Gravitational Lensing by Charged Black Holes

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The physics associated with spherically symmetric charged black holes are analyzed from the point of view of weak gravitational lensing as a means for determining the dimensionality of spacetime. In particular, the effect of charged black holes in four and five space time dimensions on the motion of photons are studied using the equations for the null geodesics and deriving the weak limit bending angle and the time delay for arrival times.

I. INTRODUCTION

Recently there has been renewed interest in the gravitational lensing by black holes [2–11] both in the weak and strong field limits. In addition the possibility that a black hole may be able to hold some non-zero electric charge has been raised by a number of authors. Charged black holes may well be the end point of the evolution of massive highly magnetized stars where the neutralization of charge is avoided through some mechanism of selective accretion. Isolated black holes may then be capable of remaining charged for sometime and may therefore be detectable through their influence on the passage of light rays in the space surrounding them.

However black holes appear in higher dimensional theories of gravity as well. The question one might ask is whether it is possible to make an observation that would distinguish the difference between black holes in four space time dimensions and those that might exist in higher dimensions but where the influence on the lower 4-D case can be felt.

In order to understand the physics that might arise as a result of the collapse to higher dimensions we undertake a study of the gravitational lensing of photons passing by charged black holes that are obtained as vacuum solutions to four dimensional Einstein theory and five-dimensional (classical) Kaluza-Klein theory. That is we compare the gravitational lensing occurring close to a Reissner-Nordström black hole to a class of charge black hole solutions in Kaluza-Klein theory that have been discussed by Liu and Wesson [1]. These higher dimensional black holes can exist without charge. However in that case the projection onto a four-dimensional spacetime of the uncharged solution is equivalent to the 4-dimensional Schwarzschild solution and no difference in gravitational light ray bending would be measured. Therefore in order to determine whether or not the higher dimensional case could exist, it is necessary that the black holes be capable of holding onto some residual electric charge and as a result produce a difference in the deflection angle and therefore a difference in the location of the lensed images.

It has already been shown by Sereno [4] that the deflection angle of a Reissner-Nordström black hole is less than that for a Schwarzschild black hole with the same mass. That is the effect of the charge is to increase the

A gravitational lensing observation alone is insufficient to determine both the charge and the dimensionality of the black hole. However should the black hole have an accretion disk of ionized material surrounding it, one can in principle determine the charge from the Lorentz force law. The electric field for both the 4D and the 5D charged black holes that we consider here takes on its flat space Coulombic configuration and therefore the charge can be determined independently of the spacetime dimensions.

II. CHARGED 5-D KALUZA-KLEIN BLACK HOLES

A number of spherically symmetric solutions to the vacuum Kaluza-Klein equations are known. However, most of them lack event horizons and therefore cannot be considered as black hole solutions. In what follows we will concentrate on a particular class of solutions that in the appropriate limit reduce to the standard 4D Schwarzschild solution. Some of the properties of these black holes have been discussed previously by Liu and Wesson [1] who referred to these objects as 5D charged black holes.

Using coordinates $(x^0, x^1, x^2, x^3, x^4) = (t, r, \theta, \phi, \psi)$ where ψ represents a spatial coordinate in the fifth dimension, the line element for the charged black holes can be written in the form

$$
\begin{aligned}
ds^2 &= \mathcal{B}(r)\mathcal{E}^{-1}(r)dt^2 - \mathcal{B}^{-1}(r)dr^2 - r^2 d\theta^2 - \\
&\quad r^2 \sin^2\theta d\phi^2 - \mathcal{E}(r)(d\psi + \mathcal{A}(r)dt)^2, \qquad (1)
\end{aligned}
$$

where $\mathcal{A}(r)$, $\mathcal{B}(r)$, $\mathcal{E}(r)$ are the potentials what can be written in the following form

$$
\mathcal{E} \equiv \frac{1-k\mathcal{B}}{1-k} = 1 + \frac{2Mk}{r} \qquad (2)
$$

$$
\mathcal{A} \equiv \frac{\sqrt{k}(\mathcal{B}-1)}{1-k\mathcal{B}} = -\frac{2M\sqrt{k}}{\mathcal{E}r} \qquad (3)
$$

$$
\mathcal{B} \equiv 1 - \frac{2M(1-k)}{r} = \mathcal{E} - \frac{2M}{r}. \qquad (4)
$$

