Composing Models

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Abstract

• We study a new composition operation on (epistemic) multiagent models and update actions that takes vocabulary extensions into account.

• This operation allows to represent partial observational information about a large model in a small model, where the small models can be viewed as representations of the observational power of agents, and about their powers for changing the facts of the world.

• Our investigation provides ways to check relevant epistemic properties on small components of large models, and our approach generalizes the use of ‘locally generated models’.
Overview: Three Simple Messages

• Models can be made small by vocabulary restriction
• Composing restricted models is easy
• Compositions of restricted models are useful

Note: an expanded version of this LOFT paper can be found in Chapter 5 of the PhD Thesis of Yanjing Wang, *Epistemic Modelling and Protocol Dynamics*, to be defended in September 2010 (available upon request from the author).
Multi-agent Models with Different Vocabularies

Fix a set of proposition letters $P$. Call a subset of $P$ a vocabulary.

Consider multi-agent models with vocabularies $Q$ taken from $P$.

Call such models restricted models.

This allows us to refine ‘knowledge about the world’ to ‘knowledge about $Q$’.
Knowing Nothing About Anything

The restricted model $E$ for knowing nothing about anything:

Formally, $(\{e\}, I, \{(e, e) \mid i \in I\}, e \mapsto \emptyset, \emptyset)$.

Compare: the non-restricted model for knowing nothing about anything, for a language over $P$ with $|P| = n$ has $2^n$ worlds.
Restricted Models for Muddy Children

Single child not knowing whether it is muddy. Voc restricted to $m_i$:

Single **muddy** child not knowing whether it is muddy:

Single **clean** child not knowing whether it is muddy:
Restricted Model Composition: Example

\[ m_1 \oplus \overline{m_1} \leftrightarrow m_2 \oplus \overline{m_2} \]
Restricted Model Composition: Definition

Restricted model composition is a product construction. The composition \( \mathcal{M} \circ \mathcal{N} \) of two restricted multi-agent models with the same agent set \( I \) is given by \((W, I, R, V, Q_M \cup Q_N)\), where the new set of worlds is given by:

\[
W = \{(w, v) \mid w \in W_M, v \in W_N, V_M(w) \cap Q_N = V_N(v) \cap Q_M\},
\]

the new accessibility relations are defined as the product of the relations on the components, in the usual product way:

\[(w, v)R_i(w', v') \text{ iff } wR_iMw' \text{ and } vR_iNv',\]

and \(V(w, v)\) agrees with \(V_M(w)\) on \(Q_M\) and with \(V_N(v)\) on \(Q_N\):

\[V(w, v) = V_M(w) \cup V_N(v).\]
Composing the Model for Three Muddy Children

$m_1$  
$m_1$  
$m_1$

$m_2$  
$m_2$

$m_3$  
$m_3$

$\odot$  
$\odot$
Structural Properties of $\hat{\circ}$

$\hat{\circ}$ is a congruence for $\hat{\circ}$:

If $M_1 \hat{\circ} M_2$ and $N_1 \hat{\circ} N_2$ then $M_1 \hat{\circ} N_1 \hat{\circ} M_2 \hat{\circ} N_2$.

Multi-agent models form a commutative monoid under $\hat{\circ}$:

$$E \hat{\circ} M \hat{\circ} M$$

$$M \hat{\circ} (N \hat{\circ} K) \hat{\circ} (M \hat{\circ} N) \hat{\circ} K$$

$$M \hat{\circ} N \hat{\circ} N \hat{\circ} M$$

This yields the well-known preordering $\leq$:

$$M \leq N$$ if there is a $K$ with $M \hat{\circ} K \hat{\circ} N$. 
⊕ is not idempotent

There are $M$ with the property that $M \circ M \neq M$. Example:

$M : \quad \overline{p} \quad p \quad p \quad \overline{p}$

$s \quad t \quad u \quad v$

$(t, u)$ is a $p$-world in $M \circ M$, but $(t, u)$ cannot reach a $\overline{p}$ world in $M \circ M$. 
Left-Simulation

A left-simulation between $\mathcal{M}$ and $\mathcal{N}$ is like a bisimulation, but with the invariance condition restricted to the vocabulary of $\mathcal{M}$, and with the zig condition omitted.

Formally, a left-simulation between $\mathcal{M}$ and $\mathcal{N}$ is a relation $C \subseteq W_M \times W_N$ such that $wCv$ implies that the following hold:

**Restricted invariance** $p \in V_M(w)$ iff $p \in V_N(v)$ for all $p \in Q_M$,

**Zag** If for some $i \in I$ there is a $v' \in W_N$ with $v \xrightarrow{i} v'$ then there is a $w' \in W_M$ with $w \xrightarrow{i} w'$ and $w'Cv'$.

$\mathcal{M}, w \leftarrow \mathcal{N}, v$: there is a left-simulation that connects $w$ and $v$.

$\mathcal{M} \leftarrow \mathcal{N}$: there is a total left-simulation between $\mathcal{M}$ and $\mathcal{N}$.

**Theorem 1** If $\mathcal{M} \leq \mathcal{N}$ then $\mathcal{M} \leftarrow \mathcal{N}$. 
$\mathcal{M}$ is propositionally differentiated if it holds for all worlds $w, w'$ of $\mathcal{M}$ that if $w$ and $w'$ have the same valuation then $w \iff w'$.

