In the sequel it will be convenient to merge π and M into a single tensor $T : [N]^r \to \mathbb{R}$ defined by $T(q_1, \dots, q_r) = \pi(q_1, \dots, q_r) M(q_1, \dots, q_r)$. Conversely, any tensor $T : [N]^r \to \mathbb{R}$ defines (up to normalization) an XOR game by setting the distribution to $\pi(q_1, \dots, q_r) = |T(q_1, \dots, q_r)|$ and the game tensor to $M(q_1, \dots, q_r) = \mathrm{sign}\big(T(q_1, \dots, q_r)\big)$.

2.4 ε-nets

Our probabilistic proof of the existence of a game for which there is a large QC-gap relies on the construction of specific ε-nets over Hermitian matrices, which we describe in this section.

Definition 4. *An ε-net for a subset $\mathcal{W}$ of a metric space $(\mathcal{V}, d)$ is a finite set $W \subseteq \mathcal{V}$ such that for every $x \in \mathcal{W}$, there exists an $s \in W$ such that $d(x, s) \leq \varepsilon$.*

It is well-known that for every $N \in \mathbb{N}$ and any $\varepsilon > 0$ there exists an ε-net S_ε for $S(\mathbb{C}^N)$ of cardinality $|S_\varepsilon| \leq (1 + 2/\varepsilon)^N$ (see e.g. [Pis99, Lemma 4.10] for a proof, which follows from a simple volume argument).

The following lemma shows that for any $\varepsilon > 0$ and $N > 1$, an ε-net over N-dimensional Hermitian matrices with Frobenius norm at most 1 can be obtained from an $\varepsilon/(4\sqrt{\ln N})$-net over (normalized, signed) N-dimensional projections. The lemma follows from a well known equivalence between the unit ball of normalized projections and the unit ball corresponding to the matrix norm derived from the Lorentz-sequence semi-norm $\ell_{2,1}$. We give a self-contained proof below.

Lemma 5. *Let $N > 1$ and $X \in B\big(\mathrm{Herm}\,(N)\big)$. Then X can be decomposed as a linear combination*

$$X = \sum_x \lambda_x X_x,$$

where each $X_x \in \overline{\mathrm{Proj}}\,(N)$ is a normalized projector and $\sum_x |\lambda_x| \leq 4\sqrt{\ln N}$.

PROOF: Let $X = \sum_i \lambda_i |u_i\rangle\langle u_i|$ be the spectral decomposition of a Hermitian matrix X with norm

$$\|X\|_F^2 = \sum_i \lambda_i^2 \leq 1. \tag{3}$$

For every $t \in [-1, 1]$, let P_t be the projector on $\text{Span}\{|u_i\rangle : \lambda_i \in [-1, t)\}$ if $t < 0$ and the projector on $\text{Span}\{|u_i\rangle : \lambda_i \in (t, 1]\}$ if $t \geq 0$. Then the following holds:

$$X = \int_{-1}^{1} \text{sign}(t)\, P_t\, dt = \int_{-1}^{1} \sqrt{\text{rank}P_t}\, \frac{\text{sign}(t)\, P_t}{\sqrt{\text{rank}P_t}}\, dt, \tag{4}$$

where the integral is taken coefficient-wise.[9] By a direct calculation,

$$\int_{-1}^{1} |t|\, \text{Tr}(P_t) dt = \frac{1}{2} \sum_i \lambda_i^2 \leq \frac{1}{2}, \tag{5}$$

where the last inequality follows from (3). Eq. (4) shows that X may be written as a non-negative linear combination of the $\text{sign}(t)\, P_t / \sqrt{\text{rank}P_t}$ with coefficients summing up to

$$\int_{-1}^{1} \sqrt{\text{rank}P_t}\, dt \leq \int_{-1/\sqrt{N}}^{1/\sqrt{N}} \sqrt{N}\, dt + \int_{-1}^{-1/\sqrt{N}} \sqrt{\text{rank}P_t}\, dt + \int_{1/\sqrt{N}}^{1} \sqrt{\text{rank}P_t}\, dt$$

$$\leq 2 + \left(2 \int_{1/\sqrt{N}}^{1} \frac{1}{t}\, dt \right)^{1/2} \left(\int_{-1}^{-1/\sqrt{N}} (-t)\, \text{rank}P_t\, dt + \int_{1/\sqrt{N}}^{1} t\, \text{rank}P_t\, dt \right)^{1/2}$$

$$\leq 2 + \sqrt{\frac{\ln N}{2}},$$

where the first inequality uses $\text{rank}P_t \leq N$ for every t, the second inequality follows from Cauchy-Schwarz and the last uses (5), together with $\text{rank}P_t = \text{Tr}P_t$. □

The following lemma gives a straightforward construction of an ε-net for the set of normalized rank-k projectors on $\mathbb{C}^N$ (see e.g. [Sza82] for more general constructions of nets on Grassmannian spaces).

Lemma 6. *For every $k \in [N]$ and any $0 < \varepsilon \leq 1$ there exists a set*

$$\mathcal{Z}_\varepsilon^k \subseteq \bigcup_{\ell=1}^{k} \overline{\text{Proj}}\,(N)_\ell$$

of size $|\mathcal{Z}_\varepsilon^k| \leq 2(5/\varepsilon)^{kN}$, such that for any $X \in \overline{\text{Proj}}\,(N)_k$ there is an $\widetilde{X} \in \mathcal{Z}_\varepsilon^k$ satisfying $\|X - \widetilde{X}\|_F \leq \varepsilon$.

PROOF: Let $\eta = \varepsilon / \sqrt{2}$ and S_η be an η-net for the unit sphere $S(\mathbb{C}^N)$ of size $|S_\eta| \leq (1 + 2/\eta)^N$; as mentioned above such a set is guaranteed to exist. For every k-element subset $T \subseteq S_\eta$ let Y_T be the projector onto the space spanned by the vectors in T and let $\overline{Y}_T = Y_T / \sqrt{\text{rank}Y_T}$. Define the set $\mathcal{Z}_\varepsilon^k$ by

$$\mathcal{Z}_\varepsilon^k = \{\overline{Y}_T : T \subseteq S_\eta,\ |T| = k\}.$$

---

[9] The coefficients of P_t are step functions, so the integral is well-defined.

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Note that for any $\overline{Y} \in \mathcal{Z}_\varepsilon^k$, we have $\mathrm{rank}(\overline{Y}) \leq k$. Moreover, by the upper bound on the size of S_η, we have

$$|\mathcal{Z}_\varepsilon^k| \leq \binom{|S_\eta|}{k} \leq \binom{(3/\eta)^N}{k} \leq 2 \left(\frac{5}{\varepsilon}\right)^{kN}.$$

Fix $X \in \overline{\mathrm{Proj}}\,(N)_k$ and let $|\phi_1\rangle, \ldots, |\phi_k\rangle \in S(\mathbb{C}^N)$ be orthonormal eigenvectors of X with eigenvalue $1/\sqrt{k}$. Let $|\psi_1\rangle, \ldots, |\psi_k\rangle \in S_\eta$ be the vectors closest to $|\phi_1\rangle, \ldots, |\phi_k\rangle$ (resp.) with respect to the Euclidean distance. Let Y be the projector on the space spanned by the $|\psi_1\rangle, \ldots, |\psi_k\rangle$ and let $\overline{Y} = Y/\sqrt{\mathrm{rank}Y}$. Clearly, $\overline{Y} \in \mathcal{Z}_\varepsilon^k$. Since Y is positive semidefinite and for every $i = 1, \ldots, k$, the vector $|\psi_i\rangle$ is an eigenvector of Y with eigenvalue 1,

$$\langle \phi_i | Y | \phi_i \rangle \geq |\langle \phi_i | \psi_i \rangle|^2 \geq 1 - \eta^2,$$

where the second inequality follows since $|\psi_i\rangle$ is closest to $|\phi_i\rangle$ in the η-net.[10] By definition of the Frobenius norm and the fact that X and $\overline{Y}$ are Hermitian, we get

$$
\begin{aligned}
\|X - \overline{Y}\|_F^2 &= \|X\|_F^2 + \|\overline{Y}\|_F^2 - 2\mathrm{Tr}(X\overline{Y}) \\
&\leq 2 - 2\mathrm{Tr}(X\overline{Y}) \\
&\leq 2\left(1 - \frac{1}{k}\sum_{i=1}^{k}\langle \phi_i | Y | \phi_i \rangle\right) \\
&\leq 2\left(1 - \left(1 - \eta^2\right)\right) \\
&= \varepsilon^2,
\end{aligned}
$$

and the lemma is proved. $\square$

Definition 7. *For every triple of integers $(k, \ell, m) \in [N]^3$ and any real number $0 < \varepsilon \leq 1$, define*

$$\mathcal{Z}_\varepsilon^{(k,\ell,m)} = \{X \otimes Y \otimes Z : (X, Y, Z) \in \mathcal{Z}_\varepsilon^k \times \mathcal{Z}_\varepsilon^\ell \times \mathcal{Z}_\varepsilon^m\},$$

and $\mathcal{Z}_\varepsilon = \bigcup_{(k,\ell,m)\in[N]^3} \mathcal{Z}_\varepsilon^{(k,\ell,m)}$.