This leads to a class of solutions that depend on the two-parameters (k, M). The electric field (Faraday) tensor $F_\alpha\beta$ has a single component

$$
F_{01} = E_{(r)} = \frac{2M\sqrt{k}}{\mathcal{E}^2 r^2},
$$

which is the static electric field component in the radial direction. Using the expressions for $\mathcal{E}$ and $\mathcal{A}$, and the condition that the electrostatic potential reduces to the Coulomb potential as the distance to the black hole approaches infinity,

$$\lim_{r\to\infty}(\mathcal{A}) = -\frac{2M\sqrt{k}}{r} \tag{5}$$

$$= -\frac{Q}{r} \tag{6}$$

we can to determine the parameter to be k

$$k = \frac{Q^2}{4M^2}.$$

Now for the metric of the the form
$g = diag(A', B', C', D')$, the coefficients are given by

$$A_{(r)} = \frac{\mathcal{B}}{\mathcal{E}} = 1 - \frac{2M}{r - \frac{Q^2}{2M}} \tag{7}$$

$$B_{(r)} = \frac{1}{\mathcal{B}} = \left(1 - \big(2M - \frac{Q^2}{2M}\big)\frac{1}{r}\right)^{-1} \tag{8}$$

$$C_{(r)} = 1 \tag{9}$$

$$D_{(r)} = \mathcal{E} = 1 + \frac{Q^2}{2Mr}, \tag{10}$$

and the electric field becomes

$$E_{(r)} = \frac{Q}{\mathcal{E}^2 r^2}.$$

Clearly this metric is different than the standard Reissner-Nordström solution. However when the charge Q becomes infinitesimally small, the solution reduces to the 5D Schwarzschild vacuum solution, which is just the 4D Schwarzschild solution with a flat fifth dimension.

III. APPROXIMATION OF THE DEFLECTION ANGLE

The general second order differential equation of the inverse radial distance from the black hole was nicely derived by J. Bodenner and C.M. Will [2]. A general four dimensional, static, spherically symmetric line element can be written as

$$ds^2 = A_{(r)}dt^2 - B_{(r)}dr^2 - C_{(r)}r^2(d\theta^2 + \sin\theta d\phi^2). \tag{11}$$

The equations of motion can be obtained either by 6 the Lagrangian or from the geodesic equations. For photons, the line element is zero. To simplify the calculations, the variation in the azimuthal angle can be set to zero since we are dealing with spherical symmetry. The constants of motion for this case are

$$\ell \equiv A\frac{dt}{d\lambda} \tag{12}$$

$$J \equiv Cr^2\frac{d\phi}{d\lambda} \tag{13}$$

$$0 = \frac{d}{d\lambda}\left(2B\frac{dr}{d\lambda}\right) + A'\left(\frac{dt}{d\lambda}\right)^2 - B'\left(\frac{dr}{d\lambda}\right)^2 - (Cr^2)'\left(\frac{d\phi}{d\lambda}\right)^2, \tag{14}$$

where ℓ and J are the energy and angular momentum of the photon and a prime stands for a derivative with respect to r. Substituting equations 12 and 13 into 14 and making the 0 $u = 1/r$ and rewriting it such that ϕ is the dependent variable, we obtain a second order differential equation for the inverse radial distance from the black hole u,

$$\frac{d^2u}{d\phi^2} + \left(\frac{C}{B}\right)u = -\frac{1}{2}u^2\frac{d}{d\lambda}\left(\frac{C}{B}\right) + \frac{\ell^2}{2J^2}\frac{d}{d\lambda}\left(\frac{C^2}{AB}\right). \tag{15}$$

It can be shown that $\ell^2/J^2 = 1/b^2$, where b is the impact parameter. Once the metric is specified, equation 15 can be approximated to find the angle of deflection.

