Chapter 1
Strategies for Belief Revision

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Abstract This paper is a contribution to dynamic doxastic logic (DDL), that is, doxastic logic supplemented with operators for belief change due to new information. Thus in addition to operators for belief (B) and doxastic commitment (K) we also have, for each pure Boolean formula φ, a propositional operator [∗φ] with the informal reading “after the agent has come to believe that φ and revised his beliefs accordingly it is the case that”. The resulting new logical landscape turns out to be overwhelmingly rich. An important question—and the topic of this paper—is how to deal with this richness.

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Chapter 2
Strategies for belief revision

2.1 Introduction

Within the KGM-paradigm of belief change, iterated revision poses no problem. But within the other major paradigm, AGM, it has been the subject of much discussion and little agreement. One reason for the lack of agreement may be that our uneducated intuitions are vague and that they pull in different directions. If so, one would expect the problem to become more tractable once formal semantic modellings have been developed.

This paper is an effort to bear out that expectation. After this first section, which provides a conceptual background (section 2.1), we consider two important modellings to which we refer as the onion semantics (section 2.2) and the nearness semantics (section 2.3). In each case we try to give a systematic treatment.

The present paper extends the author’s work in [13] and complements that in [14]. For a recent survey paper on belief change, see [11]. For a general introduction to the modal logic of belief change, see [10]. For a more technical treatment of the dynamic doxastic logic of iterated belief revision, including completeness theorems, see [15].

2.1.1 Basic dynamic doxastic logic

As used in this paper, the term “dynamic doxastic logic” (DDL for short) does not refer to any particular system of logic but rather to a branch of modal logic, involving a particular kind of object language as well as a particular kind of model.

**Language.** Our object language has the following categories of primitive symbols: (i) a denumerably infinite set of propositional letters, (ii) a functionally complete set of truth-functional (or Boolean) connectives, (iii) the doxastic operators

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1 The two paradigms are named for their originators: KGM for Katsuno, Grahne and Mendelzon [2, 4], AGM for Alchourrón, Gärdenfors and Makinson [1]. See [12],[5] and [10] for references.
**B** and **K** (unary proposition-forming propositional operators), and (iv) revision operators. Formulæ are formed in two steps. First, the *pure Boolean formulæ* are those made up exclusively of propositional letters and Boolean operators. Second, a *(general)* formula is either a pure Boolean formula, or a Boolean combination of formulæ of type **Bφ** or **Kφ** or [∗φ]θ, where φ is a pure Boolean formulæ (it is the last restriction that accounts for the “basic” in “basic DDL”.)

To this we add a convention about dual operators: for every pure Boolean formulæ φ we regard bφ and kφ and ⟨∗φ⟩θ as abbreviations for ¬**B**¬φ and ¬**K**¬φ and ¬[∗φ]¬θ, respectively.

**Frames and models.** One basic idea of possible worlds semantics is that propositions may be identified with certain sets of possible worlds (which may or may not be expressible in a certain language). Technically, this idea is implemented by the introduction of frames, constructs which come with a nonempty set of points (the universe of the frame). A proposition (in that frame) is a subset of that set. A theory (in that frame)—normally a set of propositions—is identified by the intersection of the corresponding subsets.

To do justice to this idea some topological terminology will be useful. A *Stone space* is a topological space that is both totally separated and compact. (Totally separated: for any two distinct points there there is a clopen set (set that is both closed and open) containing just one of those points. Compact: every family of open sets with the property of covering the whole space (in the sense that its union coincides with the whole space) includes a subfamily with that same property. For more on topology, see any textbook.)

In this paper we discuss two kinds of models for belief revision, both in effect deriving from David Lewis’s pioneering work [7]. They share certain basic features, which we bring together in this section. In each case we will define a frame of the type (U,T,···) where (U,T) is a Stone space. A *model* is a structure (U,T,···,V), where (U,T,···) is a frame and V is a valuation in (U,T), meaning that V is a function assigning to each point u in U a clopen set V(u). The valuation V can be extended to the set of all pure Boolean formulæ in the usual way. Writing [φ] for the extension of V we have

\[
[\mathcal{P}] = V(\mathcal{P}), \text{ if } \mathcal{P} \text{ is a propositional letter,}
\]

\[
[\phi \land \psi] = [\phi] \cap [\psi],
\]

\[
[\phi \lor \psi] = [\phi] \cup [\psi],
\]

\[
[\neg \phi] = U - [\phi],
\]

and similar conditions for other Boolean connectives.

The *propositions* of a frame (U,T,···) are the clopen sets in (U,T), while the *definable propositions* of a model (U,T,···,V) are the sets [φ], where φ is a pure Boolean formulæ.

**Truth and validity.** Our definitions of truth of a general formula in a model will have the following structure. We write (···,u) ⊨ [φ] (or perhaps just (···,u) ⊨ φ, if the reference to [φ] is understood from the context) to express that the formulæ φ is *true* at the index (···,u) in the model [φ], where u is a point in U and the three dots are to be explained:
\[(\cdots, u) \models^\mathfrak{M} \phi \text{ iff } u \in \mathcal{F}\phi], \text{ if } \phi \text{ is a pure Boolean formula,}\\(\cdots, u) \models^\mathfrak{M} \phi \land \psi \text{ iff } (\cdots, u) \models^\mathfrak{M} \phi \text{ and } (\cdots, u) \models^\mathfrak{M} \psi,\\(\cdots, u) \models^\mathfrak{M} \phi \lor \psi \text{ iff } (\cdots, u) \models^\mathfrak{M} \phi \text{ or } (\cdots, u) \models^\mathfrak{M} \psi,\\(\cdots, u) \models^\mathfrak{M} \neg \phi \text{ iff not } (\cdots, u) \models^\mathfrak{M} \phi,\\[\text{similar conditions for other Boolean operators}]\\(\cdots, u) \models^\mathfrak{M} \mathbf{B}\phi \text{ iff } \cdots,\\(\cdots, u) \models^\mathfrak{M} \mathbf{K}\phi \text{ iff } \cdots,\\(\cdots, u) \models^\mathfrak{M} [\#\phi]_\emptyset \text{ iff } \cdots.

Note that the clause for the pure Boolean formulæ depends only on the point \(u\), the world state, not on what is to replace the three dots. The three last will turn out not to depend on the world state at all—this is a hallmark of basic DDL, in contrast with nonbasic or full DDL.

More generally, we say that a formula set \(\Sigma\) \textit{implies} a formula \(\phi\) \textit{at an index} \((\cdots, u)\) \textit{in a model }\mathfrak{M}\text{ if } (\cdots, u) \models^\mathfrak{M} \sigma, \text{ for all } \sigma \in \Sigma, \text{ only if } (\cdots, u) \models^\mathfrak{M} \phi. \text{ Furthermore, we say that } \Sigma \textit{implies } \phi \textit{ in a frame }\mathfrak{F}\text{, in symbols } \Sigma \models^\mathfrak{F} \phi, \text{ if } \Sigma \textit{implies } \phi \textit{ at every index in every model on }\mathfrak{F}.

We say that \(\Sigma\) \textit{implies } \phi \textit{ with respect to a class }C\text{ of frames, in symbols } \Sigma \models^C \phi, \text{ if } \Sigma \textit{implies } \phi \textit{ in every frame in }C. \text{ In the special case that }C\text{ is the class of all frames, we say that } \Sigma \textit{implies } \phi, \text{ in symbols } \Sigma \models \phi. \text{ We say that a formula is } \textit{valid} \text{ in a frame if it is true at all indices (in the frame under all valuations). If }\mathfrak{F}\text{ is a frame we write } L(\mathfrak{F})\text{ for the set of formulæ that are valid in }\mathfrak{F} \text{ and refer to the latter as the } \textit{logic determined by }\mathfrak{F}.

\textbf{Informal remarks.} Informally, a point of the universe represents a possible \textit{world state}, that is, a possible state of the environment. An informal requirement is that it should be possible to think of whatever fills the three dots in an index \((\cdots, u)\) as a possible \textit{belief state} of the agent. Note that in basic DDL—this is not the case in full DDL—belief states are not part of world states and do not even depend on them. Consequently the points of the universe of our frames should not be thought of as possible worlds in the usual all-engulfing sense of that term.

To push our preview of developments-to-come one step further: in each of the three following sections we will define a kind of entity that, we submit, can be seen as representing a belief state. For each such entity \(e\) we will define, in the next section, further entities \(\text{bst}(e)\) and \(\text{kst}(e)\), meant to be seen as representations of the “belief set” and the “commitment set”, respectively (representing, respectively, the current beliefs of the agent and the limits for what it is possible for the agent to believe). The three last clauses of the truth-definition will share the following format, where \(u\) is a point of \(U\) as before and \(e\) is one of the promised belief states:

\[(e, u) \models \mathbf{B}\phi \text{ iff } \text{bst}(e) \subseteq \mathcal{F}\phi,\\(e, u) \models \mathbf{K}\phi \text{ iff } \text{kst}(e) \subseteq \mathcal{F}\phi,\\(e, u) \models [\#\phi]_\emptyset \text{ iff } (e', u) \models \emptyset, \text{ for all } e' \text{ such that } (e, e') \in \cdots.

