On the Logic of Lying

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Abstract

We look at lying as an act of communication, where (i) the proposition that is communicated is not true, (ii) the utterer of the lie knows that what she communicates is not true, and (iii) the utterer of the lie intends the lie to be taken as truth. Rather than dwell on the moral issues, we provide a sketch of what goes on logically when a lie is communicated. We present a complete logic of manipulative updating, to analyse the effects of lying in public discourse. Next, we turn to the study of lying in games. First, a game-theoretical analysis is used to explain how the possibility of lying makes such games interesting, and how lying is put to use in optimal strategies for playing the game. Finally, we give a matching logical analysis. Our running example of lying in games in liar's dice.

Open the window that the lies may fly out

I will tell you something. I saw two roasted fowls flying; they flew quickly and had their breasts turned to Heaven and their backs to Hell; and an anvil and a mill-stone swam across the Rhine prettily, slowly, and gently; and a frog sat on the ice at Whitsuntide and ate a ploughshare.

Four fellows who wanted to catch a hare, went on crutches and stilts; one of them was deaf, the second blind, the third dumb, and the fourth could not stir a step. Do you want to know how it was done? First, the blind man saw the hare running across the field, the dumb one called to the deaf one, and the lame one seized it by the neck.

There were certain men who wished to sail on dry land, and they set their sails in the wind, and sailed away over great fields. Then they sailed over a high mountain, and there they were miserably drowned.

A crab was chasing a hare which was running away at full speed; and high up on the roof lay a cow which had climbed up there. In that country the flies are as big as the goats are here.

Open the window that the lies may fly out.

Jacob Ludwig Grimm and Wilhelm Carl Grimm, *The Ditmarsch Tale of Wonders* [Gri44].

1 What is a Lie?

Whatever a lie is, the text on René Magritte's famous picture is not an example.

Magritte himself has tried to explain why this is not a lie:

La fameuse pipe, me l'a-t-on assez reprochée! Et pourtant, pouvez-vous la bourrer ma pipe? Non, n'est-ce pas, elle n'est qu'une représentation. Donc si j'avais écrit sous mon tableau 'ceci est une pipe', j'aurais menti!

[This famous pipe, how have I been reproached for it! But you cannot stuff it with tobacco, can you? No, for it is just an image. So had I written under my picture 'This is a pipe' I would have lied.]

In this quote, Magritte equates lying with saying things that are not true, and asserts that he—on this particular occasion—was not guilty of that.

The church father St. Augustine, who wrote at length about lying in *De Mendacio* [Aug88], holds a more subtle view on what is a lie and on what it is not.

For not every one who says a false thing lies, if he believes or opines that to be true which he says. Now between believing and opining there is this difference, that sometimes he who believes feels that he does not know that which he believes, (although he may know himself to be ignorant of a thing, and yet have no doubt at all concerning it, if he most firmly believes it:) whereas he who opines, thinks he knows that which he does not know. Now whoever utters that which he holds in his mind either as belief or as opinion, even though it be false, he lies not. For this he owes to the faith of his utterance, that he thereby produce that which he holds in his mind, and has in that way in which he produces it. Not that he is without fault, although he lie not, if either he believes what he ought not to believe, or thinks he knows what he knows not, even though it should be true: for he accounts an unknown thing for a known.

Augustine, *De Mendacio* (On Lying), ca. AD 395 [Aug88]

And on what lying is:

Wherefore, that man lies, who has one thing in his mind and utters another in words, or by signs of whatever kind. Whence also the heart of him who lies is said to be double; that is, there is a double thought: the one, of that thing which he either knows or thinks to be true and does not produce; the other, of that thing which he produces instead thereof, knowing or thinking it to be false. Whence it comes to pass, that he may say a false thing and yet not lie, if he thinks it to be so as he says although it be not so; and, that he may say a true thing, and yet lie, if he thinks it to be false and utters it for true, although in reality it be so as he utters it. For from the sense of his own mind, not from the verity or falsity of the things themselves, is he to be judged to lie or not to lie. Therefore he who utters a false thing for a true, which however he opines to be true, may be called erring and rash: but he is not rightly said to lie; because he has not a double heart when he utters it, neither does he wish to deceive, but is deceived. But the fault of him who lies, is, the desire of deceiving in the uttering of his mind; whether he do deceive, in that he is believed when uttering the false thing; or whether he do not deceive, either in that he is not believed, or in that he utters a true thing with will to deceive, which he does

not think to be true: wherein being believed, he does not deceive though it was his will to deceive: except that he deceives in so far as he is thought to know or think as he utters.

Augustine, [Aug88]

We cannot do better than to follow Augustine in assuming that intent to mislead is part of the definition of a liar. Thus, to us, lying about p is communicating p in the knowledge that $\neg p$ is the case, with the intent to be believed.