A. Schwarzschild

The Schwarzschild metric coefficients are $A = B^{-1} = 1 - 2Mu$ and $C = 1$. This will leave equation 15 as

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2. \tag{16}$$

Assuming the solution to be of the form $u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$ will enable us to approximate the solution to arbitrary order of ϵ. The solution to the homogeneous equation is $u_0 = u_N\cos\phi$, where u_N is the inverse of the Newtonian distance of closest approach. This happens to be be equal to the inverse of the impact parameter ($u_N = 1/b$). If we now set $\epsilon = Mu_N$, equation 16 can be written as

$$(u_0'' + u_o) + \epsilon(u_1'' + u_1 - 3\cos^2\phi) + \epsilon^2(u_2'' + u_2 - 6\cos\phi u_1) + \cdots = 0, \tag{17}$$

so that the equations up to second order ϵ become

$$u_1'' + u_1 = 3\cos^2\phi \tag{18}$$

$$u_2'' + u_2 = 6u_1\cos\phi \tag{19}$$

Solving these leaves us with the following expression for the inverse radial distance from the black hole.

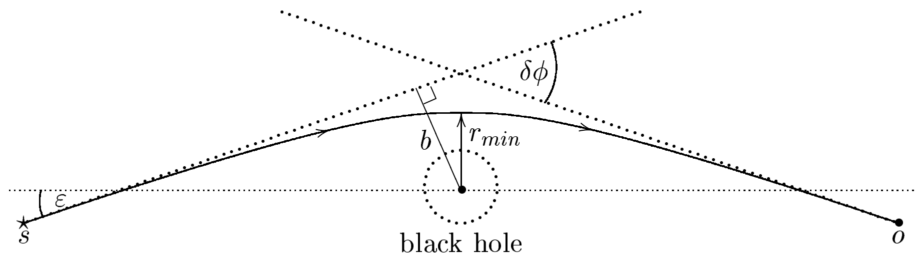

FIG. 1: Photon deflection by spacetime curvature

$$u \simeq u_N \left(\cos\phi + \frac{1}{2} M u_N (3 - \cos(2\phi)) + \frac{3}{16} M^2 u_N^2 (20\phi \sin\phi + \cos(3\phi)) \right). \tag{20}$$

For large distances we approximate the deflection angle ϕ by solving equation 20 upto second order ϵ. The inverse radial distance u 0 to zero and the deflection angle can be expected to be very small. The deflection angle can be found by solving for ε, using the angle $\phi = \pi/2 + \varepsilon$ (see figure 1). Here the fact is used that the trajectory is symmetric about $\phi = 0$, so that the deviation from straight line motion is the angle $\delta\phi = 2\varepsilon$. Since ε is very small, the trigonometric terms can be expanded and we can keep ε terms only upto first order of the zeroth order ϵ (Mu_N). One more substitution needs to be made for u_N. We want the final expression for the deflection angle to be in terms of $1/r_{min}$ and not u_N. Since u_N is only the 0 order approximation of $1/r_{min}$. The distance of closest approach can be found to occur when $\phi = 0$, and gives $1/r_{min} = u_N + M u_N^2 + 3M^2 u_N^3/16$. All this leaves us with an approximation of the deflection angle up to second order M/r_{min}

$$\delta\phi \simeq \frac{4M}{r_{min}} + \frac{M^2}{r_{min}^2} \left(\frac{15}{4}\pi - 4 \right) \tag{21}$$

B. Reissner-Nordström

The metric coefficients for Reissner-Nordström are $A = B^{-1} = 1 - 2Mu + Q^2u^2$ and $C = 1$. Following the same procedure as for the Schwarzschild case, equation (15) becomes

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2 - 2Q^2u^3, \tag{22}$$

so that the expanded form of equation (22) is

$$(u_0'' + u_o) + \epsilon(u_1'' + u_1 - 3\cos^2\phi) + \epsilon^2(u_2'' + u_2 - 6\cos\phi u_1 + 2\frac{Q^2}{M^2}\cos^3\phi) + \cdots = 0. \tag{23}$$

