The Grothendieck problem with rank constraint

Jop Briët, Fernando Mário de Oliveira Filho, and Frank Vallentin

***Abstract*— Finding a sparse/low rank solution in linear/semidefinite programming has many important applications, e.g. combinatorial optimization, compressed sensing, geometric embedding, sensor network localization. Here we consider one of the most basic problems involving semidefinite programs with rank constraints: the Grothendieck problem with rank-k-constraint. It contains the MAX CUT problem as a special case when $k = 1$. We perform a complexity analysis of the problem by designing an approximation algorithm which is asymptotically optimal if one assumes the unique games conjecture.**

I. INTRODUCTION

Given positive integers m, n, k and a matrix $A = (A_{ij}) \in \mathbb{R}^{m\times n}$, the *Grothendieck problem with rank-k-constraint* is defined as

$$\mathrm{SDP}_k(A) = \max\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}\; x_i \cdot y_j : \right.$$
$$\left. x_1,\dots,x_m \in S^{k-1}, y_1,\dots,y_n \in S^{k-1}\right\},$$

where $S^{k-1} = \{x \in \mathbb{R}^k : x \cdot x = 1\}$ is the unit sphere; the inner product matrix of the vectors $x_1,\dots,x_m,y_1,\dots,y_n$ has rank k. This problem was introduced by Briët, Buhrman, and Toner [2] in the context of quantum nonlocality where they applied it to nonlocal XOR games.

The case $k = 1$ is the classical Grothendieck problem where $x_1,\dots,x_m,y_1,\dots,y_n \in \{-1,+1\}$. It was introduced by Grothendieck [5] in the study of norms of tensor products of Banach spaces. It is an NP-hard problem: If A is the Laplacian matrix of a graph then $\mathrm{SDP}_1(A)$ coincides with the value of a maximum cut of the graph. The maximum cut problem (MAX CUT) is one of Karp's 21 NP-complete problems.

Over the last years, there has been a lot of work on algorithmic applications, interpretations and generalizations of the Grothendieck problem and the companion Grothendieck inequalities. For instance, Nesterov [10] showed that it has applications to finding and analyzing semidefinite relaxations of nonconvex quadratic optimization problems. Alon and Naor [1] showed that it has applications to constructing Szemerédi partitions of graphs and to estimating the cut norms of matrices. Khot and Naor [7], [8] showed that it has applications to kernel clustering.

When k is a constant that does not depend on the matrix size m, n there is no polynomial-time algorithm known which solves SDP_k. However, it is not known if the problem SDP_k is NP-hard when $k \geq 2$. On the other hand the *semidefinite relaxation* of $\mathrm{SDP}_k(A)$ defined by

$$\mathrm{SDP}_\infty(A) = \max\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}\; u_i \cdot v_j : \right.$$
$$\left. u_1,\dots,u_m,v_1,\dots,v_n \in S^\infty\right\}$$

can be computed in polynomial time using semidefinite programming. Here S^∞ denotes the unit sphere of the Hilbert space $l^2(\mathbb{R})$ of square summable sequences, which contains $\mathbb{R}^n$ as the subspace of the first n components.

Rietz [12] (in the context of the Grothendieck inequality) showed that SDP_1 and SDP_∞ are always within a factor of at most $(4/\pi - 1)^{-1} = 3.65979\dots$ from each other. That is, for all matrices $A \in \mathbb{R}^{m\times n}$ we have

$$\mathrm{SDP}_1(A) \leq \mathrm{SDP}_\infty(A) \leq \frac{1}{4/\pi - 1}\mathrm{SDP}_1(A).$$

Alon and Naor [1] showed that Rietz' argument gives a polynomial-time approximation algorithm for SDP_1. However Rietz' argument does not provide the best approximation factor. The best approximation factor is called *Grothendieck's constant* K_G which presently is not known exactly. It is known that it lies between $1.67695\dots$ and $\pi/(2\log(1+\sqrt{2}) = 1.78221\dots$. Krivine [9] gave the best known upper bound (which is conjectured to be tight) and Davie [4] and Reeds [11] gave the best known lower bound.