For the moral sides to the issue of lying, see the ninth of the ten commandments ('Thou shalt not bear false witness'), the fourth of the five Buddhist precepts ('I undertake the precept to refrain from false speech'), and, on the other hand, the *Analects* of Confucius, where Confucius is quoted as condoning a lie, if its purpose is to preserve social structure:

The Governor of She said to Confucius, 'In our village we have an example of a straight person. When the father stole a sheep, the son gave evidence against him.' Confucius answered, 'In our village those who are straight are quite different. Fathers cover up for their sons, and sons cover up for their fathers. In such behaviour is straightness to be found as a matter of course.' *Analects, 13.18.*

Among philosophical treatises, the quoted text of Augustine is a classic.[1] For more, see [Bok78] and [Are06] and the references therein.

Rather than dwell on the moral side of the issue of lying, in this paper we will study its logic, focussing on simple cases of lying in game situations, and on a particular kind of public announcement that may or may not be deceptive (we call these 'manipulative updates'). Thus, we abstract from the moral issues; we feel that it is important to understand why lying is tempting (why and how it pays off) before addressing the choice between condemnation and absolution.

2   The Logic of Communication and Change

The logic of communication and change presented in [BvEK06] provides means to model communicative actions and actions that change the world, and their effects in given epistemic situations. In this section we introduce

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[1]We found a site on the internet where the views of the writer *De Mendacio* on lying are expounded.

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the syntax and semantics of the logic. In the next sections we show how this machinery can be put to use to analyse manipulation in public discourse and to describe what goes on in the game of Liar's Dice.

Definition 1 (Epistemic models) *Let a set of propositional variables P and a finite set of agents Ag be given. An epistemic model is a triple $M = (W, V, R)$ where W is a set of worlds, $V : W \to \mathcal{P}(P)$ assigns a valuation to each world $w \in W$, and $R : \mathrm{Ag} \to \mathcal{P}(W^2)$ assigns an accessibility relation $\xrightarrow{i}$ to each agent $i \in \mathrm{Ag}$.*

A pair $\mathbf{M} = (M, U)$ with $U \subseteq W$ is a multiple pointed epistemic model, indicating that the actual world is among U.

[BMS03] proposes to model epistemic actions as epistemic models, with valuations replaced by preconditions. (See also: [Bal02; BM04; BMS99; Ben01; Ben02; Dit00; FHMV95; Ger99; Koo03].) [BvEK06] proposes to add substitutions for modelling change in the world (this proposal is based on [Eij04]). See also [DvdHK05]. The story of update logic with substitutions is retold for a fragment in [Koo07].

Definition 2 (Substitutions) *$\mathcal{LANG}$ substitutions are functions of type $\mathcal{LANG} \to \mathcal{LANG}$ that distribute over all language constructs, and that map all but a finite number of basic propositions to themselves. $\mathcal{LANG}$ substitutions can be represented as sets of bindings*

$$\{p_1 \mapsto \phi_1, \ldots, p_n \mapsto \phi_n\}$$

where all the p_i are different . If σ is a $\mathcal{LANG}$ substitution, then the set $\{p \in P \mid \sigma(p) \neq p\}$ is called its domain, notation $dom(\sigma)$. Use ϵ for the identity substitution. Let $\mathbf{Sub}_{\mathcal{LANG}}$ be the set of all $\mathcal{LANG}$ substitutions.

Definition 3 (Epistemic Models under a Substitution) *If*

$$M = (W, V, R)$$

is an epistemic model and σ is a $\mathcal{LANG}$ substitution (for an appropriate epistemic language $\mathcal{LANG}$), then V_M^σ is the valuation given by $\lambda p \cdot [\![\sigma(p)]\!]^M$. In other words, V_M^σ assigns to w the set of worlds w in which $\sigma(p)$ is true. For $M = (W, V, R)$, call M^σ the model given by (W, V_M^σ, R).

Definition 4 (Action models for a given language $\mathcal{L}$) *Let a finite set of agents Ag and an epistemic language $\mathcal{L}$ be given. An action model for $\mathcal{L}$*

*is a quadruple $A = (W, pre, sub, R)$ where W is a set of action states, $pre :$
$W \to \mathcal{L}$ assigns a precondition to each action state, $sub : W \to \mathbf{Sub}_{\mathcal{LANG}}$
assigns a substitution to each action state, and $R : \mathrm{Ag} \to \mathcal{P}(W^2)$ assigns an
accessibility relation $\xrightarrow{i}$ to each agent $i \in \mathrm{Ag}$.*

*A pair $\mathbf{A} = (A, S)$ with $S \subseteq W$ is a multiple-pointed action model,
indicating that the actual action that takes place is a member of S.*

The epistemic language $\mathcal{LANG}$ is defined as follows.

Definition 5 ($\mathcal{LANG}$) *Assume p ranges over set of basic propositions P,
i ranges over the set of agents Ag. The formulas of $\mathcal{LANG}$ are given by:*

$$
\begin{aligned}
\phi &::= \quad \top \mid p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid [\alpha]\phi \mid \ [A,S]\phi, \\
\alpha &::= \quad i \mid ?\phi \mid \alpha_1 \cup \alpha_2 \mid \alpha_1 ; \alpha_2 \mid \alpha^*,
\end{aligned}
$$

where (A, S) is a multiple pointed finite $\mathcal{LANG}$ (action) model.