Again there are three dots to replace, but this time it is a different three points: this time “…” should be replaced by a relation relating \(e, e'\) and \([\#\phi]\).
2.2 Onion semantics

2.2.1 Onions

David Lewis’s favourite modelling in his analysis of counterfactual conditions was that of sphere systems [7]. Later Adam Grove showed how that modelling can be used to analyze the well-known AGM theory of belief change [3]. This is the modelling used in this section, although our terminology departs in places from that of Lewis and Grove. Let \((U, T)\) be a given Stone space. An onion in \((U, T)\) is a nonempty family \(O\) of closed subsets of \(U\) that is linearly ordered by set inclusion and is closed under arbitrary intersection:

- \((\text{LIN})\) if \(X, Y \in O\) then either \(X \subseteq Y\) or \(Y \subseteq X\),
- \((\text{AINT})\) if \(O' \subseteq O\) and \(O' \neq \emptyset\), then \(\bigcap O' \in O\).

It is worth noting that Lewis’s so-called Limit Condition is satisfied in this modelling:

**Observation 2.1** Let \(P\) be a proposition and \(O\) an onion. If \(P \cap \bigcup O \neq \emptyset\), then the family of elements \(X\) in \(O\) such that \(P \cap X \neq \emptyset\) has a smallest element.

**Proof.** By compactness. ⊓⊔

Onions provide one way of representing belief states. In the onion modelling, they are the entities offered to fill out the three dots in the symbolism of section 2.1.1.

Before continuing the formal exposition, let us briefly consider the intuitions that make onions a plausible candidate for representing possible belief states. The elements of an onion \(O\), called fallbacks, are closed sets and therefore represent possible theories. The innermost fallback—there always is one!—is called the belief set, in symbols \(\text{bst}_O\). We may think of the belief set as representing the agent’s best estimate as to which of the many possible points of the model is the actual one (there is exactly one). At the other extreme is the commitment set, in symbols \(\text{kst}_O\), the set of points belonging to at least one fallback. The commitment set represents the limits as to what the agent considers possible for a point to be the actual point; everything outside the commitment set is beyond the pale.\(^2\) Note that

\[
\text{bst}_O = \bigcap O, \quad \text{kst}_O = \bigcup O.
\]

As far as the agent’s beliefs about the state of the environment goes, the belief set says it all. But in DDL the beliefs of the agent are treated as fallible, so there is always the possibility that the agent is mistaken and that the belief set may need to be replaced. This is where the other fallbacks come in. The onion is a family of increasingly weaker theories representing information that the agent finds it increasingly

\(^2\) As just defined, \(\text{kst}_O\) need not be a closed set and hence need not be a fallback. However, for the purposes of this paper it makes no difference whether the commitment set is defined as \(\bigcup O\) or as the closure of \(\bigcup O\).
difficult to question. Nevertheless, if the agent is challenged by new information, he
may be forced to give up some of his beliefs. In the onion modelling, he will do
this by falling back on the strongest fallback theory that is consistent with the new
information. Thus in one certain sense his belief change is minimal.

2.2.2 Onion frames

We say that \((U, T, Q, R)\) is an onion frame if \((U, T)\) is a Stone space, \(Q\) is a set of
onions in \((U, T)\), and \(R\) is a function assigning to each clopen set \(P\) a binary relation
\(R^P\) in \(Q\). There are the following conditions:

(i) if \((O, O') \in R^P\) then \(\bigcap O' = P \cap Z\), where \(Z\) is the smallest fallback in \(O\)
such that \(P \cap Z \neq \emptyset\),
(ii) if \((O, O') \in R^P\) then \(\bigcup O = \bigcup O'\),
(iii) \(R^P\) is serial,
(iv) \(R^P\) is functional.

Having defined the belief set and the commitment set of an onion above, we are
now able to fill in the missing clauses in the definition of truth-conditions outlined
in section 2.1.1:

\[(O, u) \Vdash B\phi \text{ iff } \bigcap O \subseteq \llbracket \phi \rrbracket,\]
\[(O, u) \Vdash K\phi \text{ iff } \bigcup O \subseteq \llbracket \phi \rrbracket,\]
\[(O, u) \Vdash [\ast\phi] \theta \text{ iff } (O', u) \Vdash \phi, \text{ for all } O' \text{ such that } (O, O') \in R^{[\phi]}\]

We note that the set of formulæ valid in all onion frames is axiomatized by the
following axiom system: (i) modus ponens as an inference rule and all tautologies
as axioms; (ii) necessitation as an inference rule and the usual Kripke schema, for
each of the operators \(B\), \(K\), and \([\ast\phi]\), where \(\phi\) is pure Boolean; and (iii) the following
special axiom schemata:

\((+0)\) \(\theta \leftrightarrow [\ast\phi] \theta\), if \(\theta\) is a pure Boolean formula,
\((+2)\) \([\ast\phi] B\phi\),
\((+3)\) \([\ast T] B\phi \rightarrow B\phi\),
\((+4)\) \(\text{b} T \rightarrow (B\phi \rightarrow [\ast T] B\phi)\),
\((+5)\) \(K\phi \rightarrow ([\ast\phi] \text{b} T)\),
\((+6)\) \(K(\phi \leftrightarrow \psi) \rightarrow ([\ast\phi] B\theta \leftrightarrow [\ast\psi] B\theta)\),
\((+7)\) \([\ast(\phi \land \psi)] B\theta \rightarrow [\ast\phi] B(\psi \rightarrow \theta)\),
\((+8)\) \([\ast\phi] B(\psi \rightarrow \theta) \rightarrow [\ast(\phi \land \psi)] B\theta\),
\((+D)\) \([\ast\phi] \theta \rightarrow [\ast\phi] \theta\),
\((+F)\) \([\ast\phi] \theta \rightarrow [\ast\phi] \theta\),
\((+K)\) \(K\phi \rightarrow [\ast\phi] K\phi\),
\((+KB)\) \(K\phi \rightarrow B\phi\).

The assumptions on which our modelling rests include three that have been present
in the AGM paradigm from the beginning, at least implicitly:
• onions are linearly ordered;
• revision relations are functional;
• the commitment set never changes.

The latter two assumptions do not seem essential to the AGM enterprise, and from an intuitive point of view they appear arbitrary. The first assumption is given up in the work of Lindström and Rabinowicz [8, 9]. Unfortunately, the theory of nonlinear belief revision is technically difficult.

### 2.2.3 The lattice of possible fallbacks

Let $O$ be an onion in some Stone space $(U, T)$. We will describe a construction for which it makes a difference whether $O$ is consistent or not. We assume that there is some ordinal $\mu$ such that in the former case $O = \{X_o : 0 < o \leq \mu\}$, whereas in the latter case $O = \{X_o : 0 \leq o \leq \mu\}$. In either case we say that $O$ is a $\mu$-onion. By convention, $X_0$ (when it exists) shall denote the empty set, so $X_1$ is always the first nonempty element of $O$, whether $O$ is consistent or not. Note that $X_\mu$ is always the commitment set of $O$.

We say that a proposition $P$ cuts an onion $O$ at $X$ if $X$ is the smallest fallback of $O$ intersecting $P$. (In other words, $X \in O$ and $P \cap X \neq \emptyset$ and, for all $Y \in O$, if $P \cap Y \neq \emptyset$ then $X \subseteq Y$.) Suppose now that $P$ is a proposition cutting an onion $O = \{X_o : 0 < o \leq \mu\}$ at $X_\pi$ (or cutting $O$ at $\pi$, for short) where thus $\pi$ is an ordinal such that $0 < \pi \leq \mu$. The set of possible new fallbacks comprises all sets of the form $X_\alpha \cup (P \cap X_\beta)$—which for simplicity we often write $X_\alpha \cup PX_\beta$—where $0 \leq \alpha \leq \mu$ and $\pi \leq \beta \leq \mu$. We say that $(\alpha, \beta)$ is, or gives, the coordinates (with respect to $O$ and $P$) of $X_\alpha \cup PX_\beta$.

In the finite case $\mu$ and $\pi$ will be finite ordinals $m$ and $p$ with $p \leq m$, and the possible fallbacks will be of the form $X_i \cup PX_j$, where $i < j$ and $p \leq j \leq m$. The coordinates of that fallback will thus be $(i, j)$.

### 2.2.4 Strategies

In our basic theory the result of an individual revision must always be an onion, but exactly which onion is not determined by either the original AGM theory or our basic theory. Indeed, an agent revising his beliefs may not have a plan for how to do so. At another extreme, an agent may well have a plan for how to proceed, even a strategy: a plan defined to meet every possible contingency. In this paper we study one possible concept of this kind.

A strategy on an infinite ordinal $\mu$, or a $\mu$-strategy for short, is a function from $\mu + 1$ to [LEFT, RIGHT], where LEFT and RIGHT are two distinct objects. (Sometimes it

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3 So under this construction, the commitment set is always a fallback. Cf. footnote above!
is convenient to identify \texttt{left} with 0 and \texttt{right} with 1. Notice that the domain of a strategy must be infinite. otherwise the agent might run out of instructions.)