Proposition 8. *For any $\varepsilon > 0$ and $N > 1$, the set $\mathcal{Z}_\varepsilon$ is a 3ε-net for the set of matrices $X \otimes Y \otimes Z$ where $(X, Y, Z) \in \overline{\mathrm{Proj}}\,(N) \times \overline{\mathrm{Proj}}\,(N) \times \overline{\mathrm{Proj}}\,(N)$, with respect to the distance function defined by the Frobenius norm.*

PROOF: Let $1 \leq k, \ell, m \leq N$ and $X \in \overline{\mathrm{Proj}}\,(N)_k$, $Y \in \overline{\mathrm{Proj}}\,(N)_\ell$ and $Z \in \overline{\mathrm{Proj}}\,(N)_m$. Let $\widetilde{X} \in \mathcal{Z}_\varepsilon^k$, $\widetilde{Y} \in \mathcal{Z}_\varepsilon^\ell$ and $\widetilde{Z} \in \mathcal{Z}_\varepsilon^m$ be the closest elements in the nets to (resp.) X, Y and Z in Frobenius distance. Using the trivial identity $A \otimes B - \widetilde{A} \otimes \widetilde{B} = A \otimes (B - \widetilde{B}) + (A - \widetilde{A}) \otimes \widetilde{B}$ twice in a row, and the triangle inequality, we can upper bound the distance $\|X \otimes Y \otimes Z - \widetilde{X} \otimes \widetilde{Y} \otimes \widetilde{Z}\|_F$ by

$$\|X \otimes Y \otimes (Z - \widetilde{Z})\|_F + \|X \otimes (Y - \widetilde{Y}) \otimes \widetilde{Z}\|_F + \|(X - \widetilde{X}) \otimes \widetilde{Y} \otimes \widetilde{Z}\|_F.$$

Since for any A, B, $\|A \otimes B\|_F = \|A\|_F \|B\|_F$, the quantity above is less than 3ε. $\square$

---

[10]Notice that for any complex unit vectors x, y, we have $\|x - y\|^2 = 2 - 2\Re(\langle x, y\rangle)$ and $|\langle x, y\rangle|^2 = (\Re(\langle x, y\rangle))^2 + (\Im(\langle x, y\rangle))^2$.

2.5 Deviation bounds

In this section we collect some useful large deviation bounds. Since our results are based on the use of Gaussian random variables, we will make repeated use of the standard tail bound for a standard normal random variable $g \sim \mathcal{N}(0,1)$:

$$\Pr\big[|g| \geq t\big] \leq 2e^{-t^2/2}, \tag{6}$$

which holds for any $t \geq 0$. We first recall a useful large deviation inequality usually credited to Bernstein.

Proposition 9 (Bernstein's inequality, see eg. Prop. 16 in [Ver10]). *Let $h_1, \ldots, h_N$ be independent centered random variables and $K > 0$ be such that $\Pr\big[|h_i| \geq t\big] \leq e^{1-t/K}$ for all i and $t \geq 0$. Then for any $a \in \mathbb{R}^N$ and $t \geq 0$,*

$$\Pr\left[\left|\sum_{i=1}^{N} a_i h_i\right| \geq t\right] \leq 2e^{-\frac{1}{4e}\min\left(\frac{t^2}{2eK^2\|a\|_2^2}, \frac{t}{K\|a\|_\infty}\right)}.$$

Corollary 10 (χ^2 tail bound). *Let $|g\rangle$ be a random vector distributed according to $\mathcal{N}(0,1)^N$. Then for every $t \geq 0$,*

$$\Pr\left[\big|\||g\rangle\|_2^2 - N\big| \geq t\right] \leq 2e^{-\frac{1}{8e}\min\left(\frac{t^2}{4eN}, t\right)}.$$

PROOF: Write $|g\rangle = g_1|1\rangle + \cdots + g_N|N\rangle$ where $g_1, \ldots, g_N$ are i.i.d. standard normal random variables. By (6), for every i the g_i satisfy that for every $t \geq 0$,

$$\Pr\big[|g_i^2 - 1| \geq t\big] = \Pr\big[g_i^2 \geq t + 1\big] + \Pr\big[g_i^2 \leq 1 - t\big]$$
$$\leq e\, e^{-(t+1)/2},$$

where the factor e in front ensures that the bound is trivial whenever the second term $\Pr(g_i^2 \leq 1 - t)$ is nonzero. Hence the random variables $h_i := g_i^2 - 1$ satisfy the hypothesis of Proposition 9 with $K = 2$, which immediately gives the claimed bound. $\qquad\square$

The following is a special case of a result due to Latała (see Corollary 1 in [Lat06]).

Corollary 11. *Let $A \in \mathrm{Herm}\,(N)$ be a Hermitian matrix and $|g\rangle \sim \mathcal{N}(0,1)^N$. Then, for any $t \geq 0$,*

$$\Pr\left[\big|\langle g|A|g\rangle - \mathrm{Tr}(A)\big| \geq t\right] \leq 2\,e^{-\frac{1}{24e}\min\left(\frac{t^2}{12e\|A\|_F^2}, \frac{t}{\|A\|_\infty}\right)}.$$

PROOF: Since A is Hermitian, it is unitarily diagonalizable: $A = UDU^\dagger$ where $D = \mathrm{diag}(\lambda_i)$, and the λ_i are its real eigenvalues. Then

$$\langle g|A|g\rangle = \sum_{i=1}^{N} \lambda_i |\langle i|U|g\rangle|^2,$$

where the g_i are the standard normal distributed coefficients of the random vector $|g\rangle$. Since the rows of U are orthogonal, the $\langle i|U|g\rangle$ are independent random variables. Moreover, since $|g\rangle$ is

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real, we have $|\langle i|U|g\rangle|^2 = \big(\Re(\langle i|U)|g\rangle\big)^2 + \big(\Im(\langle i|U)|g\rangle\big)^2$, where $\Re(\langle i|U)$ and $\Im(\langle i|U)$ are the real and imaginary parts of the unit vector $\langle i|U$ forming the i-th row of U. By rotation invariance, we have that for arbitrary $|x\rangle \in \mathbb{R}^N$, the random variable $\langle x|g\rangle$ is distributed as $\mathcal{N}(0, \||x\rangle\|^2)$. It follows from (6) that for every $i \in [N]$, we have

$$
\begin{aligned}
\Pr\Big[|\langle i|U|g\rangle|^2 \geq t\Big] &\leq \Pr\Big[\big(\Re(\langle i|U)|g\rangle\big)^2 \geq t/2\Big] + \Pr\Big[\big(\Re(\langle i|U)|g\rangle\big)^2 \geq t/2\Big] \\
&\leq 2e^{-t/(4\|\Re(\langle i|U\|)^2)} + 2e^{-t/(4\|\Im(\langle i|U_i)\|^2)} \\
&\leq 4e^{-t/4}.
\end{aligned}
$$

Hence we can apply Proposition 9 with $K = 4(\ln(4/e) + 1) \leq 6$ to obtain for any $t \geq 0$:

$$
\Pr\left[\left| \sum_{i=1}^N \lambda_i \Big(|\langle i|U|g\rangle|^2 - \mathbb{E}\big[|\langle i|U|g\rangle|^2\big] \Big) \right| \geq t \right] \leq 2\,e^{-\frac{1}{24e}\min\left(\frac{t^2}{12e\|A\|_F^2},\frac{t}{\|A\|_\infty}\right)}
$$

which proves the claim since $\mathbb{E}\big[|\langle i|U|g\rangle|^2\big] = 1$ for every $i \in [N]$ and $\sum_{i=1}^N \lambda_i = \mathrm{Tr}(A)$. $\qquad\square$

We end this section by giving analogues of the preceding concentration bounds for the case of i.i.d. Bernoulli random variables. These well-known facts will not be necessary for the proof of our main results, which are based on the use of Gaussian random variables, but can be used to prove analogue statements in the Bernoulli case (and we will indicate exactly how our proofs should be adapted in due course). We first recall Hoeffding's inequality.