We employ the usual abbreviations. In particular, $\phi_1 \vee \phi_2$ is shorthand
for $\neg(\neg\phi_1 \wedge \neg\phi_2)$, $\phi_1 \to \phi_2$ for $\neg(\phi_1 \wedge \neg\phi_2)$, $\langle\alpha\rangle\phi$ for $\neg[\alpha]\neg\phi$, $\langle A,S\rangle\phi$ for
$\neg[A,S]\neg\phi$. Note that the standard epistemic language is a sublanguage
of $\mathcal{LANG}$, with "$[i]\phi$" and "$[(\cup_{i\in\mathrm{Ag}}i)^*]\phi$" interpreted as "$K_i\phi$" and "$C\phi$",
respectively.[2]

Let MOD be the class of multiple pointed epistemic models and ACT
the class of multiple pointed finite $\mathcal{LANG}$ models. Then $\mathcal{LANG}$-update is
an operation of the following type:

$$\otimes : \mathrm{MOD} \times \mathrm{ACT} \to \mathrm{MOD}.$$

The operation $\otimes$ and the truth definition for $\mathcal{LANG}$ are defined by mutual
recursion, as follows.

Definition 6 (Update, Truth) *Given a multiple pointed epistemic model
(M, U) and an action model (A, S), we define*

$$(M, U) \otimes (A, S)$$

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[2]The reason to employ *multiple* pointed models for updating is that it allows us to
handle choice. Suppose we want to model the action of testing whether ϕ, followed by
a public announcement of the result. This involves *choice*: if the outcome of the test
is affirmative, then do this, else do that. Choice is modelled in a straightforward way
in multiple pointed action models. Once we allow multiple pointed action models, it is
reasonable to also take our epistemic models to be multiple pointed, with the multiple
points constraining the whereabouts of the actual world.

as

$$((W', V', R'), U'),$$

where

$$
\begin{aligned}
W' :=& \quad \{(w,s) \mid \quad w \in W_M, s \in W_A, M \models_w pre_s\}, \\
V'(w,s) :=& \qquad\qquad \{p \in P \mid M \models_w sub_a(p), \\
(w,s) \xrightarrow{i} (w',s') \in R' :\equiv& \qquad\qquad w \xrightarrow{i} w' \in R_M \ and \ s \xrightarrow{i} s' \in R_A, \\
U' :=& \quad \{(u,s) \mid \quad u \in U, s \in S, (u,s) \in W'\},
\end{aligned}
$$

and where the truth definition is given by:

$$
\begin{aligned}
M \models_w \top & \qquad always \\
M \models_w p \ & :\equiv \ p \in V_M(w) \\
M \models_w \neg\phi \ & :\equiv \ not \ M \models_w \phi \\
M \models_w \phi_1 \wedge \phi_2 \ & :\equiv \ M \models_w \phi_1 \ and \ M \models_w \phi_2 \\
M \models_w [\alpha]\phi \ & :\equiv \ for \ all \ w' \ with \ w \xrightarrow{\alpha} w' \ M \models_{w'} \phi \\
M \models_w [A,S]\phi \ & :\equiv \ (W',V',R') \models_{(w,s)} \phi \ for \ all \ (w,s) \in U', \\
& \qquad where \ ((W',V',R'),U') = (M,\{w\}) \otimes (A,S).
\end{aligned}
$$

In this definition $\xrightarrow{\alpha_1 \cup \alpha_2}$ is the relation $\xrightarrow{\alpha_1} \cup \xrightarrow{\alpha_2}$, $\xrightarrow{\alpha_1;\alpha_2}$ is

$$\{(x,y) \mid \exists z (x \xrightarrow{\alpha_1} z) \& (z \xrightarrow{\alpha_2} y)\},$$

and $\xrightarrow{\alpha^*}$ is the reflexive transitive closure of $\xrightarrow{\alpha}$.

In [BvEK06] it is shown that this logic is axiomatized by the axioms and inference rules of epistemic PDL, plus a set of reduction axioms that are generated from the action models by means of a process of program transformation. In the next section we will use (appropriate fragments of) this logical system to model what goes on in manipulative communication, and in Section 5 we will employ it to analyze what goes on in Liar's Dice.

3 The Logic of Lying in Public Discourse

We get lied to in the public domain, all the time, by people who have an interest in obfuscating the truth. In 1993 the tobacco company Philip Morris tried to discredit a report on *Respiratory Health Effects of Passive Smoking* by founding, through a hired intermediary, a fake citizen's group called *The Advancement of Sound Science* or TASSC, to cast doubt on it.