We say that \( s \) is \textit{skeptical} at \( o \) if \( s(o) = \texttt{left} \) but \textit{trusting} at \( o \) if \( s(o) = \texttt{right} \). Furthermore, we say that \( s \) is \textit{skeptical} or trusting if skeptical at \( o \), for all \( o \), or trusting at \( o \), for all \( o \), respectively. Occasionally we use ‘conservative’ and ‘radical’ as synonyms of ‘skeptical’ and ‘trusting’ (even though this use is at odds with traditional political \texttt{left-right} terminology).

We will now define what we shall call the \textit{fallback function} \( F \), a construction that is relative to a fixed given context: a space \((U, T)\), an onion \( O \), and a proposition \( P \) that cuts \( O \). In fact, suppose that \( O \) is a \( \mu \)-onion with elements \( X_\alpha \), and that \( \mu > 0 \). Say \( P \) cuts \( O \) at \( \pi \), where \( 0 < \pi < \mu \). For any strategy \( s \), \( F_s \) is a function defined on \( \mu + 1 \) which will give us the the fallbacks of the new onion that results when \( O \) is revised by \( P \)—the set of the latter forms the range of \( F_s \).

First define
\[
F_s(0) = \begin{cases} 
X_0 \cup PX_\pi, & \text{if } X_\pi \text{ is the smallest fallback of } O \text{ to intersect } P, \\
\emptyset, & \text{if } P \text{ does not overlap with } O.
\end{cases}
\]
(Recall that by our convention \( X_0 \), which need not be an element of \( O \), is \( \emptyset \).

Next suppose that \( F_s(o) \) has been defined, for any ordinal \( o < \mu \), and that \( F_s(o) = X_\alpha \cup PX_\beta \), where \( \alpha, \pi \leq \beta \). Then define
\[
F_s(o + 1) = \begin{cases} 
X_{\alpha + 1} \cup PX_\beta, & \text{if } \beta < \mu \text{ and } \alpha < \beta \text{ and } s(o) = \texttt{left}, \\
X_\alpha \cup PX_{\beta + 1}, & \text{if } \beta < \mu \text{ and } \alpha < \beta \text{ and } s(o) = \texttt{right}, \\
X_\alpha \cup PX_{\beta + 1}, & \text{if } \beta < \mu \text{ and } \alpha = \beta, \\
X_{\alpha + 1} \cup PX_\mu, & \text{if } \beta = \mu \text{ and } \alpha < \beta, \\
X_\mu, & \text{if } \alpha = \beta = \mu, \\
X_\mu, & \text{if } F(o) = X_\mu.
\end{cases}
\]

If \( \lambda \) is a limit ordinal such that \( \lambda \leq \mu \), define \( F_s(\lambda) = \bigcup_{o < \lambda} F_s(o) \).

It is easy to see that the set
\[
O \ast P \ (\text{mod } s) = \{ F_s(o) : 1 \leq o \leq \mu \}
\]
is an onion; let us call it the \textit{new onion defined from } \( O \) \textit{and } \( P \) \textit{by } \( s \). The new onion will have more elements than the old one but in general fewer than \( \mu + \mu = \mu \cdot 2 \). (In the finite case, when \( \mu = m \) and \( \pi = p \), for some natural numbers \( m \) and \( p \), \( O \ast P \) will have at most \( 2m - p + 1 \) elements.) Thus after \( n \) successive revisions of \( O \) the size of the current onion will be less than \( \mu \cdot (n + 1) \). Note that strategy makes no difference if \( \alpha = \beta \) or \( \beta = \mu \). Let us say that a strategy is \textit{free at } \((\alpha, \beta)\) if \( \alpha \neq \beta \) and \( \beta \neq \mu \), and otherwise \textit{not free at } \((\alpha, \beta)\).

We say that revision is \textit{implemented according to } \( s \) \textit{in a frame } \((U, T, Q, R)\) \textit{if, for all onions } \( O \in Q \) \textit{and propositions } \( P \) \textit{in } \((U, T)\),
\[
(O, O') \in R^P \text{ if and only if } O' = O \ast P \ (\text{mod } s).
\]
If \( O = \{ O_o : o \leq \mu \} \) is an onion and \( P \) a proposition in \((U, T)\), we say that \((U, T, Q^*, R^*)\) is the revision frame generated by \( s \) from \( O \) and \( P \) if \( Q^* = \bigcup_{o \in \mu} Q_o \) and \( R^* = \bigcup_{o \in \mu} R_o \), where

\[
\begin{align*}
Q_0 &= \{ O \} \quad \text{and} \quad R_0 = \emptyset, \\
Q_{\alpha+1} &= Q_\alpha \cup \{ O' : \exists O' \in Q_\alpha \exists P' \in \text{clop}(T) \ (O' = O' \ast P' \text{ (mod } s)) \}, \\
Q_\lambda &= \bigcup_{\alpha < \lambda} O_\alpha, \quad \text{if } \lambda \leq \mu \text{ is a limit ordinal}, \\
R_{\alpha+1} &= R_\alpha \cup \{ (O', O'') : \exists O' \in Q_\alpha \exists P' \in \text{clop}(T) \ (O'' = O' \ast P' \text{ (mod } s)) \}, \\
R_\lambda &= \bigcup_{\alpha < \lambda} R_\alpha, \quad \text{if } \lambda \leq \mu \text{ is a limit ordinal}.
\end{align*}
\]

For “practical” purposes the generality of our presentation is unnecessary. Nevertheless we have persisted in keeping it up till now to emphasize how fine-grained the onion approach is. In practice, most onions will be finite.

**Observation 2.2** In the finite case, \( F_\lambda(o) = X_\mu \cup PX_\beta \) implies that \( \alpha + \beta = o + \pi \), for all \( o \leq 2\mu - \pi \).

**Proof.** In the finite case there are natural numbers \( i, j, n, m, p \) playing the rôles of \( \alpha, \beta, o, \mu, \pi \). Thus the claim is that, for \( n \leq 2m - p \), if \( F_\lambda(n) = X_i \cup PX_j \), then \( n + p = i + j \). The claim is readily proved by induction on \( n \). \( \Box \)

**Observation 2.3** Let \( m \) and \( p \) be positive natural numbers such that \( p \leq m \). Suppose that \( O \) is an onion with \( m \) fallbacks and \( P \) is a proposition cutting \( O \) at \( p \). Then the new onion \( O \ast P \) will have at most \( 2m - p + 1 \) elements.

**Proof.** It is helpful to represent the implementation of a strategy with respect to a particular onion and a particular proposition as a path in the lattice of possible new fallbacks. The path, which is strictly ascending, begins at the new belief set \( PX_p \) and ends at the commitment set \( X_m \) (which is invariant according to the present theory).

Every point of the path is a new fallback of type \( X_i \cup PX_j \) (that is, of coordinate \((i, j)\)), where \( 0 \leq i \leq m \) and \( p \leq j \leq m \) and \( i < j \). Evidently a path is of maximal length if every possible jump of just one step in each coordinate is actually taken. This means that a maximal chain consists of \( 2m - p + 1 \) elements. \( \Box \)

As an aid to visualization we offer two illustrations. In Figure 2.1 \( O \) is an onion and \( P \) is a proposition. Furthermore, \( O \) consists of one belief set \( (X_1) \) and six fallbacks \((X_2 - X_7)\). If \( O \) is revised by \( P \), the new belief set will be the set \( P \cap X_3 \), or \( PX_3 \) for short.

In Figure 2.2 we have an implicit survey of all possible belief systems definable in terms of \( O \). Each of two extremes has been indicated by a thick line. On the one hand, the skeptical (or conservative) strategy is represented by an onion with the elements:

\[
PX_3, \\
X_1 \cup PX_3, \\
X_2 \cup PX_3, \\
X_3, \\
X_3 \cup PX_4, \\
X_4.
\]
Fig. 2.1 Illustration of updating a belief set by proposition $P$ in the onion semantics, where $P$ cuts the onion at $X_3$.

\[ X_1 \cup PX_5, \]
\[ X_5, \]
\[ X_3 \cup PX_6, \]
\[ X_6, \]
\[ X_5 \cup PX_7, \]
\[ X_7. \]

On the other hand, the trusting (or radical) strategy is represented by an onion with the elements:
\[ PX_3, \]
\[ PX_4, \]
\[ PX_5, \]
\[ PX_6, \]
\[ PX_7, \]
\[ X_1 \cup PX_7, \]
\[ X_2 \cup PX_7, \]
\[ X_3 \cup PX_7, \]
\[ X_4 \cup PX_7, \]
\[ X_5 \cup PX_7, \]
\[ X_6 \cup PX_7, \]
\[ X_7. \]
Fig. 2.2 Illustration of all possible belief systems definable in terms of the onion of Figure 2.1.

Note that all maximal onions, each generated by a maximal chain through the diagram, have the same number of elements.

2.2.5 Logics

If $s$ is a strategy, then by the revision logic defined by $s$ we understand the set of formulæ that are valid in all frames in which revision is implemented according to $s$.