Since u_N is assumed to be very small, the last term involving the charge Q can be counted under the second order of Mu_N as long as Q is smaller than M by an order larger than that of
u_N. The equations up to second order ϵ are now

$$u_1'' + u_1 = 3\cos^2\phi \tag{24}$$

$$u_2'' + u_2 = 6u_1\cos\phi - 2\frac{Q^2}{M^2}\cos^3\phi. \tag{25}$$

From this we find the approximate inverse radial distance,

$$u \simeq u_N \left[\cos\phi + \frac{1}{2} M u_N \Big(3 - \cos(2\phi)\Big) + \frac{3}{16} M^2 u_N^2 \left(\Big(20 - \frac{4Q^2}{M^2}\Big)\phi\sin\phi + \Big(1 + \frac{Q^2}{3M^2}\Big)\cos(3\phi) \right) \right], \tag{26}$$

and an approximate deflection angle

$$\delta\phi \simeq \frac{4M}{r_{min}} + \frac{M^2}{r_{min}^2}\left(\frac{15}{4}\pi - 4\right) - \frac{3}{4}\frac{Q^2}{r_{min}^2}\pi \tag{27}$$

The charge brings a small correction in the second order term, causing the approximate deflection 0 be smaller than in the Schwarzschild case, which agrees with the results from [4], where Fermat's principle is used to derive this.

C. Kaluza-Klein

The equations of motion for both massive and zero-mass particles are given by the geodesic equations of the 5D spacetime. Since we are interested in the 0 of photons in four dimensions, we need to determine the geodesic equations in 4D space-time. The first step to solving these is to determine the constants of the motion, which can be most easily be accomplished by analyzing the Lagrangian associated with the metric (1). The Lagrangian is given by

$$\mathcal{L} = A_{(r)} \left(\frac{dt}{d\lambda}\right)^2 - B_{(r)} \left(\frac{dr}{d\lambda}\right)^2 - C_{(r)} r^2 \left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta \left(\frac{d\phi}{d\lambda}\right)^2\right] - D_{(r)} \left[\frac{d\psi}{d\lambda} + \mathcal{A}\frac{dt}{d\lambda}\right]^2, \quad (28)$$

using the metric coefficients 7, 8, 9 and 10. Here λ is an affine parameter along the geodesic curve. Assuming that the orbit of the test particle is confined to the plane $\theta = \frac{\pi}{2}$ with $d\theta/d\lambda = 0$ the Lagrangian (28) leads directly to three constants of motion:

$$\ell \equiv A\frac{dt}{d\lambda} - D\left[\frac{d\psi}{d\lambda} + \mathcal{A}\frac{dt}{d\lambda}\right]\mathcal{A} \quad (29)$$

$$J \equiv Cr^2\frac{d\phi}{d\lambda} \quad (30)$$

$$N \equiv D\left[\frac{d\psi}{d\lambda} + \mathcal{A}\frac{dt}{d\lambda}\right] \quad (31)$$

The constant of motion N must be proportional to the charge e of the test particle in order to recover the Lorentz force law in the appropriate limiting case (see [1]).

Because we are considering photons as test particles, the line element and therefore the Lagrangian $\mathcal{L}$ must vanish. The test particle charge is also zero, which leaves only two non trivial constants of motion since N is zero in this case.

Therefore, after substituting in the 0 of motion and the 0 expressions for the 1 coefficients, the radial equation of motion becomes

$$\left(\frac{dr}{d\lambda}\right)^2 = \left(1 + \frac{Q^2}{2Mr}\right)\ell^2 - \frac{J^2}{r^2}\left[1 - \left(2M - \frac{Q^2}{2M}\right)\frac{1}{r}\right].$$

This equation can be written in the form

$$\left(\frac{dr}{d\lambda}\right)^2 - \mathcal{V}^2(r) = \ell^2$$

where $\mathcal{V}$ can be interpreted as the effective 0, which is given by

$$\mathcal{V}^2(r) = \left(\frac{Q^2}{2M} - 2M\right)\frac{J^2}{r^3} + \frac{J^2}{r^2} - \frac{Q^2\ell^2}{2Mr},$$

which clearly has a $1/r$ term when $Q \neq 0$. The effective potential looks like that for massive test particles and therefore indicates that stable photon orbits are possible, which is unlike the $4D$ solutions. Returning to the weak lensing case where r is always well outside of the the region close to the black hole we expect to obtain hyperbolic orbits and will now proceed to derive a deflection angle for such trajectories.