Proposition 12 (Hoeffding's inequality). *Let $h_1, \ldots, h_N$ be independent centered random variables such that for every $i \in [N]$, we have $\Pr\big[h_i \in [a_i, b_i]\big] = 1$. Then for any $t \geq 0$,*

$$
\Pr\left[\left| \sum_{i=1}^N h_i \right| \geq t \right] \leq 2e^{-2t^2/\sum_i (b_i - a_i)^2}.
$$

Corollary 13 (Projections of Bernoulli vectors). *Let ε_{ij}, $i, j \in [N]$ be i.i.d. Bernoulli random variables, and $a \in \mathbb{R}^N$. Then*

$$
\Pr\left[\left| \sum_{j=1}^N \left(\sum_{i=1}^N a_i \varepsilon_{ij} \right)^2 - N\|a\|_2^2 \right| > t \right] \leq 2e^{-\frac{1}{4e}\min\left(\frac{t^2}{8e\|a\|_2^4 N},\frac{t}{2\|a\|_2^2}\right)}.
$$

PROOF: For any $j \in [N]$ let $\eta_j = \big(\sum_{i=1}^N a_i \varepsilon_{ij}\big)^2 - \|a\|_2^2$. The η_j are independent centered random variables, and by Proposition 12 they satisfy a tail bound as required by Proposition 9, with $K = 2\|a\|_2^2$. The corollary follows. $\qquad\square$

Finally, we state without proof an analogue of Corollary 11 which applies to Bernoulli random variables, and is a special case of a result of Hanson and Wright [HW71].

Theorem 14. *There exists a constant $D > 0$ such that the following holds. Let $A \in \mathrm{Herm}\,(N)$ be a Hermitian matrix and ε_i i.i.d. Bernoulli random variables. Then, for any $t \geq 0$,*

$$
\Pr\left[\left| \sum_{i,j} A_{ij} \varepsilon_i \varepsilon_j - \mathrm{Tr}(A) \right| \geq t \right] \leq 2\,e^{-D\min\left(\frac{t^2}{\|A\|_F^2},\frac{t}{\|A\|_\infty}\right)}.
$$

2.6 Grothendieck's inequality

We use the following version of Grothendieck's inequality [Gro53]. The constants involved come from [Haa87] and [BMMN11].

Theorem 15 (Grothendieck's inequality). *There exists a universal constant $K_G^{\mathbb{R}} < 1.783$ such that the following holds. Let N and d be positive integers. Then, for any matrix $M \in \text{Mat}(N)$ with real coefficients and any complex unit vectors $x_1, \ldots, x_N, y_1, \ldots, y_N \in S(\mathbb{C}^d)$, we have*

$$\Big| \sum_{i,j=1}^{N} M_{ij} \langle x_i, y_j \rangle \Big| \leq K_G^{\mathbb{R}} \max_{\chi, v:[N] \to \{-1,1\}} \sum_{i,j=1}^{N} M_{ij} \chi(i) v(j), \tag{7}$$

If we allow χ, v on the right-hand side of (7) to take values in the set of all complex numbers with modulus (at most) 1, then the constant $K_G^{\mathbb{R}}$ may be replaced by the complex Grothendieck constant $K_G^{\mathbb{C}} < 1.405$.

3 Unbounded gaps

This section is devoted to the proof of Theorem 1. The theorem is proved in two steps. In the first step we associate a three player XOR game G to any 3-tensor T, and relate the quantum-classical gap for that game to spectral properties of T. We emphasize that the game $G = G(T)$ is *not* defined from T in the most straightforward way (using T as the game tensor), but through a more delicate transformation, based on the use of the Fourier transform, which is exposed in Section 3.1.

Proposition 16. *Let n be an integer and let $N = 2^n$. Let T be any 3-tensor of dimensions $N^2 \times N^2 \times N^2$. Then there exists a 3-player XOR game $G = G(T)$ such that*

$$\frac{\beta^*(G)}{\beta(G)} \geq \frac{1}{2N^{3/2}} \frac{\|T\|_{3,3}}{\|T\|_{2,2,2}}.$$

Moreover, in the game G there are N^2 questions to each player, and there is a entangled strategy which achieves the claimed violation and uses only N-dimensional Pauli observables.

In the second step we show the existence of a tensor T such that $\|T\|_{3,3} / \|T\|_{2,2,2}$ is large.

Proposition 17. *There is a constant $C > 0$ such that for any integer n and $N = 2^n$ there exists a 3-tensor T of dimensions $N^2 \times N^2 \times N^2$ such that*

$$\frac{\|T\|_{3,3}}{\|T\|_{2,2,2}} \geq CN^2 \log^{-5/2} N.$$

Theorem 1 trivially follows from the two propositions above. While we have not made the constants in the preceding propositions completely explicit, it is not hard to extract numerical values from our proofs; in particular we give precise estimates for all our probabilistic arguments. Proposition 16 is proved in Section 3.1, and Proposition 17 is proved in Section 3.2.

3.1 Pauli XOR games

Let T be a complex 3-tensor of dimensions $N^2 \times N^2 \times N^2$, where $N = 2^n$ and n is an arbitrary integer. Based on T we define a three-player XOR game $G = G(T)$ with the following properties:

1. There are N^2 questions per player,

2. The best classical strategy for game $G(T)$ achieves a bias of at most $N^{9/2}\|T\|_{2,2,2}$,

3. There is a entangled strategy which uses only Pauli matrices as observables and entanglement of local dimension N per player and achieves a bias of at least $(N^3/2)\|T\|_{3,3}$.

Properties 2. and 3. imply that in game $G(T)$, the ratio between the entangled and classical biases is at least

$$\frac{\beta^*(G)}{\beta(G)} \geq \frac{1}{2N^{3/2}}\frac{\|T\|_{3,3}}{\|T\|_{2,2,2}},$$

proving Proposition 16.

Let T be a $N^2 \times N^2 \times N^2$ tensor. Since $\|T\|_{2,2,2} = \|T^\dagger\|_{2,2,2}$, the triangle inequality gives $\|(T + T^\dagger)/2\|_{2,2,2} \leq \|T\|_{2,2,2}$, and similarly $\|i(T - T^\dagger)/2\|_{2,2,2} \leq \|T\|_{2,2,2}$. By writing $T = (T + T^\dagger)/2 - i(i(T - T^\dagger)/2)$ and using the triangle inequality for the norm $\|\cdot\|_{3,3}$, we see that among the two tensors $(T + T^\dagger)/2$ and $i(T - T^\dagger)/2$, one must result in a ratio of the $\|\cdot\|_{3,3}$ norm to the $\|\cdot\|_{2,2,2}$ norm that is at least half of what it was for T. Hence we may assume that T, when seen as an $N^3 \times N^3$ matrix, is also Hermitian. In order to associate an XOR game to T, we first define coefficients indexed by Pauli matrices $P, Q, R \in \mathcal{P}_n$ as follows

$$M_{P,Q,R} := \widehat{T}(P,Q,R) = \sum_{(i,i'),(j,j'),(k,k')\in[N]^2} T_{(i,i'),(j,j'),(k,k')}\overline{P_{i,i'}Q_{j,j'}R_{k,k'}}.$$

Since both T and the Pauli matrices are Hermitian, the coefficients $M_{P,Q,R}$ are real as well. In order to obtain an XOR game $G = G(T)$, it suffices to normalize the resulting sequence according to its ℓ_1 norm (note that this normalization has no effect on the ratio of the biases that is considered in Proposition 16). This results in a game with N^2 questions per player, indexed by the Pauli matrices. Since we are ultimately only concerned with the ratio $\|T\|_{3,3}/\|T\|_{2,2,2}$, without loss of generality we assume that the transformation made above (making T Hermitian) resulted in the $\|\cdot\|_{3,3}$ norm being divided by a factor at most 2, and the $\|\cdot\|_{2,2,2}$ norm remaining unchanged.