How many revision logics are there? Do different strategies ever define the same logic? The author has not been able to answer these questions. An answer of sorts can be given, however, if we move to a slightly more expressive object language and restrict our semantics to well-ordered onions. Let $[-]$ be a new operator on formulæ with the following evaluation clause to be added to the definition of truth:

$$(O,u) \models [-]\phi \text{ if and only if } (O',u) \models \phi, \text{ where } O' = (O - \{\cap O\}) \cup \{\cup O\}.$$  

\[ Define the consequence relation defined by $s$ as the set of pairs $(\Sigma,\phi)$ such that, for all frames $\mathfrak{F}$ in which revision is implemented according to $s$, $\Sigma \models^{\mathfrak{F}} \phi$. It is possible to prove that the consequence relations defined by different strategies are always incomparable.
Here the new onion $O'$ is the old onion $O$ without the innermost fallback (the old belief set). Note that the definition ensures that the new onion will never be empty: the commitment set always remains.

A suggestive reading of $[-]\phi$ is “next $\phi$”, provided that the word “next” is not taken in a temporal sense but in the sense of second best. In fact, the belief set of the new onion is the next fallback under the inclusion relation in the old onion. Note that since the old onion is well-ordered, the new onion exists and will also be well-ordered. We refer to the new operator as \textit{minimal contraction}. It is different from standard notions of contraction and not interdefinable with any of them in the finitary object language.

The following result is not the strongest possible but enough to give an idea of the richness of the topic:

\textbf{Theorem 2.4.} The logics in the enriched language defined in the restricted semantics by any two different $\omega$-strategies are incomparable.

\textbf{Proof.} Assume that $s_1$ and $s_2$ are two different $\omega$-strategies. Let $r + 1$ be the first argument for which $s_1$ and $s_2$ disagree. In other words, $r$ is the smallest number such that, while $s_1(n) = s_2(n)$ for all $n \leq r$, $s_1(r + 1) \neq s_2(r + 1)$. We assume without loss of generality that $s_1(r + 1) = \text{LEFT}$ and $s_2(r + 1) = \text{RIGHT}$ (so $s_1$ is skeptical at $r + 1$ while $s_2$ is trusting at $r + 1$).

Roughly the idea behind the proof is this. We will try to characterize, by the use of formulæ in the object language, a situation in which the two strategies differ. To achieve this we must first try to characterize a certain onion. After that we shall have to find a way to bring out the difference between the strategies. The onion we have in mind will depend on two parameters, $p$ and $m$, which in turn depend on the given strategies.

Let $r$ be as above, and let “card” stand for “cardinality”. Let $k$ and $l$ be the natural numbers such that

\[
\begin{align*}
\text{card}\{n : n \leq r \& s_1(n) = \text{LEFT}\} &= k, \\
\text{card}\{n : n \leq t \& s_2(n) = \text{LEFT}\} &= k, \\
\text{card}\{n : n \leq t \& s_1(n) = \text{RIGHT}\} &= l, \\
\text{card}\{n : n \leq t \& s_2(n) = \text{RIGHT}\} &= l.
\end{align*}
\]

Note that $k + l = r$. (Cf. Observation 2.2 above.) Fix numbers $p$ and $m$ such that

$k < p + l < m$.

This can certainly be done, for example, by putting $p = r + 1$ and $m = 2(r + 1)$.

Let $P, Q_1, \ldots, Q_m$ be $m + 1$ distinct propositional letters. We define a formula set $\Sigma$ as the set made up of the following formulæ:

(a) $kQ_n$, for all positive $n \leq m$,  
(b) $K\neg(Q_i \land Q_j)$, for all positive $i, j \leq m$ such that $i \neq j$,  
(c) $K(Q_1 \lor \cdots \lor Q_m)$,  
(d) $[-]^{p}B(Q_1 \lor \cdots \lor Q_{n+1})$, for all nonnegative $n < m$,  
(e) $[*P]BQ_p$. 

Note that $\Sigma$ is a finite set. The following observation reveals why $\Sigma$ was defined the way it was:

Claim A. Let $\mathfrak{M}$ be a model, $O$ an onion in $Q$ and $u$ a point in $U$. Write $P$ for $[F^2]$ and $Q_1, \cdots, Q_m$ for $[Q_1], \cdots, [Q_m]$. Then $\Sigma$ is satisfied at $(O,u)$ in $\mathfrak{M}$ if and only if it holds that $O$ has exactly $m$ distinct fallbacks, and that $P$ cuts $O$ at $p$.

Proof. Suppose that $(O,u) \models^{\mathfrak{M}} \sigma$, for all $\sigma \in \Sigma$. The truth of all formulæ under (a)-(c) at $(O,u)$ shows that $(Q_1, \cdots, Q_m)$ is a partitioning of the commitment set. Since all instances of (d)—notice that $B(Q_1)$ and $B(Q_1 \lor \cdots \lor Q_m)$ are special cases—are true at $(O,u)$, it is clear that $X_n = Q_1 \cup \cdots \cup Q_n$, for all positive $n \leq m$, are the fallbacks of $O$ (and in that order). That $P$ cuts $O$ at $p$ follows from the truth at $(O,u)$ of (e). The converse of the claim is obvious. ⊓⊔

Our task: to show that the strategies $s_1$ and $s_2$ define different logics. Consider the following two formulæ:

$$\phi_1 = [\ast F ] \neg \left[ \neg F \land Q_{k+1} \right],$$
$$\phi_2 = [\ast F ] \neg \left[ \left( F \land Q_{p+l+1} \right) \right].$$

Let $\tilde{\mathfrak{M}}_1$ and $\tilde{\mathfrak{M}}_2$ be frames implementing $s_1$ and $s_2$, respectively.

Claim B. $\Sigma \models^{\tilde{\mathfrak{M}}_1} \phi_1$ but $\Sigma \not\models^{\tilde{\mathfrak{M}}_2} \phi_2$, and $\Sigma \models^{\tilde{\mathfrak{M}}_2} \phi_2$ but $\Sigma \not\models^{\tilde{\mathfrak{M}}_1} \phi_1$.

Proof. Let $\tilde{\mathfrak{M}}_a = (U, T, Q, R)$ be a frame where $a$ is ambiguous between 1 and 2. Suppose that $\Sigma$ is satisfied at an index $(O,u)$ in a model $\mathfrak{M}_a$ on $\tilde{\mathfrak{M}}_a$. Let $O'$ be the unique onion such that $(O,O') \in R[PF]$. With the help of Claim A we get a good picture of the situation: the elements of the new onion $O'$ are of the form $X_i \cup PX_j$, where $1 \leq i, j \leq m$ and $i \leq j$ and $p \leq j < m$; the belief set is $PX_p$. Among the fallbacks is the set $X_k \cup PX_{p+l}$, where $k$ and $l$ are as above. Recall that $k < p + l < m$.

Now, if in this context revision is implemented by the strategy $s_1$ the next larger fallback after $X_k \cup PX_{p+l}$ will be $X_{k+1} \cup PX_{p+l}$. But if instead revision is implemented by $s_2$ then the next larger fallback will be $X_k \cup PX_{p+l+1}$. To express this more precisely, let $F$ be the fallback function of section 2.2.4 above. Then

$$F_{s_1}(r) = F_{s_2}(r) = X_k \cup PX_{p+l},$$
$$F_{s_1}(r+1) = X_{k+1} \cup PX_{p+l},$$
$$F_{s_2}(r+1) = X_k \cup PX_{p+l+1}.$$

Let $O''$ be the onion resulting from $r$ applications of minimal contraction. The belief set of $O''$ is $X_k \cup PX_{p+l}$. Up to this point, the value of $a$ has not mattered. But at this point there is a difference between $s_1$ and $s_2$:

$$(O'', u) \models^{\mathfrak{M}_1} b(\neg F \land Q_{k+1}),$$
$$(O'', u) \not\models^{\mathfrak{M}_2} b(F \land Q_{p+l+1}),$$
$$(O'', u) \not\models^{\mathfrak{M}_1} b(\neg F \land Q_{k+1}),$$
$$(O'', u) \models^{\mathfrak{M}_2} b(F \land Q_{p+l+1}).$$

Thus on the one hand $\Sigma \models^{\tilde{\mathfrak{M}}_1} \phi_1$ and $\Sigma \not\models^{\tilde{\mathfrak{M}}_2} \phi_2$, and on the other $\Sigma \not\models^{\tilde{\mathfrak{M}}_1} \phi_1$ and $\Sigma \models^{\tilde{\mathfrak{M}}_2} \phi_2$. ⊓⊔
We have already remarked that $\Sigma$ is a finite set. Let $\sigma_\circ$ be a conjunction of all the formulæ in $\Sigma$. It follows from Claim B that $\sigma_\circ \rightarrow \phi_1$ is a thesis of $L(\overline{\sigma}_1)$ but not of $L(\overline{\sigma}_2)$, while $\sigma_\circ \rightarrow \phi_2$ is a thesis of $L(\overline{\sigma}_2)$ but not of $L(\overline{\sigma}_1)$. Hence $L(\overline{\sigma}_1) \notin L(\overline{\sigma}_2)$ and $L(\overline{\sigma}_2) \notin L(\overline{\sigma}_1)$.