With the metric coefficients 7, 8, 9 and **??**, we find

$$\frac{d^2u}{d\phi^2} + u = \frac{Q^2}{4Mb^2} + \frac{3}{2}\left(2M - \frac{Q^2}{2M}\right)u^2 \quad (32)$$

$$= \alpha + \beta u^2. \quad (33)$$

Since we are going to approximate the solution to this equation only at distances much larger than the impact parameter, it can again be written in terms of a perturbation parameter ϵ, such that $u'' + u = \alpha + \epsilon(\beta u^2/\epsilon)$. The homogeneous solution to this equation is $u_0 = \alpha + u_N\cos\phi$. If we now set $\epsilon = u_N\beta$, equation (33) becomes $u'' + u = \alpha + \epsilon(u_N u^2)$. Now expanding u in terms of a power series of ϵ, we get the equations to first and second order ϵ to be

$$u_1'' + u_1 = \frac{1}{b}\left(\alpha + u_N\cos\phi\right)^2 \quad (34)$$

$$u_2'' + u_2 = \frac{2}{b}\left(\alpha + u_N\cos\phi\right)u_1. \quad (35)$$

After eliminating all excess terms of higher order ϵ, we end up with the following expression for the inverse radial distance,

$$u \simeq u_N\Bigg[\cos\phi + \frac{1}{2}Mu_N\Big(3 - \frac{Q^2}{4M^2} - (1 - \frac{Q^2}{4M^2})\cos(2\phi)\Big) + \frac{3}{16}M^2u_N^2\Big((20 - \frac{2Q^2}{M^2} + \frac{5Q^4}{4M^4})\phi\sin\phi + (1 - \frac{Q^2}{2M^2} + \frac{Q^4}{16M^4})\cos(3\phi)\Big)\Bigg], \quad (36)$$

from which the deflection angle can be found to be

$$\delta\phi \simeq \frac{4M}{r_{min}} - \frac{Q^2}{2Mr_{min}} + \frac{M^2}{r_{min}^2}\left(\frac{15}{4}\pi - 4\right) + \frac{Q^2}{r_{min}^2}\left(1 - \frac{3}{8}\pi\right) + \frac{Q^4}{16M^2r_{min}^2}\left(\frac{15}{4}\pi - 3\right). \quad (37)$$

The charge already appears in the first order correction term would and thus has a significant effect on the deflection angle.

IV. EXACT DEFLECTION ANGLES

The deflection angles can also be calculated exactly by finding an expression for the polar angle in terms of the radial distance. Following §8.5 of [12] we find that the total deflection angle can be found by solving an integral in terms of the four dimensional metric coefficients.

$$\delta\phi = 2\int_{r_{min}}^{\infty} \frac{\sqrt{B_{(r)}}}{r\sqrt{\left(\frac{r}{r_{min}}\right)^2\left(\frac{A_{(r_{min})}}{A_{(r)}}\right) - 1}} - \pi \quad (38)$$

This can also be used for zero-charge test particles in the Kaluza-Klein theory since the fifth 0 is flat in this