The fact that property 1. above holds is clear, by definition. Next we prove that property 2. holds. Let $\chi, \upsilon, \zeta : \mathcal{P}_n \to \{-1, 1\}$ be an optimal classical strategy. Define the matrices $X = \sum_{P\in\mathcal{P}_n} \chi(P)P$, $Y = \sum_{Q\in\mathcal{P}_n} \upsilon(Q)Q$ and $Z = \sum_{R\in\mathcal{P}_n} \zeta(R)R$. Then X, Y and Z are Hermitian, and

$$\|X\|_F^2 = \mathrm{Tr}(X^\dagger X) = \sum_{P,P'\in\mathcal{P}_n} \chi(P)\chi(P')\mathrm{Tr}(P^\dagger P') = N\sum_{P\in\mathcal{P}_n}\chi(P)^2 = N^3,$$

and the same holds for Y and Z. The classical bias can be bounded as

$$\beta(G) = \sum_{P,Q,R\in\mathcal{P}_n} \widehat{T}(P,Q,R)\,\chi(P)v(Q)\zeta(R)$$

$$= \sum_{P,Q,R\in\mathcal{P}_n} \langle T, P\otimes Q\otimes R\rangle\,\chi(P)v(Q)\zeta(R)$$

$$\leq \max_{X,Y,Z\in B(\mathrm{Herm}(N),N^{3/2})} \langle T, X\otimes Y\otimes Z\rangle$$

$$\leq N^{9/2}\|T\|_{2,2,2}.$$

Finally, we prove property 3. by describing a good entangled strategy for $G(T)$. We simply let the observable corresponding to question P (resp. Q, R) be the n-qubit Pauli matrix P (resp. Q, R). Let $|\Psi\rangle$ be a shared entangled state. The bias of the corresponding strategy is

$$\sum_{P,Q,R} \widehat{T}(P,Q,R)\,\langle\Psi|P\otimes Q\otimes R|\Psi\rangle \;=\; N^3\langle\Psi|T|\Psi\rangle \;=\; N^3\|T\|_{3,3},$$

where for the last equality we chose $|\Psi\rangle$ an eigenvector of T with largest eigenvalue.

Remark. *In our construction, the only properties of the Pauli matrices that we use is that they form a family of Hermitian matrices that each square to identity and are pairwise orthogonal with respect to the Hilbert-Schmidt inner product on* $\mathrm{Herm}\,(N)$. *Any other such family would lead to a completely analogous construction (in which the player's observables in the entangled strategy are replaced by the corresponding elements).*

3.2 Constructing a good tensor T

In this section we prove Proposition 17 by giving a probabilistic argument for the existence of a tensor T with good spectral properties. Let N be an integer, and $|g\rangle$ the (random) N^3-dimensional vector

$$|g\rangle := \sum_{i,j,k=1}^{N} g_{ijk}|i\rangle|j\rangle|k\rangle \sim \mathcal{N}(0,1)^{N^3},$$

where the g_{ijk} are i.i.d. $\mathcal{N}(0,1)$ random variables. We define a tensor T depending on the g_{ijk}, and then prove bounds on the $\|\cdot\|_{3,3}$ and $\|\cdot\|_{2,2,2}$ norms of T that hold with high probability over the choice of the g_{ijk}. Let

$$T := \sum_{i\neq i',j\neq j',k\neq k'} g_{ijk}\,g_{i'j'k'}\,|i,j,k\rangle\langle i',j',k'|. \tag{8}$$

T is a real $N^3\times N^3$ symmetric matrix that equals $|g\rangle\langle g|$ with some coefficients zeroed out, including those on the diagonal. Hence T is very close to a rank 1 matrix and it should therefore be no surprise that its spectral norm is large, as we show in Section 3.2.1 below. More work is needed to upper bound the $\|\cdot\|_{2,2,2}$ norm of T. In particular, we note that zeroing out the diagonal coefficients is essential to getting a good bound on $\|T\|_{2,2,2}$. While we show in Section 3.2.2 that with

high probability over $|g\rangle$ we have $\|T\|_{2,2,2} = O(N \log^{5/2} N)$, it is not hard to see that in expectation we already have $\||g\rangle\langle g|\|_{2,2,2} = \Omega(N\sqrt{N})$ (indeed, simply choose $X = Y = Z = I/\sqrt{N}$ in the definition of $\|\cdot\|_{2,2,2}$). Zeroing out some entries of $|g\rangle\langle g|$ approximately preserves the spectral norm, but decreases its norm as a trilinear operator by almost a factor $\sqrt{N}$.

Remark. *The same construction, with the normal random variables g_{ijk} replaced by i.i.d. Bernoulli random variables, can be used to obtain similar results.[11] Indeed, Lemma 18 below holds trivially in that case, and to obtain the analogue of Lemma 19 it suffices to replace the use of Corollary 10 and Corollary 11 in the proof of Lemma 22 by Corollary 13 and Theorem 14 respectively.*

3.2.1 A lower bound on the spectral norm

A lower-bound on the spectral norm of T as defined in (8) follows easily from the fact that it is, by definition, very close to a rank-1 matrix. We show the following.

Lemma 18. *For any $\tau > 0$ and all large enough N it holds that*

$$\|T\|_{3,3} \geq N^3 - \tau N^2$$

with probability at least $1 - e^{-\Omega(\tau^2)}$.

PROOF: Define $|\Psi\rangle = N^{-3/2}|g\rangle$. By Corollary 10, for any $\delta > 0$ we have

$$\Pr\left[\left|\sum_{i,j,k} g_{ijk}^2 - N^3\right| \leq \delta N^3\right] \geq 1 - 2e^{-\delta^2 N^3/(32e^2)}.$$

Provided this holds,

$$\left|\||\Psi\rangle\|^2 - 1\right| = \frac{1}{N^3}\left|\sum_{i,j,k} g_{ijk}^2 - 1\right| \leq \delta. \tag{9}$$

Another application of Corollary 10, together with a union bound, shows that the probability that there exists an $i \in [N]$ such that $\sum_{j,k} g_{ijk}^2 \leq (1+\delta)N^2$ is at least $1 - 2Ne^{-\delta^2 N^2/(32e^2)}$. Provided this holds,

$$\sum_i \left(\sum_{j,k} g_{ijk}^2\right)^2 \leq (1+\delta)^2 N^5 \tag{10}$$

and the same holds symmetrically for j or k. This lets us bound

$$\langle \Psi|T|\Psi\rangle = \frac{1}{N^3} \sum_{i\neq i', j\neq j', k\neq k'} g_{ijk}^2 g_{i'j'k'}^2$$

$$\geq \frac{1}{N^3}\left(\left(\sum_{i,j,k} g_{ijk}^2\right)^2 - \sum_i \left(\sum_{j,k} g_{ijk}^2\right)^2 - \sum_j \left(\sum_{i,k} g_{ijk}^2\right)^2 - \sum_k \left(\sum_{i,j} g_{ijk}^2\right)^2\right)$$

$$\geq \frac{(1-\delta)^2 N^6 - 3(1+\delta)^2 N^5}{N^3} \geq (1 - 2\delta)\, N^3,$$

---

[11]We thank Ignacio Villanueva for asking this question.

where the second inequality uses (9) and (10), and the last holds for large enough N. Hence, using (9) once more,

$$\|T\|_{3,3} \geq \frac{\langle \Psi|T|\Psi \rangle}{\||\Psi\rangle\|^2} \geq (1-2\delta)\,N^3(1+\delta)^{-1} \geq (1-4\delta)N^3$$

for small enough δ. The claimed bound follows by setting $\delta = \tau/(4N)$. □

3.2.2 Upper-bounding $\|T\|_{2,2,2}$

In this section we give an upper bound for $\|T\|_{2,2,2}$ that holds with high probability over the choice of T, where T is as in (8), a 3-tensor of dimensions $N^2 \times N^2 \times N^2$. Recall that

$$\|T\|_{2,2,2} = \max_{X,Y,Z \in B(\mathrm{Herm}(N))} |\langle T, X \otimes Y \otimes Z \rangle|.$$

We prove the following.

Lemma 19. *There exist universal constants $d, D > 0$ such that for all large enough N, we have*

$$\|T\|_{2,2,2} \leq DN(\ln N)^{5/2}$$

with probability at least $1 - e^{-dN}$ over the choice of $|g\rangle$.

We note that if T was a random tensor with entries i.i.d. standard normal, then a result by Nguyen et al. [NDT10] would show that $\|T\|_{2,2,2} = O(N\sqrt{\log N})$ holds with high probability. However, the entries of our tensor T are not independent, and we need to prove a bound tailored to our specific setting.