It is perhaps worth remarking that there are $[-]$-free, if more complicated, formulæ that could have played the rôles of $\phi_1$ and $\phi_2$ in the preceding proof, for example,

$$\psi_1 = [\ast P][\ast (\neg P \land Q_{k+1}) \lor (P \land Q_{p+1})]b(\neg P \land Q_{k+1}).$$

$$\psi_2 = [\ast P][\ast (\neg P \land Q_{k+1}) \lor (P \land Q_{p+1})]b(P \land Q_{p+1}).$$

**Brief aside on minimal contraction.** The new operator $[-]$ may appear artificial, but it is not without interest. If an agent is challenged, not by new information but by a demand that he relinquish at least some belief or other (“something’s gotta go!”), then minimal contraction makes some sense. This is an idea that is difficult to model in finitary logic but may be worth a try. In his paper [6] Levesque has introduced an epistemic operator $E$ with the reading “the agent knows exactly that”. Changing “knows” to “believes” we might introduce a doxastic operator $D$ with the truth-condition

$$(O,u) \models D\phi \text{ iff } \bigcap O = \llbracket \phi \rrbracket.$$ 

In the resulting logic

$$D\phi \rightarrow ([-]\theta \leftrightarrow [\ast \neg \phi]\theta)$$

becomes valid. This does not make the minimal contraction operator definable, but it suggests that it mixes well with $D$ and other operators.

### 2.2.6 Which strategies?

**Total strategies.** The strategies we have considered so far are total in the sense of corresponding to maximal branches through the lattice of possible fallbacks. If we think of the branching as growing from the root $PZ$ upwards, then, to pursue the terminology of the preceding section, we will think of a strategy as skeptical or trusting to the extent that it is drawn towards the left or the right, respectively.

This terminology allows a strategy to be both trusting and skeptical. Suppose for example that $s$ is a strategy and $\pi$ and $\eta$ are two ordinals such that $\pi < \eta$ and $s(\pi) = 1$, for all ordinals $\alpha < \pi$, while $s(\alpha) = 0$, for all ordinals $\alpha$ such that $\pi \leq \eta$. Then, in a sufficiently large onion $\{X_o\}_{o}$, if $X_\pi$ is the cutting fallback of a certain proposition $P$, the new onion will include $\{PX_o : \pi \leq o \leq \eta\} \cup \{X_\eta \cup PX_\eta : o \leq \eta\}$. Such a strategy may be called an ITBS-strategy—initially trusting but basically skeptical. Similarly there are ISBT-strategies—initially skeptical but basically trusting.

Among the total strategies, two extreme cases stand out:

- the **totally conservative** or **totally skeptical** strategy (the strategy of total skepsis) which always returns the value 0 (that is, $s(\alpha) = 0$, for all $\alpha$),
• the totally radical or totally trusting strategy (the strategy of total trust) which always returns 1 (that is, $s(o) = 1$, for all $o$).

The former defines the new onion as the set

$$\{ Y : (Y = X_\alpha \cup PX_\pi & 0 \leq \alpha \leq \pi) \text{ or } (Y = X_\gamma \text{ or } Y = X_\gamma \cup PX_{\gamma+1} & \pi \leq \gamma \leq \mu) \},$$

while the latter defines it as the set

$$\{ Y : (Y = PX_\alpha & \pi \leq \alpha \leq \mu) \text{ or } (Y = X_\alpha \cup PX_\mu & 0 \leq \alpha \leq \mu) \}. $$

Those two strategies are periodic with period 0 and 1, respectively. It is natural to refer to periodic strategies by the period, so we will refer to these two as (0) and (1). They have the shortest period possible, which gives them a simplicity and naturalness that make them stand out as interesting possibilities. For every other strategy one would have to ask, why employ that particular strategy? Every other periodic strategy seems arbitrary, even if the period is short; for example, why pick (01) or (0001) over other alternatives? And for strategies that are not periodic, the prospects seem even worse.

The problem is perhaps that strategies as defined here do not depend on the plausibility of the new information (the proposition with which revision is to be performed) or the reliability of its source but rather what might be called the personality or the mood of the agent performing the revision. In a sense, by going in the direction of this paper we seem to have reached one limitation of the present modelling.

**Eclectic strategies.** One feature of our strategies is that they preserve a maximum of information about the old onion $O$ and the input $P$. But there are also what may be called eclectic strategies: recipes for revision modelled on a strategy but in which some of the information is given up. In eclectic strategies only some but not all of the fallbacks recommended by a strategy are retained in the new onion.

Most eclectic strategies seem highly arbitrary. Perhaps the most reasonable among them is one that may be called the simple moderate strategy or middle-of-the-way, defined by replacing the first two clauses in the induction step of the definition of fallback function by the single clause

$$F_s(o + 1) = X_{\alpha+1} \cup PX_{\beta+1} $$

(where $s$ is thus supposed to be free to choose). Note that, when this strategy is free to choose, then $X_{\alpha+1} \cup PX_{\beta}$ and $X_{\alpha} \cup PX_{\beta+1}$ will not normally be elements of the new onion.

A simple example will illustrate the character of the strategies mentioned. Let $O = \{X_1, X_2, X_3\}$ be an onion with three distinct elements, where $X_1$ is the belief set and $X_3$ is the commitment set; thus $X_1 \subset X_2 \subset X_3$. Assume that $P$ is a proposition such that the sets $P \cap X_1$, $P \cap X_2$ and $P \cap X_3$ are distinct and nonempty. Then the new onion resulting from revising $O$ by $P$ by a certain strategy has elements as follows:

• the totally skeptical strategy: $PX_1, X_1 \cup PX_2, X_2 \cup X_3, X_3,$
• the middle-of-the-way: \( PX_1, X_1 \cup PX_2, X_2 \cup PX_3, X_3 \),
• the totally trusting strategy: \( PX_1, PX_2, PX_3, X_1 \cup PX_3, X_2 \cup PX_3, X_3 \).

It should be noted that we have no special concept of “expansion”. Classically, one expands one’s beliefs by adding belief in a certain proposition to the set of beliefs one already has, in this way obtaining a new belief set that is consistent if and only if the old belief set is consistent with the proposition in question. But there are many ways of defining a binary operation \( + \), operating on a belief state (onion) \( O \) and a proposition \( P \), in such a way that \( \text{bst} (O + P) = P \cap \text{bst} O \). The most natural among the latter is perhaps \( O + P = O \cup \{ P \cap \bigcap O \} \).

Strategy sets. There is of course nothing to rule out that the agent operates with a set of strategies rather than a single strategy. Thus suppose that \( S \) is a strategy set (set of strategies). Then a formula \( [\* \phi] \theta \) would be true in a model at an index \((O, u)\) if and only if \( \theta \) were true at all indices \((O', u)\), where \( O' \) results by the application of one of the strategies in \( S \); that is to say, for all onions \( O' \) such that, for some \( s \in S, O' = O * \square \phi \) (\text{mods}). For example, the set of ITBS-strategies and the set of ISBT-strategies mentioned above are possible strategy sets in this sense.

In the present setting this possibility may not seem very interesting. But in a richer setting—with more information about the agent, the new information and what is at stake—it might be. But we will not pursue this topic.

Formal results. One would think that with an infinite—even uncountable—number of strategies to choose from, there would be many viable candidates. But our brief inspection of the field suggests that only two of them are really interesting, the (totally) skeptical and the (totally) trusting one.

Theorem 2.5. The skeptical strategy is axiomatized by adding the following two schemata to the axiom system in section 2.2.2:

\[
\begin{align*}
(\text{s1}) & \quad [\* \psi] B \phi \rightarrow ([\* \phi] B \neg \psi \rightarrow ([\* \phi] B \psi \leftrightarrow [\* \psi] B (\phi \rightarrow \theta)), \\
(\text{s2}) & \quad [\* \psi] B \neg \phi \rightarrow ([\* \phi] B \neg \psi \rightarrow ([\* \phi] B \psi \leftrightarrow [\* \psi] B \theta)).
\end{align*}
\]

The trusting strategy is axiomatized by adding the following two schemata instead:

\[
\begin{align*}
(\text{t1}) & \quad K (\phi \land \psi) \rightarrow ([\* \phi] [\* \psi] B \theta \leftrightarrow [\* (\phi \land \psi)] B \theta), \\
(\text{t2}) & \quad K \neg (\phi \land \psi) \rightarrow ([\* \phi] [\* \psi] B \theta \leftrightarrow [\* \psi] B \theta).
\end{align*}
\]

Similar results have been proved by Jonathan Zvesper in [17]. The author’s proofs, obtained independently, were published in [14].

2.3 Nearness semantics

One of David Lewis’s many modellings in [7] is in terms of what he called “comparative similarity”: the concept of a possible world \( j \) being at least as similar to a possible world \( i \) as is a possible world \( k \) (in symbols \( j \preceq i \)). Another modelling, popular in the computer science community, is in terms of “preference” between
possible worlds: \( j \) is preferred to or regarded as equal to \( k \), both viewed from the standpoint of \( i \). In this section we will introduce a similar modelling but with two differences: we will be dealing with total states of an environment, not possible worlds, and we will do so from the standpoint of a certain subset of the environment, not a single point.