Exxon-Mobile used the same organisation to spread disinformation about global warming.[3] Their main ploy: hang the label of 'junk science' on peer-reviewed scientific papers on smoking hazards or global warming, and promote propaganda disguised as research as 'sound science'. It worked beautifully for a while, until the *New York Times* exposed the fraud [Mon98]. As a result, many educated people who should know better are still in doubt about the reality of global warming, or think the issues are just too hard for them to understand.

> It has frequently been noted that the surest result of brainwashing in the long run is a peculiar kind of cynicism, the absolute refusal to believe in the truth of anything, no matter how well it may be established. In other words, the result of a consistent and total substitution of lies for factual truth is not that the lie will now be accepted as truth, and truth be defamed as lie, but that the sense by which we take our bearings in the real world—and the category of truth versus falsehood is among the mental means to this end—is being destroyed.

> Hannah Arendt, "Truth and Politics", 1967 [Are06].

Below we will give a logic of possibly deceptive public speech acts, to model the effects of lying in politics. Our modelling shows that Arendt is right about the grave consequences of lying in politics.

First, take the protypical example of lying about p. Picture an initial situation where agent a knows that p, and agent a knows that that agents b, c do not know that p. One way to picture this initial situation is like this:

[3]See http://www.exxonsecrets.org/html/orgfactsheet.php?id=6.
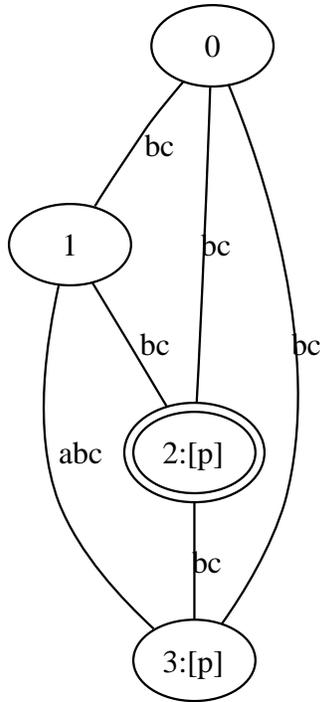
Here the double oval indicates that 1 is the actual world. Note that agent a knows that p, but agents b, c also consider it possible that agent a believes $\neg p$, or that a does not know whether p.

In typical examples of bearing witness in court, the situation is often a bit different. In cases of providing an alibi, for example, the question 'Was the accused at home with you during the evening of June 6th?' is posed on the understanding that the witness is in a position to know the true answer, even if nobody can check that she is telling the truth.

Let us assume that everyone knows that a knows whether p. The picture now becomes:
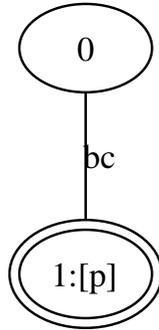
Assume agent a sends a group communication to b, c to the effect that $\neg p$. Would this be a correct communication model for the lie that $\neg p$?
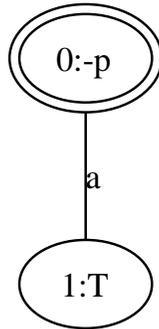

It is easy to see that this cannot be right. The result of this update is a model that has no actual worlds, i.e., an inconsistent model. The actual worlds of an update are pairs (w, a) where w is an actual world of the input epistemic model and a an actual action of the update model, and w satisfies the precondition of a. Since the actual world has p true, and the precondition of the actual action is $\neg p$, there are no such pairs.

Rather, the misleading communication should be modelled as a KD45 action model, as follows:
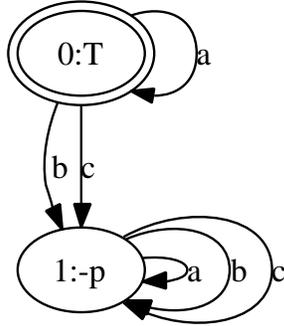
The misleading agent a knows that no truthful communication is being made, but the two agents b, c mistakenly believe that $\neg p$ is being truthfully asserted. Updating the initial model with this gives:
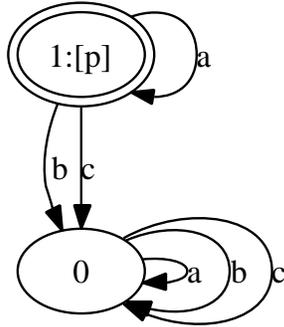

This is a model where a knows that p, where b, c mistakenly believe that $\neg p$, and where b, c also believe that a believes (knows) that $\neg p$. Note that the model is KD45: beliefs are still consistent ($[i]\phi \rightarrow \langle i \rangle \phi$ holds in the model), but the model is not truthful anymore (there are ϕ and i for which $[i]\phi \rightarrow \phi$ does not hold). The postulate of truthfulness has been replaced by the weaker postulate of consistency (the D postulate $[i]\phi \rightarrow \langle i \rangle \phi$).

This way to model lying suggests a natural generalization of the well-studied concept of a public announcement. In the logic of public announcements [Pla89; Ger99], a public announcement $!\phi$ is always taken to be a *true* statement. Unfortunately, this is not completely realistic.

