NOTE ON CHEBYSHEV POLYNOMIALS

In this note we show that Chebyshev polynomials are completely bounded in the sense of \[ABP19\]. As an immediate corollary, using the characterization of quantum query algorithms from \[ABP19\] and a well-known result of Nisan and Szegedy \[NS94\], we recover the quantum algorithm for the OR\(_n\) function restricted to strings of Hamming weight at most 1, as implied by Grover.

For each \(k \in \mathbb{N} \cup \{0\}\) the Chebyshev polynomial \(T_k \in \mathbb{R}[x]\) is the degree-\(k\) polynomial defined recursively by

\[
T_0(x) = 1, \\
T_1(x) = x, \\
T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x).
\]

Define the \(n\)-variate polynomials \(p_k \in \mathbb{R}[x_1, \ldots, x_n]\) by

\[
(1) \quad p_k(x_1, \ldots, x_n) = T_k \left( \frac{x_1 + \cdots + x_n}{n} \right).
\]

1. Main lemma

**Lemma 1.1.** For each \(k \geq 2\) there exists a \(k\)-linear form \(F_k\) on \(\mathbb{R}^n\) such that \(\|F_k\|_{cb} \leq 1\) and \(F_k(x, \ldots, x) = p_k(x)\) for each \(x \in \{-1, 1\}^n\).

**Proof:** Define the bilinear form \(F_2\) on \(\mathbb{R}^n\) and linear forms \(f_1^1, \ldots, f_1^n\) on \(\mathbb{R}^n\) by

\[
(2) \quad F_2(x, y) = \mathbb{E}_{i \in [n]} \left[ x_i \left( \frac{2 \mathbb{E}_{j \in [n]} [y_j] - y_i}{f_1(y)} \right) \right],
\]

where the expectations are over uniformly random indices in \([n]\). For \(k \geq 2\), recursively define the \((k+1)\)-linear form \(F_{k+1}\) and \(k\)-linear forms \(f_k^1, \ldots, f_k^n\) by

\[
(3) \quad F_{k+1}(x, y, z) = \mathbb{E}_{i \in [n]} \left[ x_i \left( \frac{2 F_k(y, z) - y_i f_{k-1}^i(z)}{f_k(y, z)} \right) \right],
\]

for \(x, y \in \mathbb{R}^n\) and \(z \in (\mathbb{R}^n)^{k-2}\).
We first show by induction on $k$ that $F_k(x, \ldots, x) = p_k(x)$ for every $x \in \{-1, 1\}^n$. Since $x_i^2 = 1$ for $x_i \in \{-1, 1\}$, it is easy to see from (2) that
\[
F_2(x, x) = 2(\mathbb{E}_{i \in [n]}[x_i])p_1(x) - 1 = p_2(x).
\]
Let $k \geq 2$ and assume that the claim holds for $k$. Below, the number of repetitions of $x$ in a sequence $(x, \ldots, x)$ will vary but be clear from the context. Again using that $x_i^2 = 1$, it follows from (3) that
\[
F_{k+1}(x, \ldots, x) = 2\mathbb{E}_{i \in [n]}[x_i]F_k(x, \ldots, x) - \mathbb{E}_{i \in [n]}[f_{k-1}^i(x, \ldots, x)].
\]
By the induction hypothesis, that $F_k(x, \ldots, x) = p_k(x)$, we find that
\[
\mathbb{E}_{i \in [n]}[f_{k-1}^i(x, \ldots, x)] = \mathbb{E}_{i \in [n]}\left[2F_{k-1}(x, \ldots, x) - x_if_{k-1}^i(x, \ldots, x)\right]
\]
\[
= 2p_{k-1}(x) - \mathbb{E}_{i \in [n]}[x_if_{k-1}^i(x, \ldots, x)]
\]
\[
= 2p_{k-1}(x) - F_{k-1}(x, \ldots, x)
\]
\[
= 2p_{k-1}(x) - p_{k-1}(x) = p_k(x).
\]
Hence,
\[
F_{k+1}(x, \ldots, x) = 2\mathbb{E}_{i \in [n]}[x_i]p_k(x) - p_{k-1}(x) = p_{k+1}(x),
\]
which proves the claim.

Next we show that $\|F_k\|_{cb} \leq 1$. To this end, we first show that for every $k, d \in \mathbb{N}$, vector $v \in \mathbb{C}^d$ and collection of contractions $X = ((X_i^1)_{i=1}^n, \ldots, (X_i^k)_{i=1}^n)$ in $\mathbb{C}^{d \times d}$, we have
\[
\mathbb{E}_{i \in [n]}\left[\|f_k^i_d(X)v\|_2^2\right] \leq \|v\|_2^2,
\]
where $(f_k^i)_d$ is the “lifted” version of the $k$-linear form $f_k^i$ as in (3).

We again induct on $k$. For $k = 1$, the expectation (4) reduces to
\[
\mathbb{E}_{i \in [n]}\left[\|2\mathbb{E}_{j \in [n]}[X_j] - X_i\|_2^2\right].
\]
The above square norm equals
\[
4\mathbb{E}_{i,k \in [n]}\langle X_jv, X_kv \rangle - 2\mathbb{E}_{j \in [n]}\left[\langle X_jv, X_i v \rangle\right] - 2\mathbb{E}_{k \in [n]}\left[\langle X_i v, X_kv \rangle\right] + \|X_i v\|_2^2.
\]
The expectation over $i$ in (5) thus causes the first three terms in (6) to cancel. The result follows since each $X_i$ is a contraction.