### 2.3.1 Nearness relations and nearness frames

Let \((U, T)\) be a Stone space. By a nearness relation in \((U, T)\) we shall understand a certain kind of subset of \(\text{cl}(T) \times U \times U\) to be defined below. If \(\leq\) is a given nearness relation, the following terminology applies to closed subsets \(X\) of \(U\) and points \(u, v\) of \(U\). If \((X, u, v) \in \leq\) then \(u\) is at least as close to (or near) \(X\) as \(v\), in symbols \(u \leq_X v\).

If \((X, u, v) \in \leq\) but \((X, v, u) \notin \leq\) then \(u\) is closer to (or nearer) \(X\) than \(v\), in symbols \(u \prec_X v\). And if \((X, u, v) \in \leq\) and \((X, v, u) \in \leq\) then \(u\) and \(v\) are equally close to (or near) \(X\), in symbols \(u \sim_X v\).

If \(X\) is a closed set, then we say that a nearness relation \(\leq\) is focussed on \(X\) if the following conditions are satisfied:

1. \(\leq\) is nonempty,
2. for all closed sets \(Y\) and points \(u, v\), if \((Y, u, v) \in \leq\) then \(X = Y\),
3. for all \(u \in X\) there is some point \(v\) such that \((X, u, v) \in \leq\) or \((X, v, u) \in \leq\).

Notice that if \(\leq\) and \(\leq'\) are different nearness relations then it is in general possible that \(u \leq_X v\) but not \(u \leq'_X v\), even if they are both focussed on \(X\). If \(\leq\) is a nearness relation focussed on a closed set \(X\), then it is natural to regard the set \(\{(u, v) : u \leq_X v\}\) as a binary relation, and it is also natural to use the symbol \(\leq_X\) to denote that relation.

We shall write \(\text{subfield}(\leq)\) for the the field of \(\leq_X\); that is,

\[
\text{subfield}(\leq) = \text{field}(\leq_X) = \{u : \exists v((X, u, v) \in \leq\text{ or } (X, v, u) \in \leq)\}.
\]

**Observation 2.6** If \(\leq\) is a nearness relation that is focussed on \(X\), then \(X \subseteq \text{subfield}(\leq)\).

We now list five conditions characterizing our conception of nearness. First, we assume that \(\leq_X\) is a partial ordering of \(U\), that is, reflexive, transitive, and connected in its field:

- **(refl)** \(u \leq_X u\) for all \(u \in \text{subfield}(\leq)\),
- **(trans)** if \(u \leq_X v\) and \(v \leq_X w\) then \(u \leq_X w\),
- **(conn)** if \(u \neq v\) then either \(u \leq_X v\) or \(v \leq_X u\) or both, for all \(u, v \in \text{subfield}(\leq)\).

Next, if \(Y\) is any subset of \(U\), we say that an element \(u\) is \(\leq_X\)-minimal in \(Y\) if \(u \in Y\) and, for all \(v \in Y\), if \(v \in \text{field}(\leq_X)\) then \(u \leq_X v\). We write \(\min_{\leq_X}(Y)\) for the set of \(\leq_X\)-minimal elements in \(Y\). (If none exist, then \(\min_{\leq_X}(Y)\) is empty.)

**Observation 2.7** \(X = \min_{\leq_X}(X)\).
Finally, we will also assume that the following separation condition is satisfied:

\( (\text{sep}) \) If it is not the case that \( v \preceq_X u \), then there is a closed set that includes the set \( \{ x \in U : x \preceq u \} \) but does not contain \( v \).

If we introduce the notation \( u \not\preceq_X v \) to express that it is not the case that \( u \preceq_X v \), we can express \((\text{sep})\) in a more compact form:

If \( v \not\preceq_X u \), then \( \{ w \in U : w \preceq_X u \} \subseteq Y \) and \( v \notin Y \), for some closed set \( Y \).

Note that if \( v \not\preceq_X u \) then either \( u \prec_X v \) or else either \( u \) or \( v \) is not in the field of \( \preceq_X \). For each closed subset \( X \) of \( U \), a nearness relation \( \preceq \) focussed on \( X \) determines a belief set \( \text{bst}(\preceq) \) and a commitment set \( \text{kst}(\preceq) \):

\[
\begin{align*}
\text{bst}(\preceq) &= X, \\
\text{kst}(\preceq) &= \text{subfield}(\preceq).
\end{align*}
\]

A nearness frame is a structure \((U, T, S, R)\), where \((U, T)\) is a Stone space, \( S \) is a set of focussed nearness relations, and \( R \) is a function assigning to each clopen set \( P \) a binary relation \( R^P \) over \( S \). Each element \( \preceq \) of \( S \) is required to satisfy the four conditions \((\text{refl}), (\text{trans}), (\text{conn}), \) and \((\text{sep})\) above. In addition, there are also the following conditions:

\( (i) \) if \( (\preceq, \preceq') \in R^P \), where \( \preceq \) is focussed on \( X \) and \( \preceq' \) on \( X' \), then \( X' = \min_{\preceq} P \),

\( (ii) \) if \( (\preceq_X, \preceq_X') \in R^P \) then \( \text{subfield}(\preceq) = \text{subfield}(\preceq') \),

\( (iii) \) if \( (X, u, v) \in \preceq \) for some \( v \in P \), then \( (\preceq, \preceq') \in R^P \) for some \( \preceq' \in S \) that is focussed on \( \min_{\preceq} P \),

\( (iv) \) if \( (\preceq, \preceq') \in R^P \) and \( (\preceq, \preceq'') \in R^P \), then \( \preceq' = \preceq'' \).

Truth-conditions under a valuation, where \( \preceq \) is any focussed nearness relation in \( S \) and \( u \) is any point in \( U \):

\[
\begin{align*}
(\preceq, u) \vdash \text{B} \phi & \text{ iff } \text{bst}(\preceq) \subseteq \| \phi \|, \\
(\preceq, u) \vdash \text{K} \phi & \text{ iff } \text{kst}(\preceq) \subseteq \| \phi \|, \\
(\preceq, u) \vdash \text{+} \phi \theta & \text{ iff } (\preceq', u) \vdash \theta, \text{ for all relations } \preceq' \text{ such that } (\preceq, \preceq') \in R^{\| \phi \|}.
\end{align*}
\]

2.3.2 From nearness to onions and back

Let \( \preceq \) be a focussed nearness relation. For each \( u \in \text{subfield}(\preceq) \), define

\[
N_u = \{ v \in U : v \preceq_X u \}.
\]

**Observation 2.8** \( N_u \) is a closed set, for each point \( u \in \text{subfield}(\preceq) \).

**Proof.** For every point \( v \) such that \( v \not\preceq_S u \) let \( Y_v \) be a closed set such that \( N_u \subseteq Y_v \) and \( v \notin Y_v \); the existence of \( Y_v \) is guaranteed by \((\text{sep})\). Writing \( Y \) for \( \cap \{ Y_v : u \not\preceq_S v \} \) and noting that \( Y \) is closed, we conclude that \( N_u \subseteq Y \). Suppose now for a reductio argument that \( N_u \neq Y \). Then there is some point \( w \) such that
By (2), \( w \notin Y_w \). But \( w \in Y_w \) by (1).  

**Observation 2.9** \( X = \bigcap \{N_w : w \notin X\} \), for all closed subsets \( X \) of \( \text{subfield}(\leq) \).

**Proof.** First assume that \( w \notin X \), for some point \( w \in \text{subfield}(\leq) \). Suppose that \( u \) is a point in \( X \). Then \( u \in \text{min}_X X \), by Observation 2.7. Since \( w \in \text{subfield}(\leq) \) this means that \( u \leq_X w \). Hence \( u \in N_w \). This argument proves that \( X \subseteq N_w \). Consequently, \( X \subseteq \bigcap \{N_w : w \notin X\} \), which is part of what we want to establish. For the remaining part note that the set \( \{N_w : w \notin X\} \) comprises all closed extensions of \( X \). Therefore \( X = \bigcap \{N_w : w \notin X\} \).  

By the corresponding onion \( \mathcal{O}(\leq) \) we will understand the closure under (aint) of the set \( \{N_u : u \in \text{subfield}(\leq)\} \). In other words, the fallbacks of \( \mathcal{O}(\leq) \) will be all nontrivial intersections of sets of the form \( N_u \):

\[
\mathcal{O}(\leq) = \{\bigcap \{N_u : u \subseteq Y\} : Y \neq \emptyset \& Y \subseteq \text{subfield}(\leq)\}.
\]

That \( \mathcal{O}(\leq) \) really is an onion follows from Observation 2.8:

**Observation 2.10** \( \mathcal{O}(\leq) \) is an onion.

Similarly, let \( O \) be a given onion. By the corresponding nearness relation we will understand the set

\[
\mathcal{R}(O) = \{(\bigcap O, u, v) : u \in \bigcup O \& \forall X \in O \ (v \in X \Rightarrow u \in X)\}.
\]

It is obvious that our choice of terminology is justified:

**Observation 2.11** \( \mathcal{R}(O) \) is a nearness relation focussed on \( \bigcap O \).

The intimate connexion between nearness relations and onions is brought out by the following fact:

**Proposition 2.1.** If \( \leq \) is a focussed nearness relation, then \( \leq = \mathcal{R}(\mathcal{O}(\leq)) \). Similarly, if \( O \) is an onion, then \( O = \mathcal{O}(\mathcal{R}(O)) \).

