Logic, Information and Knowledge

Jan van Eijck

ESSLLI Logic in Action Course, Wednesday August 3, 2011
Abstract

Today’s lecture deals with the logic of knowledge as based on information, including changes in knowledge which result from observations of facts, or communication between agents knowing different things. This area is called epistemic logic, and its main difference with the earlier systems of Chapters 2, 3 and 4 is that we can also express facts about knowledge of one or more agents in the logical language itself. This ‘social’ perspective occurs in many settings: knowing what others do or do not know determines our actions. Another central theme of this chapter is "change": successive information processing steps change what agents know, and this, too, is essential to understanding the logic of language use and other cognitive tasks.
Very Brief History

David Lewis  Jaakko Hintikka  Robert Aumann
Very Brief History

David Lewis  Jaakko Hintikka  Robert Aumann
Very Brief History

David Lewis  Jaakko Hintikka  Robert Aumann

Joe Halpern  Jan Plaza  A. Baltag  Johan van Benthem
The Muddy Children Puzzle

$a$ clean, $b$, $c$ and $d$ muddy.

at least one of you is muddy

$\circ$ $\bullet$ $\bullet$ $\bullet$
The Muddy Children Puzzle

$a$, $b$, $c$, and $d$ muddy.

at least one of you is muddy

who knows his state?
The Muddy Children Puzzle

$a$ clean, $b$, $c$ and $d$ muddy.

at least one of you is muddy

who knows his state?
The Muddy Children Puzzle

*a clean, b, c and d muddy.*

at least one of you is muddy

who knows his state?

who knows his state now?
The Muddy Children Puzzle

\[a\] clean, \[b\], \[c\] and \[d\] muddy.

\begin{array}{ccccc}
\hline
& a & b & c & d \\
\hline
\text{at least one of you is muddy} & \bigcirc & \bullet & \bullet & \bullet \\
\text{who knows his state?} & N & N & N & N \\
\text{who knows his state now?} & N & N & N & N \\
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\end{array}
# The Muddy Children Puzzle

*a* clean, *b*, *c* and *d* muddy.

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at least one of you is muddy

who knows his state?

who knows his state now?

who knows his state now?
The Muddy Children Puzzle

*a* clean, *b*, *c* and *d* muddy.

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at least one of you is muddy

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The Muddy Children (2)

\(a, b, c\) clean, \(d\) muddy.

\( \bullet \) at least one of you is muddy
The Muddy Children (2)

\[ a, b, c \text{ clean, } d \text{ muddy.} \]

\[ \begin{array}{cccc}
  a & b & c & d \\
  \circ & \circ & \circ & \bullet \\
\end{array} \]

at least one of you is muddy

who knows his state?
The Muddy Children (2)

\( a, b, c \) clean, \( d \) muddy.

\[
\begin{array}{cccc}
\text{a} & \text{b} & \text{c} & \text{d} \\
\end{array}
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The Muddy Children (2)

\[ a, b, c \text{ clean, } d \text{ muddy.} \]

\[
\begin{array}{cccc}
\text{at least one of you is muddy} & a & b & c & d \\
\text{who knows his state?} & \bullet & N & N & N & Y \\
\text{who knows his state now?} & & & & \\
\end{array}
\]
### The Muddy Children (2)

*a, b, c clean, d muddy.*

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The Muddy Children (3)

\[ a, b \text{ clean, } c, d \text{ muddy.} \]

\[
\begin{array}{cccc}
 a & b & c & d \\
\odot & \odot & \bullet & \bullet \\
\end{array}
\]

at least one of you is muddy
The Muddy Children (3)

\(a, b\) clean, \(c, d\) muddy.

\[
\begin{array}{cccc}
a & b & c & d \\
\circ & \circ & \bullet & \bullet \\
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at least one of you is muddy who knows his state?
The Muddy Children (3)

\(a, b\) clean, \(c, d\) muddy.

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at least one of you is muddy

who knows his state?
The Muddy Children (3)

\[a, b\] clean, \[c, d\] muddy.

at least one of you is muddy
who knows his state?
who knows his state now?
The Muddy Children (3)

\[ a, b \text{ clean, } c, d \text{ muddy.} \]

\[
\begin{array}{cccc}
\text{at least one of you is muddy} & \circ & \circ & • & • \\
\text{who knows his state?} & N & N & N & N \\
\text{who knows his state now?} & N & N & Y & Y
\end{array}
\]
The Muddy Children (3)

\[ \text{a, b clean, c, d muddy.} \]

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at least one of you is muddy

who knows his state?  \(\text{N N N N N}\)

who knows his state now?  \(\text{N N Y Y}\)

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The Muddy Children (3)

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Individual Ignorance

You have to finish a paper, and you are faced with a choice: do it today, or put it off until tomorrow.