A more realistic version of public announcements leaves open the possibility of deceit, as follows. A possibly deceptive public announcement ϕ is a kind of 'if then else' action. In case ϕ is true, the announcement is a public update with ϕ, in case ϕ is false, the public is deceived into taking ϕ as true. The manipulative update with p, in a setting where the public consists of a, b, c, looks like this:
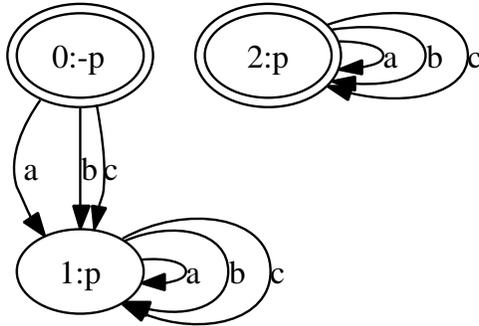
There are two actual actions, one for the situation where p is true — in this case, the public is duly informed — and one for the situation where p is false — in this case the public is misled to believe that p. This action model can be simplified, as follows:
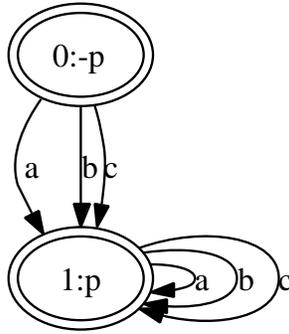
Call this the two-pointed manipulative update for ϕ. We will refer to this action model as U_ϕ. The variation on this action model where only action 0 is actual will be referred to as U_ϕ^0. This action model denotes the lie with ϕ. The variant with only action 1 actual will be referred to as U_ϕ^1. This action models denotes the public announcement with ϕ.

Let us introduce operations for these actions. The manipulative update with ϕ is denoted $\ddagger\phi$, and its two variants $\textrm{¡}\phi$ (for the lie that ϕ) and $!\phi$ (for the public announcement that ϕ).

Now it turns out that the logic of individual belief and manipulative update, has a simple axiomatisation in terms of reduction axioms, just like the logic of individual knowledge and public announcement.

$$\phi \quad ::= \quad p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid B_i\phi \mid [\ddagger\phi_1]\phi_2 \mid [\textrm{¡}\phi_1]\phi_2 \mid [!\phi_1]\phi_2$$

Interpretation as sketched above:

- $[\ddagger\phi]\psi$ is true in a model M at a world w if ψ is true in both $(w,0)$ and $(w,1)$ of updated model $M \otimes U$.

- $[\textrm{¡}\phi]\psi$ is true in a model M at a world w if ψ is true in $(w,0)$ of updated model $M \otimes U^0$.

- $[!\phi]\psi$ is true in a model M at a world w if ψ is true in $(w,1)$ of updated model $M \otimes U^1$.

Complete axiomatisation: the usual KD45 axioms for B_i, modus ponens, necessitation for B_i, $\ddagger\phi$, $\textrm{¡}\phi$ and $!\phi$, and reduction axioms for the $[\ddagger\phi]$, $[\textrm{¡}\phi]$, $[!\phi]$ modalities:

$$[\ddagger\phi]\psi \quad\leftrightarrow\quad [\textrm{¡}\phi]\psi \wedge [!\phi]\psi$$

This defines the effect of $[\ddagger\phi]$ in terms of those of $[!\phi]$ and $[\textrm{¡}\phi]$. Next, we have the usual reduction axioms for public announcement:

$$
\begin{aligned}
{[!\phi]}p &\leftrightarrow \phi \to p \\
{[!\phi]}\neg\psi &\leftrightarrow \phi \to \neg[!\phi]\psi \\
{[!\phi]}(\psi_1 \wedge \psi_2) &\leftrightarrow [!\phi]\psi_1 \wedge [!\phi]\psi_2 \\
{[!\phi]}B_i\psi &\leftrightarrow \phi \to B_i[!\phi]\psi
\end{aligned}
$$

Finally, the reduction axioms for lying:

$$
\begin{aligned}
{[\textrm{¡}\phi]}p &\leftrightarrow \neg\phi \to p \\
{[\textrm{¡}\phi]}\neg\psi &\leftrightarrow \neg\phi \to \neg[\textrm{¡}\phi]\psi \\
{[\textrm{¡}\phi]}(\psi_1 \wedge \psi_2) &\leftrightarrow [\textrm{¡}\phi]\psi_1 \wedge [\textrm{¡}\phi]\psi_2 \\
{[\textrm{¡}\phi]}B_i\psi &\leftrightarrow \neg\phi \to B_i[!\phi]\psi
\end{aligned}
$$

The final axiom of this list is the most interesting: it expresses that believing ψ after a lie that ϕ amounts to the belief that a public announcement of ϕ implies ψ, conditioned by $\neg\phi$.