**Proof.** To prove the first claim of the observation, suppose that \( \leq \) is a nearness relation focussed on some closed set \( X \). Assume that \( u, v \in \text{subfield}(\leq) \). If \( u \leq_X v \), then, for any fallback \( Y \) in the onion \( \mathcal{O}(\leq) \), if \( v \in Y \) also \( u \in Y \). Hence \((X, u, v) \in \leq \). Conversely, if \( u \not\leq_X v \) then by connectedness of \( \leq_X \) and the fact that \( u \) and \( v \) are in the field of \( \leq_X \), we have \( v \leq_X u \). Thus \( N_v \) is a fallback of \( \mathcal{O}(\leq) \) containing \( v \) but not \( u \), and hence \((X, u, v) \notin \leq \). These two remarks prove the first claim. The second claim follows from Observation 2.9.  

⊙
2.3.3 Revision in the nearness modelling

Because of the close relationship between the nearness and the onion modelings, the same strategies for revision are available in the former as in the latter. Nevertheless, the idioms of the two modelings are different, and this may influence informal analysis. That is to say, the way we naturally think about revision may differ. For example, onions and simple parts of onions are the natural ingredients in a discussion of revision in the onion modelling. But even though onions can be defined in the nearness idiom, there are ways in which the discussion about revision by a proposition could be carried out in a much more restricted language and therefore much more simply.

Thus given a nearness relation \( \preceq \) and a proposition \( P \), suppose that whether the new relation holds between \( u \) and \( v \) (that is, \( u \preceq S' v \)) shall depend only on whether it holds under the old relation (that is, \( u \preceq S v \)) and whether the points belong to \( P \) or not and, within \( P \), whether they belong to \( \min_{S} P \) or not. In other words, we envisage a semi-formal metalanguage in which the primitive propositions are of the form

- \( u \preceq S v \),
- \( w \in \min_{S} P \),
- \( w \in P \),

while the nonprimitive propositions are truth-functional combinations of primitive ones.

Since \( \min_{S} P \) is a subset of \( P \), our language partitions the set of ordered pairs \((u, v)\), where \( u \) and \( v \) are in the field of \( \preceq \), into nine equivalence classes (which we will call “boxes”):

<table>
<thead>
<tr>
<th>( u \in \min_{S} P )</th>
<th>( v \in \min_{S} P )</th>
<th>( v \in P - \min_{S} P )</th>
<th>( v \notin P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>( u \in P - \min_{S} P )</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>( u \notin P )</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

Let us say that the ordered pair “\( (u, v) \) is in box \( n \)”, schematically \( (u, v) \in \text{box}(n) \), if \( u \) satisfies the condition to the left of \( n \), and \( v \) the condition above \( n \) (where \( 1 \leq n \leq 9 \)). Note that “\( (u, v) \in \text{box}(n) \)” may be seen as an abbreviation of a proposition in the meta-language: “\( (u, v) \in \text{box}(1) \)” abbreviates “\( u \in \min_{S} P \land v \in \min_{S} P' \)”, etc. Implicit in our approach is the following “principle of uniformity within boxes” which is by no means uncontroversial but also is not totally unnatural:

(§) if \( (u, v) \in \text{box}(n) \) and \( (u', v') \in \text{box}(n) \), then \( u \preceq_{S'} v \) if and only if \( u' \preceq_{S'} v' \).

We now raise the question how a new relation can be defined in terms of our meagre metalanguage. The definition should be of the form

\[ u \preceq_{S'} v \text{ if and only if } \bigvee_{1 \leq n < 9} ((u, v) \in \text{box}(n) \land \gamma(n)), \]
where \( \gamma \) is a function from the set of (the numbers of) boxes to the set of truth-functions of the condition \( u \leq_S v \). Recall that there are four truth-functions in terms of one variable: tautology, contradiction, identity and negation. Evidently, each function \( \gamma \) defines a new nearness relation and so, implicitly, a recipe for revision. The number of such recipes is \( 4^9 = 262,144 \), a very large number. However, as we shall see, the number quickly decreases as we impose certain simple restrictions.

The principles mentioned may not be uncontroversial, but in the nearness perspective they have a certain claim to naturalness. In any case they have the effect of drastically reducing the number of possible recipes for revision.

One immediate restriction is to exclude the fourth truth-function (negation) from consideration: given that it is revision we are discussing, it seems intuitively unreasonable to consider a revision recipe which for two points \( u \) and \( v \) in a certain box \( n \) specifies that \( u \leq_S v \) if and only if \( u \not\leq_S v \). This restriction means that the functions \( \gamma \) that we are prepared to consider have their ranges included in the set \{truth, falsity, identity\}. There are “only” \( 3^9 = 19,683 \) such functions, still an unmanageable number.

But as just remarked, we intend to bring down the number much further.

To begin with, let us assume as a basic principle—not controversial as much as simplifying—that the field of the old relation coincides with the field of the new relation:

\[
(\natural 0) \text{ field}(\leq_S) = \text{field}(\leq_{S'}).
\]

In the conditions considered below we assume that \( u \) and \( v \) are parameters ranging over the field of \( \leq_S \).

The fundamental idea concerning revision by \( P \) in the nearness modelling is that the new belief set be made up of the nearest, and only the nearest, \( P \)-points: that \( \min_{\leq_{S'}} U = \min_{\leq_S} P \). This intuition is also captured by the following two conditions:

\[
(\natural 1) \quad \text{if } u \in \min_{\leq_S} P, \text{ then } u \leq_{S'} v,
\]

\[
(\natural 2) \quad \text{if } u \in \min_{\leq_S} P \text{ and } v \notin \min_{\leq_S} P, \text{ then } u \prec_{S'} v.
\]

Another idea, not fundamental but possible, is that revision by \( P \) should not affect the relation of nearness either within the set of \( P \)-points or within the set of non-\( P \)-points. Formally:

\[
(\natural 3) \quad \text{if } u \in P \text{ and } v \in P, \text{ then } u \leq_S v \text{ if and only if } u \leq_{S'} v,
\]

\[
(\natural 4) \quad \text{if } u \notin P \text{ and } v \notin P, \text{ then } u \leq_S v \text{ if and only if } u \leq_{S'} v.
\]

Yet another idea, again not fundamental but possible, is that revision by \( P \) should not improve the position of non-\( P \)-points over that of \( P \)-points. That is, according to this intuition, if \( u \not\leq_S v \) before revision by \( P \), where \( u \notin P \) and \( v \in P \), then \( u \not\leq_{S'} v \) after revision. Formally:

\[
(\natural 5) \quad \text{if } u \notin P \text{ and } v \in P, \text{ then } u \not\leq_S v \text{ if } u \not\leq_{S'} v,
\]

\[
(\natural 6) \quad \text{if } u \in P \text{ and } v \notin P, \text{ then } u \not\leq_S v \text{ only if } u \not\leq_{S'} v.
\]

With the help of these principles we can fill in a number of the boxes on the diagram above. For convenience we use the following code:
• if \( u \leq_S v \), for all points \( u \) and \( v \) in a certain box, then we put \textit{ALWAYS} in it;
• if \( u \not\leq_S v \), for all points \( u \) and \( v \) in a certain box, then we put \textit{NEVER} in it;
• if \( u \leq_S v \) iff \( u \not\leq_S v \), for all points \( u \) and \( v \) in a certain box, but it is neither the case that, for all points \( u' \) and \( v' \) in that box, \( u' \leq_S v' \), nor the case that, for all points \( u' \) and \( v' \) in that box, \( u' \not\leq_S v' \), then we put \textit{DEPENDS} in it.

First, condition (\( \sharp 1 \)) tells us that the entry in boxes 1, 2 and 3 is \textit{ALWAYS}, and condition (\( \sharp 2 \)) that in 4 it is \textit{NEVER}. Next, condition (\( \sharp 3 \)) concerns boxes 1, 2, 4 and 5. While agreeing with (\( \sharp 1 \)) on boxes 1 and 2 and with condition (\( \sharp 2 \)) on box 4, it yields the entry \textit{DEPENDS} in box 5. Condition (\( \sharp 4 \)) puts \textit{DEPENDS} in box 9, and condition (\( \sharp 5 \)) puts \textit{NEVER} in box 7. In this way we arrive at the following result:

<table>
<thead>
<tr>
<th>( u \in \min_{\leq_S} P )</th>
<th>( v \in \min_{\leq_S} P )</th>
<th>( v \in P - \min_{\leq_S} P )</th>
<th>( v \notin P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>\textit{ALWAYS}</td>
<td>\textit{ALWAYS}</td>
<td>\textit{NEVER}</td>
<td>\textit{DEPENDS}</td>
</tr>
<tr>
<td>\textit{ALWAYS}</td>
<td>\textit{ALWAYS}</td>
<td>\textit{NEVER}</td>
<td>\textit{DEPENDS}</td>
</tr>
<tr>
<td>\textit{ALWAYS}</td>
<td>\textit{ALWAYS}</td>
<td>\textit{NEVER}</td>
<td>\textit{DEPENDS}</td>
</tr>
</tbody>
</table>

Thus we are left with two gaps and nine possible ways of filling them.\(^5\) However, some of those ways are not really open to us. Here is why not.