Let $k \geq 1$ and assume the claim for $k$. Let $X = (X_i^1)_{i=1}^n$ and let $Y = ((X_i^2)_{i=1}^n, \ldots, (X_i^k)_{i=1}^n)$. By definition of $f_{k+1}^i$, we then have that
\[
(f_{k+1}^i)_d(X, Y) = 2(F_{k+1})_d(X, Y) - X_i^1(f_k^i)_d(Y).
\]
Define \( A = (F_{k+1})_d(X, Y) \) and \( B_i = X_i^i(f^i_k)_d(Y) \), so that the above equals \( 2A - B_i \). Observe that by definition of \( F_{k+1} \), we have
\[
\mathbb{E}_{i \in [n]}[B_i] = (F_{k+1})_d(Y, X) = A.
\]
Hence,
\[
\mathbb{E}_{i \in [n]}[\| (f^i_k)_d(X) v \|_2^2] = \mathbb{E}_{i \in [n]}[\| (2A - B_i) v \|_2^2]
\]
\[
= \mathbb{E}_{i \in [n]}[4\| Av \|_2^2 - 2\langle Av, B_i v \rangle - 2\langle B_i v, Av \rangle + \| B_i v \|_2^2]
\]
\[
= \mathbb{E}_{i \in [n]}[\| B_i v \|_2^2]
\]
\[
= \mathbb{E}_{i \in [n]}[\| X_i^i(f^i_k)_d(Y) v \|_2^2]
\]
\[
\leq \mathbb{E}_{i \in [n]}[\| (f^i_k)_d(Y) v \|_2^2]
\]
\[
\leq \| v \|_2^2,
\]
where the first inequality follows from the fact that \( X_i \) is a contraction and and the second inequality follows by the induction hypothesis. This proves (4).

Let \( X, X, Y \) and \( v \) be as above. Then, by Jensen’s inequality and (4),
\[
\|(F_k)_d(X, Y) v \|_2 = \left\| \mathbb{E}_{i \in [n]}[X_i^i(f^i_{k-1})_d(Y)] v \right\|_2
\]
\[
\leq \mathbb{E}_{i \in [n]}[\| (f^i_{k-1})_d(Y) v \|_2]
\]
\[
\leq \left( \mathbb{E}_{i \in [n]}[\| (f^i_{k-1})_d(Y) v \|_2^2] \right)^{1/2}
\]
\[
\leq \| v \|_2
\]
showing that \( \| F_k \|_{cb} \leq 1 \).

2. Obtaining Grover’s algorithm.

**Notation.** For \( i \in [n] \), let \( e_i \in \{-1, 1\}^n \) be the vector with \(-1\) on the \( i \)-th position and 1s otherwise. Let \( \text{OR}_n \) be an \( n \)-bit function defined as: \( \text{OR}_n(x) = 1 \) if and only if \( x = 1^n \). Let \( |x| = \sum_i x_i \).

Nisan and Szegedy [NS94] showed that the Chebyshev polynomials can be used to find low-degree polynomials that approximate \( \text{OR}_n \). A slight modification of their argument allows us to recover a the existence of a \( O(\sqrt{n}) \)-quantum query algorithm for \( \text{OR}_n \) restricted to strings of Hamming weight at most 1, as implied by Grover.

**Lemma 2.1.** Let \( D = \{e_i\}_{i \in [n]} \cup \{1^n\} \) and let \( \text{OR}_n : D \to \{-1, 1\} \). There exists a \( O(\sqrt{n}) \)-query quantum algorithm that, on input \( x \), outputs a sign with expected value \( \text{OR}(x) \), with error at most \( 1/4 \).
Proof: Let $d = 2\pi / 5 \cdot \sqrt{n}$. Here, we show that

(7) \[ |T_d\left(\frac{|x|}{n}\right) - \text{OR}(x)| \leq 1/4 \quad \text{for all } x \in D.\]

To see this, first observe that for $x = 1^n$, we have $T_d(|x|/n) = 1$ and $\text{OR}_n(1^n) = 1$, so Eq. (7) is satisfied. Let $x = e_i$ for some $i \in [n]$. By the definition of Chebyshev polynomials, we have

$$T_d\left(\frac{|x|}{n}\right) = T_d\left(1 - \frac{2}{n}\right) = \cos\left(d \arccos\left(1 - \frac{2}{n}\right)\right).$$

By the Taylor series expansion of $\arccos(1 - z)$ (around the point $z = 0$), we have $\arccos(1 - z) \geq \sqrt{2z}$. This implies that $d \arccos(1 - 2/n) \geq 2\pi / 5 \cdot \sqrt{n} \cdot \sqrt{4/n} = 4\pi / 5$. Using the monotonicity and negativity of $\cos(\phi)$ for $\phi \in (\pi/2, \pi)$, we have

$$T_d\left(\frac{|x|}{n}\right) = \cos\left(d \arccos\left(1 - \frac{2}{n}\right)\right) \leq \cos(4\pi / 5) \leq -\frac{3}{4}.$$

In particular, for such $x$s the value of $\text{OR}_n(x) = -1$, so Eq. (7) is satisfied.

The proof of the lemma follows from Eq. (1) and Lemma 1.1. \hfill \square

REFERENCES