Since all these axioms have the form of equivalences, completeness of the calculus of manipulation and individual belief follows from a reduction argument, as in the case of public announcements with individual knowledge:

Theorem 7 *The calculus of manipulation and individual belief is complete.*

Another way to see that the logic is complete is by means of the observation that this is the special case of the LCC logic where updates are restricted to

manipulations, announcements and lies, and where epistemic programs are restricted to individual accessibilities. Then apply the equivalence between $[A, s]K_i\phi$ and $\mathrm{pre}(A, s) \rightarrow \bigwedge_{\{t|s\overset{i}{\rightarrow}t\}} K_i[A, t]\phi)$.

Interestingly, our logic of manipulation is closely related to the variation on public announcement that is used in [Ger07] to analyse the 'surprise exam puzzle', where public announcement of ϕ is defined as an operation that restricts the epistemic alternatives of the agents to the worlds where ϕ is true, i.e., M^ϕ is the model where each R_i gets replaced by R_i^ϕ given by $R_i^\phi(w) = [\![\phi]\!] \cap R_i(w)$. Using $\dagger\phi$ for this alternative announcement, the corresponding reduction axiom is $[\dagger\phi]B_i\psi \leftrightarrow B_i(\phi \rightarrow [\dagger\phi]\psi)$. This same definition and reduction axiom can be found in [Koo07], but neither author cites the other on this.

At first sight, this alternative semantics for announcement takes us outside of the framework sketched in Section 2 above. However, if $\dagger\phi$ is alternative announcement, then we have:

Proposition 8 $M, w \models [\dagger\phi]\psi$ *iff* $M, w \models [\ddagger\phi]\psi$.

Alternative announcement turns out to be the same as manipulative updating, and our analysis can be viewed as a decomposition of alternative announcement into public lying and (regular) public announcement.

Regular public announcements can be expressed in terms of manipulative updating:

Proposition 9 $\vdash [!\phi]\psi \leftrightarrow (\phi \rightarrow [\ddagger\phi]\psi)$.

The proof is by induction on ψ.

It is the case that the logic of public announcement and the logic of manipulation have the same expressive power: this follows from the fact that they both reduce to multi-modal KD45. But note that the logic of manipulative updating has greater 'action expressivity' than the logic of public announcement: the logic of $[!\phi]$ has no means to express an operation mapping S5 models to KD45 models, and $[\ddagger\phi]$ is such an operation.

As an example of reasoning with the calculus, we use the axioms to show that a manipulative update followed by a belief is equivalent to a belief followed by the corresponding public announcement:

Proposition 10 $\vdash [\ddagger\phi]B_i\psi \leftrightarrow B_i[!\phi]\psi$.

Proof.

$$\begin{aligned}
[\ddagger\phi]B_i\psi \quad &\leftrightarrow \quad [\mathord{\text{\textexclamdown}}\phi]B_i\psi \wedge [!\phi]B_i\psi \\
&\leftrightarrow \quad (\neg\phi \to B_i[!\phi]\psi) \wedge (\phi \to B_i[!\phi]\psi) \\
&\leftrightarrow \quad B_i[!\phi]\psi
\end{aligned}$$

$\square$

An important difference between manipulative update and public announcement shows up when we work out the preconditions of inconsistency after an update. For announcements we get:

Proposition 11 $\vdash [!\phi]\bot \leftrightarrow \neg\phi$.

Proof.

$$\begin{aligned}
[!\phi]\bot \quad &\leftrightarrow \quad [!\phi](p \wedge \neg p) \\
&\leftrightarrow \quad [!\phi]p \wedge [!\phi]\neg p \\
&\leftrightarrow \quad [!\phi]p \wedge (\phi \to \neg[!\phi]p \\
&\leftrightarrow \quad (\phi \to p) \wedge (\phi \to \neg p) \\
&\leftrightarrow \quad \neg\phi
\end{aligned}$$

$\square$

This shows that a public announcement with ϕ leads to an inconsistent state iff the negation of ϕ is true. Similarly, it is easy to work out that a public lie that ϕ leads to an inconsistency iff ϕ is true, i.e., we can derive

Proposition 12 $\vdash [\mathord{\text{\textexclamdown}}\phi]\bot \leftrightarrow \phi$.

Using this we can work out the preconditions for inconsistency after a manipulative update:

Proposition 13 $\vdash [\ddagger\phi]\bot \leftrightarrow \bot$.

Proof.

$$\begin{aligned}
[\ddagger\phi]\bot \quad &\leftrightarrow \quad [!\phi]\bot \wedge [\mathord{\text{\textexclamdown}}\phi]\bot \\
&\overset{\text{Prop 12}}{\leftrightarrow} \quad \neg\phi \wedge \phi \\
&\leftrightarrow \quad \bot
\end{aligned}$$

$\square$

This means that a manipulative update in a consistent state will *never* lead to inconsistency.

The following proposition about public announcements can be proved by induction on ϕ:

Proposition 14 $\vdash [!\bot]\phi \leftrightarrow \top$.