• \textit{NEVER} is not an option for box 6.
  
  \textit{Argument.} Suppose that \( u \leq_S v \), for some points \( u \) and \( v \) such that \( u \notin P \) and \( v \in P - \min_{\leq_S} P \). Thus if we had entered \textit{NEVER} in box 6 we must conclude that \( u \not\leq_S v \). But this result would contradict condition (\( \sharp 6 \)), which requires \( u \leq_S v \). Hence \textit{NEVER} in box 6 is not an option if we seek a revision rule that is generally applicable—even in a situation where elements \( u \) and \( v \) are as specified. \( \Box \)

• \textit{ALWAYS} is not an option for box 8.
  
  \textit{Argument.} The argument is similar to the one just given against \textit{NEVER} in box(6), with condition (\( \sharp 5 \)) playing the rôle of (\( \sharp 6 \)). \( \Box \)

• The combination \textit{ALWAYS} for box 6 and \textit{DEPENDS} for box 8 is not an option.

\textit{Argument.} Suppose that \( u \) and \( v \) are points such that \( u, v \in P - \min_{\leq_S} P \) and \( x, y \notin P \) and

\[ x \leq_S u, \quad y \leq v. \]

With \textit{ALWAYS} in box 6 we infer

\[ u \leq_S x, \quad u \leq_S y, \quad v \leq_S x, \quad v \leq_S y. \]

\( ^5 \) To make quite clear what is going on, let us list the instructions generating the above table:

box(1): if \( u \in \min_{\leq_S} P \) and \( v \in \min_{\leq_S} P \) then \( u \leq_S v \),
box(2): if \( u \in \min_{\leq_S} P \) and \( v \in P - \min_{\leq_S} P \) then \( u \leq_S v \),
box(3): if \( u \in \min_{\leq_S} P \) and \( v \not\in P \) then \( u \leq_S v \),
box(4): if \( u \in P - \min_{\leq_S} P \) and \( v \in \min_{\leq_S} P \) then \( u \not\leq_S v \),
box(5): if \( u \in P - \min_{\leq_S} P \) and \( v \in P - \min_{\leq_S} P \) then \( u \leq_S v \) if and only if \( u \leq_S v \),
box(7): if \( u \not\in P \) and \( v \in \min_{\leq_S} P \) then \( u \not\leq_S v \),
box(9): if \( u \not\in P \) and \( v \not\in P \) then \( u \leq_S v \) if and only if \( u \leq_S v \).
Consequently \( u \preceq_{S'} y \preceq_{S'} v \preceq_{S'} x \preceq_{S'} u \). Hence \( u \sim_{S'} y \sim_{S'} v \sim_{S'} x \sim_{S'} u \) by transitivity—the four elements \( u, v, x, y \) are all equivalent under the new relation \( \preceq_{S'} \). But this is not possible if we also have \textsc{depends} in box 8. Thus as in the two preceding cases, the suggestion we are considering is not an option if we want a decision rule that can be applied to every possible situation.

- If \textsc{never} is the entry for box 8, then \textsc{always} must be the entry for box 6.

\textit{Proof.} Suppose that \( u \in P - \min_{\preceq_S} P \) and \( v \notin P \). Then \textsc{never} in box 8 requires that \( v \not\preceq_{S'} u \). By connectedness then \( u \preceq_{S'} v \). Hence trivially \( u \preceq_{S'} v \).

Thus we find that the only possibilities for filling in the gaps in boxes 6 and 8 are \((\textsc{always}, \textsc{never})\) and \((\textsc{depends}, \textsc{depends})\). We have reduced the 9 possibilities to just 2! But now reduction has come to a halt: the two remaining possibilities are both viable. We discuss them in turn.

\begin{itemize}
  \item \textbf{6—always; 8—never.} In this case there are two conditions in addition to the general conditions (\(\#0-6\)):
    \begin{enumerate}
      \item if \( u \in P - \min_{\preceq_S} P \) and \( v \notin P \), then \( u \preceq_{S'} v \),
      \item if \( u \notin P \) and \( v \in P - \min_{\preceq_S} P \), then \( u \preceq_{S'} v \).
    \end{enumerate}
    Note that they may be compressed into one additional condition:
    \begin{enumerate}
      \item if \( u \in P - \min_{\preceq_S} P \) and \( v \notin P \), then \( u \preceq_{S'} v \),
    \end{enumerate}

The description of the new relation \( \preceq_{S'} \) is straight-forward. After revision, the points are divided into just two categories. First there are all the \( P \)-points, related to one another under the new relation as they were under the old one. Then there are all the non-\( P \)-points, also related to one another as they were under the old one. And each of the former is strictly nearer the new belief set than each of the latter.

\begin{itemize}
  \item \textbf{6—depends; 8—depends.} Here the are two other additional conditions:
    \begin{enumerate}
      \item if \( u \in P - \min_{\preceq_S} P \) and \( v \notin P \), then \( u \preceq_S v \) if and only if \( u \preceq_{S'} v \),
      \item if \( u \notin P \) and \( v \in P - \min_{\preceq_S} P \), then \( u \preceq_S v \) if and only if \( u \preceq_{S'} v \).
    \end{enumerate}
    Given conditions (\(\#3-4\)), these conditions can be replaced by one single condition:
    \begin{enumerate}
      \item \ if \( u, v \notin \min_{\preceq_S} P \) then \( u \preceq_{S'} v \) if and only if \( u \preceq_{S'} v \).
    \end{enumerate}

Also in this case it is simple to describe the elements in the field after revision. Again there are two categories. First there are all the \( P \)-points that were minimal in \( P \) under the old relation and now are minimal \textit{tout court} under the new relation. Then there are all the other points, with the old relation intact as far as their internal relationship goes. And all the former are nearer the new belief set than the latter.

Condition (\(\#\)) yields what Van Benthem has termed the “revolutionary policy”, condition (\(b\)) his “Machiavellian policy” (see \cite{16}). They are closely related to the two main strategies of section 2.2.6 above: given an onion, the trusting strategy induces the former while the skeptical strategy induces the latter.
2.3.4 Comments

The result of the analysis in this section depends on the chosen metalanguage. We ended up with two revision policies which are “reasonable”, in some sense of the word. But it cannot claim to be the only one possible, even within a nearness type of modelling. And with a richer metalanguage we might come across other possibilities. In general, the richer the metalanguage, the more possibilities there would be. Suppose for example that we define the concept of \( n \)-minimality, where \( n \) is a positive integer:

\[
\begin{align*}
\min_{\leq n}^1 P &= \min_{\leq n} P, \\
\min_{\leq n}^{n+1} P &= \{u : u \in \min_{\leq n} (P - \bigcup_{i \leq n} \min_{\leq n}^i P)\}.
\end{align*}
\]

It is clear that in terms of these new concepts we would be able to define an endless number of “\( n \)-revolutionary” and “\( n \)-Machiavellian” policies.

2.4 Conclusion

As remarked in the introduction, our raw uneducated intuitions about belief revision are vague and ambiguous, which makes theorizing difficult. It seems desirable, then, to try to re-formulate our intuitions in a formal setting, which can then be discussed in a semi-formal meta-language jargon. This is what we have done here, first in the context of the onion modelling and then in the context of a nearness modelling. In each case the pre-theoretical intuitions become influenced by the new setting—they become “re-educated”. This is always the case with formalization. But \textit{a priori} it is quite possible that different formalizations will bring out different outcomes. Therefore it is of some interest that our analyses suggest the same result in both cases: there are really only two strategies or policies for revision that stand out among the great number of possibilities.

One might object that this result is not surprising, given that the two modellings under consideration are conceptually equivalent. But an objection formulated as baldly as that overlooks a methodological point that deserves to be considered before it is dismissed. Two modellings may be equivalent at a deeper level and yet appear different to the viewer; they may emphasize different aspects of the underlying theory and thereby encourage different paths of thinking. This is why mathematicians have sometimes been able to solve a problem by translating it from one area of mathematics into another. And this is no doubt one reason why David Lewis was concerned to work out so many different but equivalent modellings in [7].

Let us end by noting how little is assumed in the kind of theory considered here. For example:

- We don’t know the personality of the agent. Is he in general conservative or radical, skeptical or trusting?
• We don’t know the mood of the agent on a particular occasion. Is he constant in his ways, or is he sometimes this way, sometimes that way?
• We don’t know what kind of agent we are dealing with. Is he simple (a human being or a robot), or is “he” a collective (a committee (with a structure) or a crowd (without one))?
• We don’t know about the incoming information except that it has been accepted and must be integrated into the agent’s beliefs. But how reliable is it, where does it come from, how important is it?

Incorporating features such as these might make for a more interesting theory.

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