The aim of this paper is to provide to initiate an analysis for SDP_k by generalizing Rietz' argument. So far the only cases which were studied are the cases $k = 1$ (see the discussion above) and the case $k = 2$ (see the discussion below. We summarize our results in the following theorem.

Theorem. *For all matrices $A \in \mathbb{R}^{m\times n}$ we have*

$$\mathrm{SDP}_k(A) \leq \mathrm{SDP}_\infty(A) \leq \frac{1}{2\gamma(k) - 1}\mathrm{SDP}_k(A),$$

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where

$$\gamma(k) = \frac{2}{k}\left(\frac{\Gamma((k+1)/2)}{\Gamma(k/2)}\right)^2 = 1 - \Theta(1/k),$$

and

$$\frac{1}{2\gamma(k)-1} = 1 + \Theta(1/k),$$

and there is a randomized polynomial-time approximation algorithm for SDP_k *achieving this ratio. On the other hand, under the assumption of the unique games conjecture there is no polynomial-time algorithm which approximates* SDP_k *with an approximation ratio less than* $1/\gamma(k)$.

The first three values of $\frac{1}{2\gamma(k)-1}$ are:

$$\begin{aligned}
\frac{1}{2\gamma(1)-1} &= \frac{1}{4/\pi - 1} = 3.65979\ldots,\\
\frac{1}{2\gamma(2)-1} &= \frac{1}{\pi/2 - 1} = 1.75193\ldots,\\
\frac{1}{2\gamma(3)-1} &= \frac{1}{16/(3\pi) - 1} = 1.43337\ldots.
\end{aligned}$$

Haagerup [6] gave an approximation ratio of $1.40491\ldots$ for the case $k = 2$ by following Krivine's proof.

In Section II we present the approximation algorithm together with its analysis. Our main contribution is the analysis of a rounding scheme which can deal with rank-k-constraints in semidefinite programs. For this we use the Wishart distribution from multivariate statistics. We believe this analysis is of independent interest and will turn out to be useful in different contexts, e.g. for approximating low dimensional geometric embeddings.

The proof of the unique games conjecture hardness of approximating SDP_k can be found in Briët, Oliveira, Vallentin [3].

The main result of Briët, Buhrmann and Toner [2] shows that the gap between SDP_k and SDP_∞ given in our theorem is asymptotically correct up to lower order terms.

II. APPROXIMATION ALGORITHM

The following proof follows the idea of Alon and Naor [1, Section 4] which in turn relies on ideas of Rietz [12]

Proof: The randomized polynomial-time approximation algorithm which we use to prove the theorem is the following three-step process.

1) By solving $\mathrm{SDP}_\infty(A)$ we obtain the vectors $u_1, \ldots, u_m, v_1, \ldots, v_n \in S^{m+n-1}$.
2) Choose $Z = (Z_{ij}) \in \mathbb{R}^{k\times(m+n-1)}$ so that every matrix entry Z_{ij} is distributed independently according to the standard normal distribution with mean 0 and variance 1: $Z_{ij} \sim N(0,1)$.
3) Set $x_i = Zu_i/\|Zu_i\| \in S^{k-1}$ with $i = 1, \ldots, m$, and $y_j = Zv_j/\|Zv_j\| \in S^{k-1}$ with $j = 1, \ldots, n$.