Our first step consists of showing that the supremum in the definition of $\|T\|_{2,2,2}$ can be restricted to a supremum over projector matrices, at the cost of the loss of a logarithmic factor in the bound.[12]

Lemma 20. *Let $|g\rangle$ be a vector in $\mathbb{R}^{N^3}$ and let T be the associated tensor, as in (8). Then*

$$\|T\|_{2,2,2} \leq 64\,(\ln N)^{3/2} \max \big|\langle g|X \otimes Y \otimes Z|g\rangle - \mathrm{Tr}(X \otimes Y \otimes Z)\big|, \tag{11}$$

where the maximum is taken over all triples $(X,Y,Z) \in \overline{\mathrm{Proj}}\,(N)^3$.

PROOF: Let $X, Y, Z \in B\big(\mathrm{Herm}\,(N)\big)$ be traceless Hermitian matrices such that $\|T\|_{2,2,2} = \langle T, X \otimes Y \otimes Z \rangle$. Because of the specific form of T, we may assume without loss of generality that the diagonal entries of X, Y, Z are all zero: they do not contribute to the inner-product, and setting them to zero cannot increase the Frobenius norm of X, Y or Z. Under this condition, we also get $\langle T, X \otimes Y \otimes Z \rangle = \langle g|X \otimes Y \otimes Z|g\rangle$, by definition of T. Decompose X, Y, Z as in Lemma 5, giving

$$X = \sum_x \alpha_x X_x, \qquad Y = \sum_y \beta_y Y_y \qquad \text{and} \qquad Z = \sum_z \gamma_z Z_z,$$

---

[12]We thank Gilles Pisier for suggesting the use of this decomposition.

where $\|(\alpha_x)_x\|_1, \|(\beta_y)_y\|_1, \|(\gamma_z)_z\|_1 \le 4\sqrt{\ln N}$ and $X_x, Y_y, Z_z \in \overline{\mathrm{Proj}}\,(N)$. Note that

$$0 = \mathrm{Tr}(X \otimes Y \otimes Z) = \sum_{x,y,z} \alpha_x \beta_y \gamma_z \, \mathrm{Tr}(X_x \otimes Y_y \otimes Z_z).$$

By linearity and Hölder's inequality, we have

$$\langle g|X \otimes Y \otimes Z|g\rangle - \mathrm{Tr}(X \otimes Y \otimes Z)$$
$$= \sum_{x,y,z} \alpha_x \beta_y \gamma_z \left(\langle g|X_x \otimes Y_y \otimes Z_z|g\rangle - \mathrm{Tr}(X_x \otimes Y_y \otimes Z_z) \right)$$
$$\le 64(\ln N)^{3/2} \max_{x,y,z} \left| \langle g|X_x \otimes Y_y \otimes Z_z|g\rangle - \mathrm{Tr}(X_x \otimes Y_y \otimes Z_z) \right|,$$

proving the lemma. $\qquad\square$

Our next step is to show that we may further restrict the maximum on the right-hand side of (11) to a maximum over projectors taken from the ε-net $\mathcal{Z}_\varepsilon$ given in Definition 7.

Lemma 21. *Let $|g\rangle$ be a vector in $\mathbb{R}^{N^3}$, T the associated tensor and $\varepsilon > 0$. Then*

$$\|T\|_{2,2,2} \le 64 \, (\ln N)^{3/2} \left(\max \left| \langle g|X \otimes Y \otimes Z|g\rangle - \mathrm{Tr}(X \otimes Y \otimes Z) \right| + 3\varepsilon \left(N^{3/2} + \| |g\rangle \|_2^2 \right) \right), \quad (12)$$

where the maximum is taken over all $X \otimes Y \otimes Z \in \mathcal{Z}_\varepsilon$.

PROOF: Fix a triple $(X, Y, Z) \in \overline{\mathrm{Proj}}\,(N)^3$. By Proposition 8, there exists an $\tilde{X} \otimes \tilde{Y} \otimes \tilde{Z} \in \mathcal{Z}_\varepsilon$ such that

$$\left\| X \otimes Y \otimes Z - \tilde{X} \otimes \tilde{Y} \otimes \tilde{Z} \right\|_F \le 3\varepsilon.$$

By the Cauchy-Schwarz inequality, we have

$$\left| \langle g|X \otimes Y \otimes Z|g\rangle - \langle g|\tilde{X} \otimes \tilde{Y} \otimes \tilde{Z}|g\rangle \right| = \left| \langle g|X \otimes Y \otimes Z - \tilde{X} \otimes \tilde{Y} \otimes \tilde{Z}|g\rangle \right|$$
$$\le \left\| X \otimes Y \otimes Z - \tilde{X} \otimes \tilde{Y} \otimes \tilde{Z} \right\|_F \, \| |g\rangle\langle g| \|_F.$$

Another application of the Cauchy-Schwarz inequality and the definition of the Frobenius norm give

$$\left| \mathrm{Tr}(X \otimes Y \otimes Z - \tilde{X} \otimes \tilde{Y} \otimes \tilde{Z}) \right| = \left| \langle I, X \otimes Y \otimes Z - \tilde{X} \otimes \tilde{Y} \otimes \tilde{Z} \rangle \right|$$
$$\le N^{3/2} \left\| X \otimes Y \otimes Z - \tilde{X} \otimes \tilde{Y} \otimes \tilde{Z} \right\|_F.$$

Hence the lemma follows from Lemma 20. $\qquad\square$

We upper-bound the right-hand side of (12) by first showing that for any fixed triple $(k, \ell, m) \in [N]^3$ and $X \otimes Y \otimes Z \in \mathcal{Z}_\varepsilon^{(k,\ell,m)}$, this quantity is bounded with high probability over the choice of $|g\rangle$. We conclude by applying a union bound over the net $\mathcal{Z}_\varepsilon = \bigcup_{(k,\ell,m)\in[N]^3} \mathcal{Z}_\varepsilon^{(k,\ell,m)}$.

Lemma 22. *There exist constants $C, c > 0$ such the following holds. For any $0 < \varepsilon \leq N^{-3}$ and $\tau \geq CN \ln(1/\varepsilon)$, the probability over the choice of $|g\rangle$ that there exists an $X \otimes Y \otimes Z \in \mathcal{Z}_\varepsilon$ such that*

$$\left| \langle g | X \otimes Y \otimes Z | g \rangle - \operatorname{Tr}(X \otimes Y \otimes Z) \right| > \tau \tag{13}$$

is at most $e^{-c\tau}$.

PROOF: Fix a triple $(k, \ell, m) \in [N]^3$, and assume that $k \geq \max\{\ell, m\}$, the other cases being reduced to this one by permutation of the indices. Since $k + \ell + m \leq 3k$, we have

$$\left| \mathcal{Z}_\varepsilon^{(k,\ell,m)} \right| \leq 8 \left(\frac{5}{\varepsilon} \right)^{(k+\ell+m)N} \leq e^{3kN \ln(5/\varepsilon)+3}. \tag{14}$$

We distinguish two cases.

Case 1: $\ell m > k$. Fix an $X \otimes Y \otimes Z \in \mathcal{Z}_\varepsilon^{(k,\ell,m)}$. By definition of the nets $\mathcal{Z}_\varepsilon^j$,

$$\|X \otimes Y \otimes Z\|_F \leq 1 \quad \text{and} \quad \|X \otimes Y \otimes Z\|_\infty = \frac{1}{\sqrt{k\ell m}}.$$

Hence, by Corollary 11 there exists a constant $c' > 0$ such that for any $\tau > 0$

$$\Pr_{|g\rangle} \left[\left| \langle g | X \otimes Y \otimes Z | g \rangle - \operatorname{Tr}(X \otimes Y \otimes Z) \right| \geq \tau \right] \leq e^{-c' \min\{\tau^2, \tau\sqrt{k\ell m}\}}. \tag{15}$$

Our assumption $\ell m > k$ implies $\sqrt{k\ell m} > k$, hence the probability above is at most $e^{-c' \min\{\tau^2, k\tau\}}$. Using the bound (14) on the size of $\mathcal{Z}_\varepsilon^{(k,\ell,m)}$, by a union bound there exists a $C' > 0$ such that for any $\tau \geq C' N \ln(1/\varepsilon)$ the probability that there exists an $X' \otimes Y' \otimes Z' \in \mathcal{Z}_\varepsilon^{(k,\ell,m)}$ such that

$$\left| \langle g | X' \otimes Y' \otimes Z' | g \rangle - \operatorname{Tr}(X' \otimes Y' \otimes Z') \right| \geq \tau$$

is at most $e^{-\Omega(\tau)}$.