In other words, if $w \not\iff w'$ then this difference shows up as a difference in the valuations of $w$ and $w'$.

**Theorem 2** If $\mathcal{M}$ is propositionally differentiated, then $\mathcal{M} \models N$ implies $\mathcal{M} \leq N$.

The full paper has an example showing that the theorem may fail for models that are not propositionally differentiated.
Expansion to Larger Vocabulary

Expansion of this model to $m_1, m_2$:
Vocabulary Expansion, Formally

Let $Q^I$ be the universal ignorance model for $Q$, i.e. $Q^I = (W, I, R, V, Q)$ with $W = \mathcal{P}(Q)$, $R_i = W^2$, $V = \text{id}$.

If $M = (W, I, R, V, Q)$ is a restricted static model and $Q_1$ is a set of proposition letters, then we define the expanded model for the larger vocabulary $Q \cup Q_1$ as follows:

$M \triangleleft Q_1 = M \oplus Q_1^I$. 
Theorem 3 (Preservation) If a pointed model \((M, s)\) is decomposable into models

\((M_0, s_0), \ldots, (M_n, s_n)\)

with disjoint vocabularies

\(Q_0, Q_1, \ldots, Q_n,\)

then for any \(i:\)

\(M_i, s_i \equiv_{Q_i} M, s.\)

Therefore for any \(\phi\) in \(PDL_{Q_i,Ag}:\)

\(M_i, s_i \models \phi \iff M, s \models \phi.\)

This means that any properties of the large model that can be stated in a local vocabulary can be checked locally.
We say $\mathcal{M}$ is *locally generated* if, for every agent $i$, there is a set of boolean formulas $\Phi_i$ (the set of local observables) based on $Q_M$ such that for all $w, w' \in W_M$:

$$w \sim_i w' \text{ iff for all } \varphi \in \Phi_i, \mathcal{M} \models_w \varphi \iff \mathcal{M} \models_{w'} \varphi$$

Intuitively, a model is locally generated if the local observables of the agents determine the epistemic relations in the model.

Example: the $n$-Muddy Children model is locally generated by set of observables $\Phi_1, \ldots, \Phi_n$, where

$$\Phi_i = \{m_j \mid j \in I, j \neq i\}.$$
Theorem 4 (Decomposition by agents) Let a set of agents
\[ \text{Ag} = \{1, 2, \ldots n\} \]
be given.

If \( M = (W, Q, \text{Ag}, \sim, V) \) is locally generated by \( \Phi_1, \ldots, \Phi_n \), then there are models \( M_1, \ldots, M_n \) and \( M_0 \) such that:

- \( M \leq (M_0 \uplus M_1 \uplus \cdots \uplus M_n) \);
- \( |W_{M_j}| \leq |W| \) and \( M_i \) is a bisimulation contracted model;
- \( Q_{M_j} = \{ p \in Q_M \mid p \text{ appears in } \Phi_j \} \) for \( j > 0 \).

Another possible decomposition of locally generated models is by issues. Example: Our earlier Muddy Children decomposition. See Yanjing’s thesis.
Decomposition by agents of the 3-Muddy Children model, for first agent:

\[ \Phi_1 = \{m_2, m_3\}. \]

The model \( M_1 \) looks like this:

![Diagram of the 3-Muddy Children model for the first agent]
Not locally generated, but decomposable:
Two muddy children who each have a mirror, but they do not know that of each other.
Update with Vocabulary Expansion: Public Announcement

\[ m_1 \quad 1 \quad \overline{m_1} \]

\[ \{m_2\} \]

\[ m_1m_2 \quad 1 \quad \overline{m_1m_2} \]

\[ m_1\overline{m_2} \quad 1, 2 \quad \overline{m_1m_2} \]

\[ \overline{m_1m_2} \quad 1 \]
!(m₁ ∨ m₂)
Update of Other Component

\[ m_2 \xrightarrow{\{m_1\}} \overline{m_2} \quad \text{and} \quad \overline{m_1m_2} \xrightarrow{1,2} \overline{m_1m_2} \]
Composition of update results
Interaction of ⊠ and ⊗

**Theorem 5** If $A$ is propositionally differentiated then:

$$(\mathcal{M} \oplus \mathcal{N}) \otimes A \Leftrightarrow (\mathcal{M} \otimes A) \oplus (\mathcal{N} \otimes A).$$

And without conditions on the action models, with the appropriate notion of $\oplus$ for action models:

**Theorem 6** $\mathcal{M} \otimes (A \oplus B) \Leftrightarrow (\mathcal{M} \otimes A) \oplus (\mathcal{M} \otimes B)$. 
Further Work

- Extend DEMO with $\Box$, in order to allow epistemic model checking of large models on local components.

- Characterize models in terms of their composition. (Example: what do models that are composed from only two-world components look like? Answered in the full paper.)

- Study the combination of communicative actions and vocabulary expansion. Example task: axiomatize the strong Kleene logic of public announcement $!\phi$ and vocabulary expansion $\#p$, where $\#p$ is interpreted as the model changing operation $\mathcal{M} \mapsto \mathcal{M} \leftarrow \{p\}$.

- Work out obvious connections with awareness logics, and with work on the dynamics of awareness.