In the case of manipulatively updating with an inconsistency, the result is not an inconsistent model, but a model where all accessibilities have vanished. In the particular case of an i-belief, we get:

Proposition 15 $\vdash [\ddagger\bot]B_i\phi \leftrightarrow \top$.

Proof.

$$
\begin{aligned}
[\ddagger\bot]B_i\phi \quad &\leftrightarrow \quad [!\bot]B_i\phi \wedge [\text{¡}\bot]B_i\phi \\
&\leftrightarrow \quad \top \wedge B_i[!\bot]\phi \\
&\leftrightarrow \quad B_i[!\bot]\phi \\
&\overset{\text{Prop 14}}{\leftrightarrow} \quad B_i\top \\
&\leftrightarrow \quad \top
\end{aligned}
$$

$\square$

After an a manipulative update with an inconsistency, the public will no longer be able to distinguish what is false from what is true.

Finally, the following proposition spells out under what conditions our 'sense by which we take our bearings in the real world' is destroyed. This happens exactly when we are manipulated into acccepting as truth what flatly contradicts our firm belief:

Proposition 16 $\vdash [\ddagger\phi]B_i\bot \leftrightarrow B_i\neg\phi$.

Proof.

$$
\begin{aligned}
[\ddagger\phi]B_i\bot \quad &\leftrightarrow \quad [!\phi]B_i\bot \wedge [\text{¡}\phi]B_i\bot \\
&\leftrightarrow \quad (\phi \rightarrow B_i[!\phi]\bot) \wedge (\neg\phi \rightarrow B_i[!\phi]\bot) \\
&\leftrightarrow \quad (\phi \rightarrow B_i\neg\phi) \wedge (\neg\phi \rightarrow B_i\neg\phi) \\
&\leftrightarrow \quad B_i\neg\phi
\end{aligned}
$$

$\square$

The full logic of manipulative updating, to which we now turn, consists of epistemic PDL extended with manipulative updates, lies and announcements:

$$\alpha \quad ::= \quad i \mid ?\phi \mid \alpha_1; \alpha_2 \mid \alpha_1 \cup \alpha_2 \mid \alpha^*$$
$$\phi \quad ::= \quad p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid [\alpha]\phi \mid [\ddagger\phi_1]\phi_2 \mid [¡\phi_1]\phi_2 \mid [!\phi_1]\phi_2$$

Again, there is a complete axiomatisation: the axioms and rules of PDL, necessitation for $[\ddagger\phi]$, $[¡\phi]$, $[!\phi]$, and the following reduction axioms for the three update modalities.

The definition of $\ddagger$ in terms of $¡$ and $!$ is as before:

$$[\ddagger\phi]\psi \quad \leftrightarrow \quad [¡\phi]\psi \wedge [!\phi]\psi$$

Reduction axioms for public announcement are as follows:

$$
\begin{aligned}
[!\phi]p &\leftrightarrow \phi \rightarrow p \\
[!\phi]\neg\psi &\leftrightarrow \phi \rightarrow \neg[!\phi]\psi \\
[!\phi](\psi_1 \wedge \psi_2) &\leftrightarrow [!\phi]\psi_1 \wedge [!\phi]\psi_2 \\
[!\phi][i]\psi &\leftrightarrow [?\phi; i][!\phi]\psi \\
[!\phi][?\chi]\psi &\leftrightarrow [?\phi; ?\chi][!\phi]\psi \\
[!\phi][\alpha_1; \alpha_2]\psi &\leftrightarrow [!\phi][\alpha_1][\alpha_2]\psi \\
[!\phi][\alpha_1 \cup \alpha_2]\psi &\leftrightarrow [!\phi]([\alpha_1]\psi \wedge [\alpha_2]\psi) \\
[!\phi][\alpha^*]\psi &\leftrightarrow [\alpha'^*][!\phi]\psi
\end{aligned}
$$

$$\text{where } \alpha' \text{ such that } [!\phi][\alpha]\psi \leftrightarrow [\alpha'][!\phi]\psi$$

It can be shown by an inductive argument that for every epistemic program α, every announcement $!\phi$, and every postcondition ψ an epistemic program α' exists such that $[!\phi][\alpha]\psi \leftrightarrow [\alpha'][!\phi]\psi$,