Case 2: $k \geq \ell m$. Fix an $X \otimes Y \otimes Z \in \mathcal{Z}_\varepsilon^{(k,\ell,m)}$. Since X, Y and Z are normalized projectors,

$$\operatorname{Tr}(X \otimes Y \otimes Z) \leq \sqrt{k\ell m} \leq k \leq N.$$

Write the spectral decompositions of X, Y and Z as

$$X = \frac{1}{\sqrt{k}} \sum_p |x_p\rangle\langle x_p|, \qquad Y = \frac{1}{\sqrt{\ell}} \sum_q |y_q\rangle\langle y_q| \qquad \text{and} \qquad Z = \frac{1}{\sqrt{m}} \sum_r |z_r\rangle\langle z_r|,$$

where the indices p, q, r run from 1 to at most k, ℓ, m, respectively. For any vectors $|y\rangle, |z\rangle \in \mathbb{C}^N$, define the N-dimensional vector

$$|g(y, z)\rangle := (I \otimes \langle y| \otimes \langle z|)|g\rangle = \sum_{p,q,r} g_{pqr} \overline{y_q z_r} |p\rangle,$$

where in this last expression coordinates are taken with respect to the canonical basis of $\mathbb{C}^N$. With this definition it holds that for any vector $|x\rangle \in \mathbb{C}^n$,

$$\langle x | g(y, z) \rangle = ((\langle x| \otimes \langle y| \otimes \langle z|)|g\rangle \quad \text{and} \quad \||g(y, z)\rangle\| \leq \|g\| \|y\| \|z\|. \tag{16}$$

By rotation invariance of the Gaussian distribution, if $|y\rangle, |z\rangle$ have norm 1 then $|g(y, z)\rangle$ is distributed according to $\mathcal{N}(0, 1)^N$. Since $|x_1\rangle, |x_2\rangle, \ldots$ are a (possibly incomplete) orthonormal basis, we have

$$|\langle g|X \otimes Y \otimes Z|g\rangle| = \frac{1}{\sqrt{k\ell m}} \sum_{p,q,r} |\langle x_p|g(y_q, z_r)\rangle|^2$$

$$\leq \sqrt{\frac{\ell m}{k}} \max_{|y\rangle, |z\rangle \in B(\mathbb{C}^N)} \big\||g(y, z)\rangle\big\|_2^2$$

$$\leq \max_{|y\rangle, |z\rangle \in B(\mathbb{C}^N)} \big\||g(y, z)\rangle\big\|_2^2, \tag{17}$$

where for the last inequality we used that $\sqrt{\ell m/k} \leq 1$ (which follows from our assumption $k \geq \ell m$). We now upper bound the maximum in Eq. (17) in terms of a maximum over vectors taken from the ε-net S_ε. To this end, notice that for any unit vectors $y, \tilde{y}, z, \tilde{z}$, we have

$$|g(y, z)\rangle = |g(y - \tilde{y}, z)\rangle + |g(\tilde{y}, z)\rangle$$
$$= |g(y - \tilde{y}, z - \tilde{z})\rangle + |g(y - \tilde{y}, \tilde{z})\rangle + |g(\tilde{y}, z - \tilde{z})\rangle + |g(\tilde{y}, \tilde{z})\rangle.$$

Let $\tilde{y}, \tilde{z} \in S_\varepsilon$ be the closest vectors to y and z, respectively, so that $\|y - \tilde{y}\|, \|z - \tilde{z}\| \leq \varepsilon$. Using the decomposition above followed by the Cauchy-Schwarz inequality and the second bound from Eq. (16) we obtain

$$\big\||g(y, z)\rangle\big\|_2^2 = \langle g(y, z)|g(y, z)\rangle \leq 15\varepsilon \big\||g\rangle\big\|_2^2 + \big\||g(\tilde{y}, \tilde{z})\rangle\big\|_2^2.$$

It follows that the maximum in Eq. (17) is bounded from above by

$$\max_{|y\rangle, |z\rangle \in S_\varepsilon} \big\||g(y, z)\rangle\big\|_2^2 + 15\,\varepsilon \big\||g\rangle\big\|_2^2, \tag{18}$$

Applying Corollary 10, there exists a $c'' > 0$ such that for any $\tau > 0$ the squared norm $\big\||g(y, z)\rangle\big\|_2^2$ appearing in (18) is greater than $N + \tau$ with probability at most $e^{-c'' \min\{\tau^2/N, \tau\}}$. Since $|S_\varepsilon| \leq e^{2\ln(1/\varepsilon)N}$, a union bound lets us upper bound the maximum on the right hand side of (18), showing that there exists a $C'' > 0$ such that for all $X \otimes Y \otimes Z \in \mathcal{Z}_\varepsilon^{(k, \ell, m)}$ and for all $\tau \geq C'' N \ln(1/\varepsilon)$ the bound

$$|\langle g|X \otimes Y \otimes Z|g\rangle| \leq 15\,\varepsilon(N^3 + N\tau) + \tau$$

holds with probability at least $1 - e^{-c''' \tau}$ over the choice of $|g\rangle$, for some $c''' > 0$. (Here we again used Corollary 10 to upper-bound $\big\||g\rangle\big\|_2^2 \leq N^3 + N\tau$ with probability at least $1 - e^{-\Omega(\tau)}$.)

The lemma follows for some $c, C > 0$ by combining the two cases analyzed above and performing a union bound over all N^3 triples (k, ℓ, m). □

We are now in a position to prove Lemma 19.

PROOF OF LEMMA 19: Let $\varepsilon = N^{-3}$ and $\tau = CN \ln(1/\varepsilon)$, where C is the constant appearing in the statement of Lemma 22. That lemma shows that the bound

$$\big|\langle g|X \otimes Y \otimes Z|g\rangle - \mathrm{Tr}(X \otimes Y \otimes Z)\big| \leq CN \ln(1/\varepsilon)$$

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holds except with probability at most $e^{-cCN\ln(1/\varepsilon)}$. Moreover, by Corollary 10, there is a $C' > 0$ such that

$$3\varepsilon \left(N^{3/2} + \big\| |g\rangle \big\|_2^2 \right) \leq 4\varepsilon N^3,$$

except with probability at least $1 - e^{-C'N}$. Combining these two bounds with the estimate of Lemma 21 proves the lemma, provided d is chosen small enough and D large enough. □

We end this section by explaining how Lemma 18 and Lemma 19 are combined to prove Proposition 17.

PROOF OF PROPOSITION 17: Setting $\tau = N/2$, Lemma 18 shows that for all large enough N, a random tensor T constructed as in (8) satisfies $\|T\|_{3,3} \geq N^3/2$ with probability at least $1 - e^{-\Omega(N^2)}$. Lemma 19 shows that $\|T\|_{2,2,2} \leq DN(\ln N)^{5/2}$ will hold with probability at least $1 - e^{-dN}$ over the choice of T. By the union bound, provided N is large enough both inequalities hold simultaneously with probability at least $1 - e^{-\Omega(N)}$, proving the proposition. □

4 Upper bounds on violations

4.1 Bounds in terms of the number of questions

In this section we prove Theorem 2, which we restate here for convenience.

Theorem 2. *For any 3-player XOR game G in which there are at most Q possible questions to the third player,*

$$\beta^*(G) \leq \sqrt{Q}\, K_G^{\mathbb{R}}\, \beta(G),$$

where $K_G^{\mathbb{R}} < 1.783$ is the real Grothendieck constant.

The two main ingredients in the proof are a useful technique of Paulsen and Grothendieck's inequality. Paulsen's technique (see [Pau92, Proposition 2.10]) lets us "decouple" the third player from the other two players and turn his part of the entangled strategy into a classical one at a loss of a factor $\sqrt{Q}$ in the overall bias.[13] Slightly more precisely, the proof goes as follows. By grouping the game tensor and the observables of the first two players together, the entangled bias takes the form

$$\beta^*(G) = \langle \psi | \sum_{k=1}^{Q} M_k \otimes C_k | \psi \rangle,$$

where the C_k are the third player's observables in an optimal entangled strategy. The decoupling technique relies on a collection of i.i.d. $\{-1, 1\}$-valued symmetrically distributed Bernoulli random variables $\varepsilon_1, \ldots, \varepsilon_Q$ which are used to split the above sum into two sums. Using the fact that $\mathbb{E}[\varepsilon_k \varepsilon_\ell] = \delta_{k\ell}$, the above expression can be written as

$$\mathbb{E}\left[\left(\langle \psi | \sum_{k=1}^{Q} M_k \otimes (\varepsilon_k I) \right) \left(\sum_{\ell=1}^{Q} \varepsilon_\ell I \otimes C_\ell | \psi \rangle \right) \right].$$

---

[13]This technique is based on so-called Rademacher averaging, a well-know method in the field of Banach spaces.