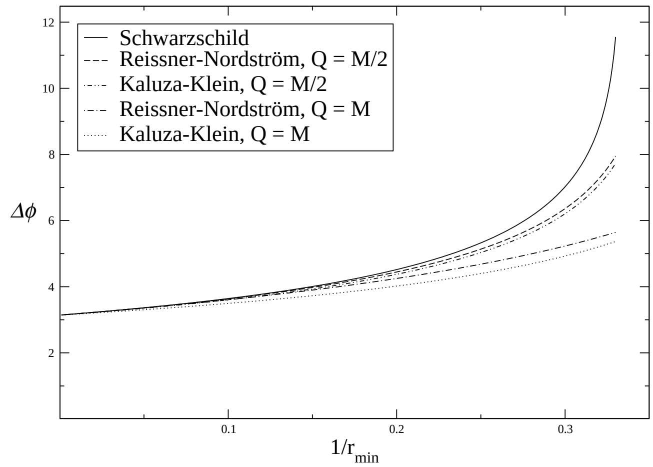

FIG. 2: Total deflection angle $\Delta\phi$ (in radians) as a function of the inverse distance of closest approach $1/r_{min}$ (in Schwarzschild radii).

case. A plot of the deflection angle $\Delta\phi = \pi + \delta\phi$ shows that the deflection decreases as the charge on the black hole increases. This effect is much more dramatic for the five dimensional Kaluza-Klein solution than it is for the Reissner-Nordström solution. When the charge on the black hole is zero, both the Reissner-Nordström and Kaluza-Klein solutions reduce to the Schwarzschild solution.

V. APPROXIMATE TIME DELAY

The deviation from flat space travel time that occurs as a result of the curved trajectories of photons in the vicinity a black hole can be estimated for all three geometries. Following §8.7 of [12], we find that when isotropic coordinates are used, the exact time delay is given by

$$t_{(r,r_{min})} = \int_{r_{min}}^{r} \left(\frac{B_{(r)}/A_{(r)}}{1 - \frac{A_{(r)}}{A_{(r_{min})}} \left(\frac{r_{min}}{r}\right)^2} \right)^{1/2} dr. \tag{39}$$

A. Schwarzschild and Kaluza-Klein

To obtain isotropic coordinates for the Schwarzschild line element, we can let $r \to \rho(1 + M/2\rho)^2$. Using this, the new metric coefficients and thus the time delay can be calculated. To first order M/ρ we have

$$t_{(\rho,\rho_{min})} \simeq \sqrt{\rho^2 - \rho^2_{min}} + 2M \ln\left(\frac{\rho + \sqrt{\rho^2 - \rho^2_{min}}}{\rho_{min}}\right) + M\sqrt{\frac{\rho - \rho_{min}}{\rho + \rho_{min}}}. \tag{40}$$

The same coordinate transformation can be used to obtain isotropic coordinates for the Kaluza-Klein line element, the only difference being that M gets replaced by $j = 1/2(2M - Q^2/2M)$. Now, the metric coefficients become

$$A_{(\rho)} = \frac{(1 - \frac{j}{2\rho})^2}{(1 + \frac{j}{2\rho})^2 + \frac{Q^2}{2M\rho}} \tag{41}$$

$$B_{(\rho)} = \rho^2 C_{(\rho)} = (1 + \frac{j}{2\rho})^4. \tag{42}$$

After expanding these up to second order $1/\rho$, the integrand can be broken up into

$$\frac{B_{(\rho)}}{A_{(\rho)}} \simeq 1 + \frac{4j - \frac{Q^2}{2M}}{\rho} + \frac{\frac{9j^2}{2} + \frac{jQ^2}{2M}}{\rho^2} \tag{43}$$

and

$$1 - \frac{A_{(\rho)}}{A_{(\rho_{min})}}\left(\frac{\rho_{min}}{\rho}\right)^2 \simeq \left(1 - \frac{\rho^2_{min}}{\rho^2}\right)\Bigg[1 - \frac{(2j + \frac{Q^2}{2M})\rho_{min}}{\rho(\rho + \rho_{min})} - \frac{(j^2 + \frac{jQ^2}{2M^2})}{\rho^2} - \frac{(2j + \frac{Q^2}{2M})^2\rho_{min}}{\rho^3}\left(1 + \frac{\rho^2_{min}}{\rho^2}\right)\Bigg], \tag{44}$$