The quality of the feasible solution $x_1, \ldots, x_m, y_1, \ldots, y_n$ for SDP_k is measured by the expectation

$$\mathrm{SDP}_k(A) \geq \mathbb{E}\left[\sum_{i=1}^m \sum_{j=1}^n A_{ij}\, x_i \cdot y_j\right].$$

For vectors $u, v \in S^\infty$ we define

$$E_k(u,v) = \mathbb{E}\left[\frac{Zu}{\|Zu\|} \cdot \frac{Zv}{\|Zv\|}\right],$$

where $Z = (Z_{ij})$ is a matrix with k rows and infinitely many columns whose entries are distributed independently according to the the standard normal distribution. Of course, if $u, v \in S^{m+n-1}$, then it suffices to work with finite matrices $Z \in \mathbb{R}^{k\times(m+n-1)}$. In Briët, Oliveira, Vallentin [3] it was shown that one can develop E_k as a power series

$$E_k(u,v) = \sum_{r=0}^\infty f_{2r+1}(u\cdot v)^{2r+1},$$

where all coefficients f_{2r+1} are nonnegative, where

$$\begin{aligned}
f_1 &= \frac{\partial E_k}{\partial t}(0)\\
&= \frac{k-1}{2\pi}\int_0^1\int_0^{2\pi} \frac{r(1-r^2)^{(k-1)/2}}{(1-r^2(\sin\phi)^2)^{3/2}}\, d\phi dr.\\
&= \gamma(k).
\end{aligned}$$

and where

$$\sum_{r=0}^\infty f_{2r+1} = 1.$$

For the computation of f_1 we used the Wishart distribution from multivariate statistics: Because E_k is invariant under the orthogonal group one can express the integral $E_k(u,v)$ with help of the standard Wishart distribution $W_2(k)$. This is the probability distribution of random matrices $Y = X^{\mathsf{T}}X \in \mathbb{R}^{2\times 2}$, where the entries of the matrix $X = (X_{ij}) \in \mathbb{R}^{k\times 2}$ are independently chosen from the standard normal distribution $X_{ij} \sim N(0,1)$.

Now we have

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{i=1}^m\sum_{j=1}^n A_{ij}\ x_i\cdot y_j\Big]\\
&= \sum_{i=1}^m\sum_{j=1}^n A_{ij}E_k(v_i, w_j)\\
&= f_1\sum_{i=1}^m\sum_{j=1}^n A_{ij} v_i\cdot w_j\\
&\qquad + \sum_{i=1}^m\sum_{j=1}^n A_{ij}\sum_{r=1}^\infty f_{2r+1}(v_i\cdot w_j)^{2r+1}.
\end{aligned}$$

The first summand equals $f_1\mathrm{SDP}_\infty(A)$. The second summand is bounded *in absolute value* by $(1-f_1)\mathrm{SDP}_\infty(A)$ as we argue now. Consider the $m+n$ vectors $u_i = v_i$, $u_{m+i} = w_j$. Then the $(m+n)\times(m+n)$-matrix with entries

$$\sum_{r=1}^\infty f_{2r+1}(u_i\cdot u_j)^{2r+1}$$

is positive semidefinite because of Schoenberg's theorem [13]. So there are vectors $u'_1, \ldots, u'_{m+n} \in \mathbb{R}^{m+n}$ all of squared length $1 - f_1$ so that

$$u'_i\cdot u'_j = \sum_{r=1}^\infty f_{2r+1}(u_i\cdot u_j)^{2r+1}.$$

Setting $v_i' = u_i$, $w_j' = u_{m+j}$ we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij} v_i' \cdot w_j' \leq (1 - f_1)\mathrm{SDP}_\infty(A).$$

The lower bound follows from the fact that replacing the sign of v_i to $-v_i$ also changes the sign of the complete sum from

$$\sum_{r=1}^{\infty} f_{2r+1}(v_i \cdot w_j)^{2r+1}$$

to

$$-\sum_{r=1}^{\infty} f_{2r+1}(v_i \cdot w_j)^{2r+1}$$

since all involved powers are odd. Thus for the second sum we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij} \sum_{r=1}^{\infty} f_{2r+1}(v_i \cdot w_j)^{2r+1} \geq (f_1 - 1)\mathrm{SDP}_\infty(A),$$

which finishes the proof. ■

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