After two applications of the Cauchy-Schwarz inequality, the third player's classical strategy will be a certain instantiation of the random variables ε_k appearing in the left brackets, while the factor $\sqrt{Q}$ will come from the term between the right brackets. An application of Grothendieck's inequality will let us turn the first two players' entangled strategy into a classical one at a loss of an extra constant factor in the overall bias. We proceed with the formal proof of the theorem.

PROOF OF THEOREM 2: Suppose that the game G is defined by the probability distribution π and sign tensor M. Define the game tensor $T_{ijk} = \pi(ijk)M(ijk)$. Fix an arbitrary constant $\epsilon > 0$ and let $|\psi\rangle, A_i, B_j, C_k$ be a finite-dimensional state and $\{-1, 1\}$-valued observables such that[14]

$$\beta^*(G) \leq (1 + \epsilon) \sum_{i,j,k} T_{ijk} \langle \psi | A_i \otimes B_j \otimes C_k | \psi \rangle.$$

Define for every $k \in [Q]$ the matrix $M_k = \sum_{i,j} T_{ijk} A_i \otimes B_j$. Let $\varepsilon_1, \ldots, \varepsilon_Q$ be i.i.d. $\{-1, 1\}$-valued symmetrically distributed Bernoulli random variables. Using the fact that $\mathbb{E}[\varepsilon_k \varepsilon_\ell] = \delta_{k\ell}$ and the Cauchy-Schwarz inequality, the right-hand side of the above inequality can be written as and bounded by

$$\mathbb{E}\left[\left(\langle \psi | \sum_{k=1}^{Q} M_k \otimes (\varepsilon_k I) \right) \left(\sum_{\ell=1}^{Q} \varepsilon_\ell I \otimes C_\ell | \psi \rangle \right) \right] \leq \mathbb{E}\left[\left\| \langle \psi | \sum_{k=1}^{Q} M_k \otimes (\varepsilon_k I) \right\|_2 \left\| \sum_{\ell=1}^{Q} \varepsilon_\ell I \otimes C_\ell | \psi \rangle \right\|_2 \right].$$

Another application of Cauchy-Schwarz gives that the right-hand side is bounded from above by

$$\left(\mathbb{E}\left[\left\| \langle \psi | \sum_{k=1}^{Q} M_k \otimes (\varepsilon_k I) \right\|_2^2 \right] \right)^{1/2} \left(\mathbb{E}\left[\left\| \sum_{\ell=1}^{Q} \varepsilon_\ell I \otimes C_\ell | \psi \rangle \right\|_2^2 \right] \right)^{1/2}. \tag{19}$$

The fact that the matrices $\varepsilon_\ell I \otimes C_\ell$ are unitary and $|\psi\rangle$ is a unit vector shows that the above term on the right equals $\sqrt{Q}$. Since the matrices $M_k \otimes (\varepsilon_k I)$ are Hermitian, the left term in (19) is at most

$$\max_{|\phi\rangle, \zeta:[Q]\to\{-1,1\}} \langle \phi | \sum_{k=1}^{Q} M_k \otimes (\zeta(k)I) | \phi \rangle.$$

Expanding the definition of M_k, we have shown that

$$\beta^*(G) \leq (1 + \epsilon) \sqrt{Q} \max_{|\phi\rangle, \zeta:[Q]\to\{-1,1\}} \langle \phi | \sum_{i,j,k} T_{ijk} A_i \otimes B_j \otimes (\zeta(k)I) | \phi \rangle. \tag{20}$$

The matrices $\zeta(k)I$ may be interpreted as observables corresponding to single-outcome projective measurements. The outcome of such a measurement does not depend on the particular entangled state shared with the other players nor on their measurement outcomes. The entangled bias of the game G is thus at most $(1 + \epsilon)\sqrt{Q}$ times the bias achievable with strategies in which the third player uses a classical strategy. The maximum on the right-hand side of (20) thus equals[15]

$$\max_{|\phi'\rangle, \zeta:[Q]\to\{-1,1\}} \langle \phi' | \sum_{i,j,k=1}^{Q} T_{ijk} A_i \otimes B_j \zeta(k) | \phi' \rangle.$$

---

[14]Recall that $\beta^*(G)$ is defined by taking a supremum over all finite-dimensional strategies.

[15]Another way to see this is by writing $\sum_{i,j,k} T_{ijk} A_i \otimes B_j \otimes (\zeta(k)I) = \left(\sum_{i,j,k=1}^{Q} T_{ijk} A_i \otimes B_j \zeta(k) \right) \otimes I$ and using the facts that the operator norm is multiplicative under tensor products and the identity matrix has operator norm 1.

Let $|\phi'\rangle$ and $\zeta : [Q] \to \{-1, 1\}$ be such that the maximum above is achieved. Define the Q-by-Q matrix $H_{ij} = \sum_{k=1}^{Q} T_{ijk}\zeta(k)$. Rearranging terms gives that the above maximum equals $\sum_{i,j} H_{ij}\langle\phi'|A_i \otimes B_j|\phi'\rangle$. Define the unit vectors $x_i = A_i \otimes I|\phi'\rangle$ and $y_j = I \otimes B_j|\phi'\rangle$. Clearly we have $\langle\phi'|A_i \otimes B_j|\phi'\rangle = \langle x_i, y_j\rangle$. The result now follows by applying Grothendieck's inequality (7) and expanding the definition of H_{ij}. □

4.2   Bounds in terms of the Hilbert space dimension

In this section we give a proof of Theorem 3, which we restate for convenience.

Theorem 3. *Let G be a 3-player XOR game in which the maximal entangled bias $\beta^*(G)$ is achieved by a strategy in which the third player's local dimension is d. Then*

$$\beta^*(G) \leq 3\sqrt{2d}\left(K_G^C\right)^{3/2}\beta(G),$$

where $K_G^C < 1.405$ is the complex Grothendieck constant.

As the bound in terms of the number of questions presented in the previous section, the proof of Theorem 3 relies on a decoupling technique, by which the third player is reduced to using a classical strategy, while only reducing the bias that the players achieve in the game by a factor depending on the local dimension of his share of the entangled state. We use the following version of the non-commutative Khinchine's inequality, proved with optimal constants in [HM07].

Theorem 23 (Khinchine's inequality, Proposition 2.12 in [HM07]). *Let A_i be complex $d \times d$ matrices, and ε_i i.i.d. $\{-1, 1\}$ symmetrically distributed. Then there exists a matrix random variable $\tilde{A}$ such that $\mathbb{E}[\varepsilon_i\tilde{A}] = 0$ for every i, and for every possible joint value taken by the tuple of random variables $(\varepsilon_1, \ldots, \varepsilon_d, \tilde{A})$ it holds that*

$$\left\|\sum_i \varepsilon_i A_i + \tilde{A}\right\|_\infty \leq \sqrt{3}\max\left\{\left\|\sum_i A_i A_i^\dagger\right\|_\infty^{1/2}, \left\|\sum_i A_i^\dagger A_i\right\|_\infty^{1/2}\right\}. \tag{21}$$

PROOF OF THEOREM 3: Suppose that the game G is defined by the probability distribution π and sign tensor M. Define the tensor $T_{ijk} = \pi(ijk)M(ijk)$. Fix an arbitrary constant $\epsilon > 0$ and let $|\Psi\rangle$, A_i, B_j, C_k be a finite-dimensional state and $\{-1, 1\}$-valued observables, where C_k has dimension $d \times d$ and A_i, B_j have (finite) dimension $D \times D$, such that

$$\beta^*(G) \leq (1 + \epsilon)\sum_{i,j,k=1}^{Q} T_{ijk}\langle\Psi|A_i \otimes B_j \otimes C_k|\Psi\rangle.$$

For each k, let $M_k = \sum_{i,j} T_{ijk} A_i \otimes B_j$. Let $|\Psi\rangle = \sum_i \lambda_i|u_i\rangle|v_i\rangle$ be the Schmidt decomposition, where $|u_i\rangle$ is a vector on the system held by the first two players, and $|v_i\rangle$ is on the third player's. Assume without loss of generality that the $|v_i\rangle$ span the local space of the third player. Letting $M = \sum_k M_k \otimes C_k$, the bias achieved by this strategy is $\langle\Psi|M|\Psi\rangle \geq (1 + \epsilon)^{-1}\beta^*(G)$. Decompose M as

$M = \sum_{i,j} E_{i,j} \otimes |v_i\rangle\langle v_j|$, where for every $(i,j) \in [d]^2$ $E_{i,j}$ is a $D^2 \times D^2$ matrix on Alice and Bob's systems; by definition

$$E_{i,j} = \sum_k \langle v_i|C_k|v_j\rangle M_k.$$

Since each M_k is Hermitian, we have $E_{i,j} = (E_{j,i})^\dagger$. We will need the following bound.