so that to second order j/ρ and j/ρ_{min},

$$t_{(r,r_{min})} \simeq \int_{\rho_{min}}^{\rho} \left(1 - \frac{\rho^2_{min}}{\rho^2}\right)^{-\frac{1}{2}} \Bigg[1 + \frac{(2j - \frac{Q^2}{2M})\rho_{min}}{\rho(\rho + \rho_{min})} + \frac{8j - Q^2}{2\rho} + \frac{(\frac{11}{2}j^2 - \frac{jQ^2}{M})}{\rho^2} + \frac{(8j^2 - \frac{3jQ^2}{M} + \frac{Q^4}{4M^2})\rho_{min}}{\rho^2(\rho + \rho_{min})} - \frac{(\frac{2Q^2 j}{M} - 4j^2 - \frac{Q^4}{4M^2})\rho_{min}}{\rho^3}\left(1 + \frac{\rho^2_{min}}{\rho^2}\right)\Bigg] d\rho. \tag{45}$$

The approximate time it takes a light ray to go from ρ_{min} to ρ is now

$$
\begin{aligned}
t_{(r,r_{min})} \simeq & \quad \sqrt{\rho^2-\rho_{min}^2}+ \\
& \left(2M-\frac{Q^2}{4M}\right)\ln\left(\frac{\rho+\sqrt{\rho^2-\rho_{min}^2}}{\rho_{min}}\right)+M\sqrt{\frac{\rho-\rho_{min}}{\rho+\rho_{min}}}+ \\
& \frac{1}{\rho_{min}}\left(\frac{11M^2}{4}-\frac{7Q^2}{8}+\frac{3Q^4}{64M^2}\right)\tan^{-1}\left(\frac{\sqrt{\rho^2-\rho_{min}^2}}{\rho_{min}}\right)+ \\
& \frac{1}{\rho_{min}}\left(2M^2-\frac{Q^2}{4}\right)\left[\frac{\sqrt{\rho^2-\rho_{min}^2}}{\rho+\rho_{min}}+\right. \\
& \left.\qquad \tan^{-1}\left(\frac{\rho_{min}}{\sqrt{\rho^2-\rho_{min}^2}}\right)-\frac{\pi}{2}\right]+ \\
& \frac{1}{\rho_{min}}2M^2\sqrt{\rho^2-\rho_{min}^2}\left(\frac{1}{\rho^2}+\frac{2\rho^2+\rho_{min}^2}{3\rho^3}\right) \qquad (46)
\end{aligned}
$$

B. Reissner-Nordström

In order to get the Reissner-Nordström line element in isotropic coordinates, we make the substitution $r \to \rho(1+M/\rho+(M^2-Q^2)/4\rho^2)$, so that,

$$
A_{(\rho)} = \rho\left(1+\frac{M}{\rho}+\frac{M^2-Q^2}{\rho^2}\right) \qquad (47)
$$

$$
B_{(\rho)} = \rho^2 C_{(\rho)} = \left(\frac{M^2-4\rho^2-Q^2}{(M+2\rho)^2-Q^2}\right)^2 \qquad (48)
$$

Once again, expanding upto second order $1/\rho$, we get

$$
\frac{B_{(\rho)}}{A_{(\rho)}} \simeq 1+\frac{4M}{\rho}+\frac{11M^2-3Q^2}{2\rho^2}
$$

and

$$
\begin{aligned}
1-\frac{A_{(\rho)}}{A_{(\rho_{min})}}\left(\frac{\rho_{min}}{\rho}\right)^2 \simeq & \left(1-\frac{\rho_{min}^2}{\rho^2}\right)\times \\
& \left[1-\frac{2M\rho_{min}}{\rho(\rho+\rho_{min})}-\frac{Q^2}{\rho^2}-\frac{4M^2\rho_{min}}{\rho^2(\rho+\rho_{min})}\right], \quad (49)
\end{aligned}
$$