Result of coin flip under a cup:
Multi Agent Ignorance

Suppose Alice and Bob are present, and Alice tosses a coin under a cup. The result of a hidden coin toss with the coin heads up:

\[ w : h \quad ab \quad w' : \overline{h} \]
Multi Agent Ignorance

Suppose Alice and Bob are present, and Alice tosses a coin under a cup. The result of a hidden coin toss with the coin heads up:

\[ w : h \quad b \quad w' : \overline{h} \]
Multi Agent Ignorance

Suppose Alice and Bob are present, and Alice tosses a coin under a cup. The result of a hidden coin toss with the coin heads up:

\[ w : h \quad \overset{b}{\longrightarrow} \quad w' : \overline{h} \]

Alice is taking a look under the cup, while Bob is watching.
Multi Agent Ignorance

Suppose Alice and Bob are present, and Alice tosses a coin under a cup. The result of a hidden coin toss with the coin heads up:

\[ w : h \xrightarrow{b} w' : \overline{h} \]

Alice is taking a look under the cup, while Bob is watching. Now Alice knows the outcome. Bob knows that Alice knows the outcome. Bob does not know the outcome himself.
Back to the Children
Back to the Children
Back to the Children
Back to the Children
Epistemic Situations: Card Deals

Alice, Bob and Carol, each draw a card from a stack of three cards. They know that the cards are red, white and blue. They cannot see the cards of the others.
Alice says: “I do not have white”
Alice says: “I do not have white”
Public Announcements

Jan Plaza
Public Announcements

Jan Plaza

Effect of a public announcement $\phi$: the domain gets restricted to situations where $\phi$ is true.

Compare the effect of the announcement: “I do not have white.”
The Emergence of Common Knowledge

David Lewis    Robert Aumann
Computing the Common Knowledge Relation
Computing the Common Knowledge Relation

1 —— 2 —— 3 —— 4 —— 5
Computing the Common Knowledge Relation
Computing the Common Knowledge Relation
Computing the Common Knowledge Relation
Common Knowledge: Definition

φ is common knowledge if everyone knows that φ and, moreover, everyone knows that φ is common knowledge.

\[ C\varphi \leftrightarrow (E\varphi \land EC\varphi). \]

Compare:

zeros = 0 : zeros
Cashiers, ATMs, and the Creation of Common Knowledge
Epistemic Model Checking of Muddy Children

- mu0: model where the children cannot see each other.
- mu1: model where the children can see each other.
- mu2: model after public announcement “at least one of you is muddy.”
- mu3: model after public announcement “no-one knows their state.”
- mu4: model after public announcement “no-one knows their state.”
- mu5: model after public announcement “b, c, d know their state.”
Epistemic Model Checking of Muddy Children

- mu0: model where the children cannot see each other.
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- mu5: model after public announcement “b, c, d know their state.”

http://homepages.cwi.nl/~jve/software/demolight
Effect of Public Announcement

\[ \begin{align*}
wrb & \quad rwb \\
& \quad bwr \quad brw \\
& \quad wbr \quad rbw
\end{align*} \]
Alice says “I hold the red card” privately to Bob.

Carol cannot distinguish this from the action where nothing happens.
Effect of This

Compute the result with a model product construction (Baltag cs., [1]):
Sending Email Messages

“Wouter Bos email”: message where all can see the recipient list. This is like a public announcement.

Private message $\phi$ to agent $i$: all other agents cannot distinguish this from the action where nothing happens:
A Riddle and A Protocol
The set of prisoners is \{0, \ldots, n - 1\}, with \(n \geq 2\).

The prisoners appoint one among them as the counter. We will assume prisoner 0 is appointed as counter.

All prisoners except the counter act as follows: the first time they enter the room when the light is off, they switch it on; on all next occasions, they do nothing.

The counter acts as follows: The first \(n - 2\) times that the light is on when he enters the interrogation room, he turns it off. Then the next time he enters the room when the light is on, he announces that everybody has been interrogated.

This protocol is proved correct in [2].
References