Claim 24. *For every $i \in [d]$,*

$$\max\left\{\left\|\sum_j E_{i,j}E_{i,j}^\dagger\right\|_\infty, \left\|\sum_j E_{i,j}^\dagger E_{i,j}\right\|_\infty\right\} \le 2\left(K_G^{\mathbb{C}}\right)^3 \beta(G)^2. \tag{22}$$

PROOF: Let $|\Phi\rangle$ be any vector. Then

$$\begin{aligned}
\langle\Phi|\sum_j E_{i,j}(E_{i,j})^\dagger|\Phi\rangle &= \sum_{k,k'}\sum_j \langle\Phi|M_kM_{k'}|\Phi\rangle\langle v_i|C_k|v_j\rangle\overline{\langle v_i|C_{k'}|v_j\rangle} \\
&= \sum_{k,k'} \langle\Phi|M_kM_{k'}|\Phi\rangle \langle C_k^i, C_{k'}^i\rangle \\
&\le K_G^{\mathbb{C}} \max_{a_k,b_{k'}\in B(\mathbb{C})} \left|\sum_{k,k'} \langle\Phi|M_kM_{k'}|\Phi\rangle a_k b_{k'}\right|, \tag{23}
\end{aligned}$$

where in the second equality we let C_k^i be the i-th row of C_k (in the $|v_j\rangle$ basis), which has norm 1 (since C_k as a matrix is an observable), and the last inequality is Grothendieck's inequality. Using the Cauchy-Schwarz inequality and the fact that the M_k are Hermitian, there are complex numbers $c_k \in B(\mathbb{C})$ such that the sum on the right-hand side of Eq. (23) is bounded from above by

$$\left(\langle\Phi|\sum_k M_k^* a_k\right) \cdot \left(\sum_{k'} M_{k'} b_{k'}|\Phi\rangle\right) \le \left\|\sum_k M_k c_k|\Phi\rangle\right\|_2^2$$

$$\le \left\|\sum_k M_k c_k\right\|_\infty^2.$$

For every i,j define $H_{ij} = \sum_k T_{ijk}c_k$, and let $|\Phi\rangle$ be the largest eigenvector of $\sum_k M_k c_k$, so that

$$\left\|\sum_k M_k c_k\right\|_\infty = \sum_{i,j} H_{ij}\langle\Phi|A_i \otimes B_j|\Phi\rangle.$$

Using Grothendieck's inequality, this last expression can be upper bounded as

$$\left|\sum_{i,j} H_{ij}\langle\Phi|A_i \otimes B_j|\Phi\rangle\right| \le K_G^{\mathbb{C}}\left|\sum_{i,j} H_{ij}a_i b_j\right|,$$

where now $a_i, b_j \in \mathbb{C}$ are arbitrary complex numbers with modulus at most 1. Using that T is a real tensor, and the a_i, b_j, c_k complex numbers of modulus 1, we have

$$\left|\sum_{ijk} T_{ijk}a_i b_j c_k\right| \le 2 \sup_{a_i',b_j',c_k'\in\{\pm1\}} \left|\sum_{ijk} T_{ijk}a_i' b_j' c_k'\right| \le 2\beta(G).$$

Putting everything together, we have shown

$$\left\| \sum_k M_k c_k \right\|_\infty^2 \le 2(K_G^{\mathbb{C}})^2 \beta(G)^2.$$

Using $E_{i,j} = E_{j,i}^\dagger$, this proves (22). □

Let ε_j be i.i.d. $\{\pm 1\}$-valued standard Bernoulli random variables, and for every $i \in [d]$ let $\tilde{E}_i$ the matrix random variable promised by Theorem 23, and $E_i := \sum_j \varepsilon_j E_{i,j} + \tilde{E}_i$. Combining the estimate in Claim 24 with the bound (21) from Theorem 23, we get that

$$\max_i \left\| E_i \right\|_\infty \le \sqrt{6} \, (K_G^{\mathbb{C}})^{3/2} \beta(G). \tag{24}$$

Let ε_i' be i.i.d. $\{\pm 1\}$-valued standard Bernoulli random variables independent from the ε_i such that $\mathbb{E}\left[\varepsilon_i' \tilde{E}_j\right] = 0$ for all (i, j). Starting from the (E_i), let $\tilde{E}$ be the matrix random variable promised by Theorem 23, and let $E := \sum_i \varepsilon_i' E_i + \tilde{E}$. Using the triangle inequality, the bound (21) together with (24) leads to

$$\left\| E \right\|_\infty \le 3\sqrt{2d} \, (K_G^{\mathbb{C}})^{3/2} \beta(G), \tag{25}$$

which is valid for all choices of ε_j and ε_i'. We may now write

$$\langle \Psi | M | \Psi \rangle = \sum_{i,j} \lambda_i \lambda_j \langle u_i | E_{i,j} | u_j \rangle$$

$$= \mathbb{E}_{\varepsilon,\varepsilon'} \left[\mathrm{Tr} \left(E \cdot \left(\sum_{i,j} \varepsilon_i' \varepsilon_j \lambda_i \lambda_j | u_j \rangle \langle u_i | \right) \right) \right]$$

$$\le \mathbb{E}_{\varepsilon,\varepsilon'} \left[\left\| E \right\|_\infty \left\| \sum_{i,j} \varepsilon_i' \varepsilon_j \lambda_i \lambda_j | u_j \rangle \langle u_i | \right\|_1 \right], \tag{26}$$

where for the second equality we used that $\mathbb{E}\left[\varepsilon_i' \tilde{E}\right] = 0$ for every i, and the last follows from Hölder's inequality. The norm $\left\| E \right\|_\infty$ is bounded by (25), and to conclude it suffices to note that, since

$$\sum_{i,j} \varepsilon_i' \varepsilon_j \lambda_i \lambda_j | u_j \rangle \langle u_i | = \left(\sum_j \varepsilon_j \lambda_j | u_j \rangle \right) \left(\sum_i \varepsilon_i' \lambda_i \langle u_i | \right),$$

its trace norm is at most

$$\left\| \sum_j \varepsilon_j \lambda_j | u_j \rangle \right\| \left\| \sum_i \varepsilon_i' \lambda_i | u_i \rangle \right\| \le \left(\sum_j \lambda_j^2 \right)^{1/2} \left(\sum_i \lambda_i^2 \right)^{1/2} \le 1.$$

□

5 Conclusion and open problems

We have described a probabilistic construction of a family of XOR games $G = (G_N)$ in which players sharing entanglement may gain a large, unbounded advantage over the best classical, unentangled players. For any $N = 2^n$ the game G_N has N^2 questions per player, and is such that

the ratio $\beta^*(G)/\beta(G) = \Omega(\sqrt{N}\log^{-5/2} N)$. Our results raise two immediate open questions. The first is whether this estimate is optimal: we could only prove an upper bound of $O(N)$ on the largest possible ratio (for games, such as G_N, with at most N^2 questions per player). The second is to give a *deterministic* construction of a family of games achieving a similar (or even weaker) ratio. Such a construction would be of great interest both to experimental physicists and to operator space theorists, no small feat!

In our results we measured the advantage of entangled players in a given XOR game G multiplicatively, as a function of the ratio $\beta^*(G)/\beta(G)$. Although this has become customary, if one is interested in experimental realizations it may not be the most appropriate way to measure the advantage gained by entanglement: indeed, even a very large ratio between the entangled and unentangled biases may be hard to notice if both biases are small, requiring many repetitions of the experiment in order to estimate either bias. In the case of our specific construction, one may compute that $\beta^*(G_N) = \Omega(N^{-3/2})$ and $\beta(G_N) = O(N^{-2}\log^{5/2} N)$: while the ratio of these two quantities is large, both are relatively close to 0 and may thus be difficult to differentiate through experiment. It is an interesting open problem to also obtain large separations as measured, say, by the *difference $\beta^*(G) - \beta(G)$*.

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