so that to second order M/ρ, Q/ρ, M/ρ_{min} and Q/ρ_{min},

$$
\begin{aligned}
t_{(r,r_{min})} \simeq & \int_{\rho_{min}}^{\rho}\left(1-\frac{\rho_{min}^2}{\rho^2}\right)^{-\frac{1}{2}}\left[1+\frac{2M}{\rho}+\right. \\
& \frac{M\rho_{min}}{\rho(\rho+\rho_{min})}+\frac{11M^2-3Q^2}{4\rho^2}+\frac{Q^2}{2\rho^2}+ \\
& \left.\frac{2M^2\rho_{min}}{\rho^2(\rho+\rho_{min})}+\frac{2m^2\rho_{min}}{\rho^2}\left(1+\frac{\rho_{min}^2}{\rho^2}\right)\right]d\rho. \quad (50)
\end{aligned}
$$

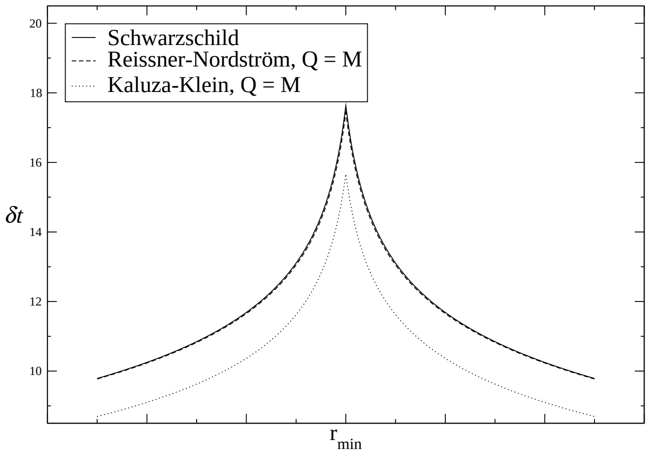

FIG. 3: The time delay due to geometrical lensing as a function of the distance of closest approach of the light ray. As a light source moves behind the black hole, the arrival time of its radiation increases. Here, the distance of closest approach (r_{min}) changes from 52 Schwarzschild radii to 2 and back again.

This leaves us with an approximate travel time

$$
\begin{aligned}
t_{(r,r_{min})} \simeq & \sqrt{\rho^2-\rho_{min}^2}+2M\ln\left(\frac{\rho+\sqrt{\rho^2-\rho_{min}^2}}{\rho_{min}}\right)+ \\
& M\sqrt{\frac{\rho-\rho_{min}}{\rho+\rho_{min}}}+ \\
& \frac{1}{\rho_{min}}\left(\frac{11M^2}{4}-\frac{Q^2}{4}\right)\tan^{-1}\left(\frac{\sqrt{\rho^2-\rho_{min}^2}}{\rho_{min}}\right)+ \\
& \frac{2M^2}{\rho_{min}}\left[\frac{\sqrt{\rho^2-\rho_{min}^2}}{\rho+\rho_{min}}+\tan^{-1}\left(\frac{\rho_{min}}{\sqrt{\rho^2-\rho_{min}^2}}\right)-\frac{\pi}{2}\right]+ \\
& \frac{2M^2}{\rho_{min}}\sqrt{\rho^2-\rho_{min}^2}\left(\frac{1}{\rho^2}+\frac{2\rho^2+\rho_{min}^2}{3\rho^3}\right) \qquad (51)
\end{aligned}
$$

Both the Kaluza-Klein and Reissner-Nordström times reduce to the same expression when $Q=0$, which corresponds to the approximate Schwarzschild time.

	Schwarzschild	Reissner-Nordström		Kaluza-Klein	
		$Q = M/2$	$Q = M$	$Q = M/2$	$Q = M$
$\delta\phi(rad)$	0.020592218	0.020531170	0.018008016	0.020348052	0.010293984
$c\delta t(km)$	23.28534541	23.28339441	22.62136541	23.27754143	20.62649894
$\delta t(msec)$	0.077669598	0.077663090	0.075454855	0.077643567	0.068800864

TABLE I: Deflection angles and time delays for a symmetric photon trajectory starting and finishing 10^4 masses from the black hole and with a distance of closest approach of 10^2 masses. Here $MG/c = 1$.

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