Reduction axioms for public lies:

$$
\begin{aligned}
[¡\phi]p &\leftrightarrow \neg\phi \rightarrow p \\
[¡\phi]\neg\psi &\leftrightarrow \neg\phi \rightarrow \neg[¡\phi]\psi \\
[¡\phi](\psi_1 \wedge \psi_2) &\leftrightarrow [¡\phi]\psi_1 \wedge [¡\phi]\psi_2 \\
[¡\phi][i]\psi &\leftrightarrow [?\neg\phi; i][!\phi]\psi \\
[¡\phi][?\chi]\psi &\leftrightarrow [?\neg\phi; ?\chi][!\phi]\psi \\
[¡\phi][\alpha_1; \alpha_2]\psi &\leftrightarrow [¡\phi][\alpha_1][\alpha_2]\psi \\
[¡\phi][\alpha_1 \cup \alpha_2]\psi &\leftrightarrow [¡\phi]([\alpha_1]\psi \wedge [\alpha_2]\psi) \\
[¡\phi][\alpha^*]\psi &\leftrightarrow [\alpha'; \alpha''^*][!\phi]\psi
\end{aligned}
$$

$$\text{where } \alpha' \text{ such that } [¡\phi][\alpha]\psi \leftrightarrow [\alpha'][!\phi]\psi$$
$$\text{and } \alpha'' \text{ such that } [!\phi][\alpha]\psi \leftrightarrow [\alpha''][!\phi]\psi$$

Again, it can be shown by an inductive argument that for every epistemic program α, every lie $¡\phi$, and every postcondition ψ, an epistemic programs α' exists such that $[¡\phi][\alpha]\psi \leftrightarrow [\alpha'][!\phi]\psi$.

The α' and α'' in the axioms for α^* can be viewed as the transformed versions of the programs α, where the update operator acts as an epistemic program transformer. To give an example, suppose $\alpha = i \cup j$, and we want to calculate the way common belief of i and j is transformed by a public lie that ϕ. Then the transformed program for $i \cup j$ becomes $?\neg\phi; i \cup j$, i.e., we have:

$$[¡\phi][i \cup j]\psi \leftrightarrow [?\neg\phi; i \cup j][!\phi]\psi.$$

Similarly for the way common belief of i and j is transformed by a public announcement: the transformed program for $i \cup j$ becomes $?\phi; i \cup j$, and we have:

$$[!\phi][i \cup j]\psi \leftrightarrow [?\phi; i \cup j][!\phi]\psi.$$

Using these transformed programs, we see that the reduction axiom for $(i \cup j)^*$ takes the shape:

$$[¡\phi][(i \cup j)^*]\psi \leftrightarrow [?\neg\phi; i \cup j; (?\phi; i \cup j)^*][!\phi]\psi.$$

This expresses that after a lie with ϕ, i and j have a common belief that ψ iff in the model before the lie it holds that along all $i \cup j$ paths that start from a $\neg\phi$ world and that pass only through ϕ worlds, $[!\phi]\psi$ is true. Note that this is a 'relativized common belief' similar to the relativized common knowledge that is needed to get a reduction style analysis going of public announcement in the presence of common knowledge.

In fact, the style of axiomatisation that we have adopted is borrowed from the reduction axioms formulated in terms of program transformations, in [BvEK06]. In the same manner as in [BvEK06] we can derive:

Theorem 17 *The calculus of manipulative updating is complete.*

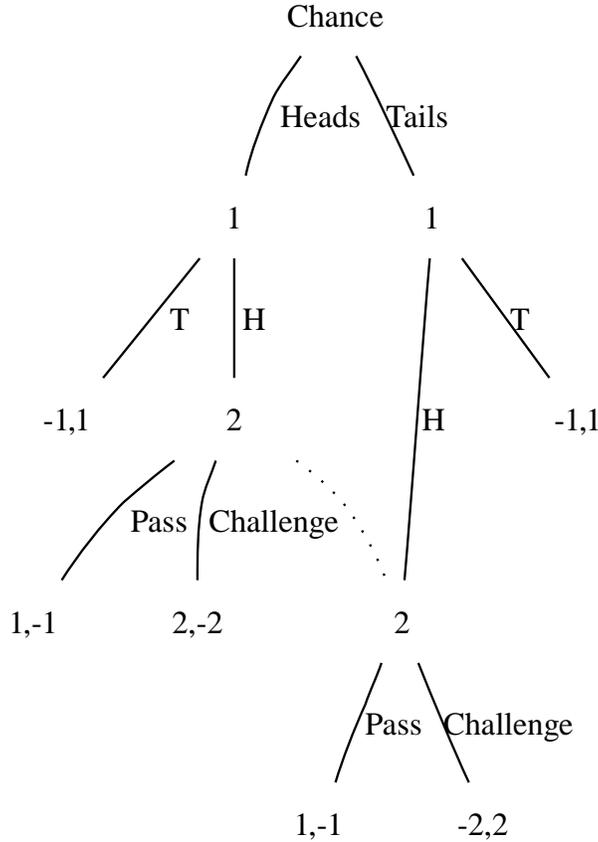# 4   Liar's Dice — Game Theoretical Analysis

In his later years as a saint, Augustine held the opinion that lying, even in jest, is wrong, but as the young and playful sinner that he was before his turn to seriousness he may well have enjoyed an occasional game of dice. We will examine a simplified version of two-person Liar's Dice, and show by means of a game-theoretical analysis that it is precisely the possibility of

lying — using private information in order to mislead an opponent — that makes the game interesting.

In our simplified version of Liar's Dice, the die is replaced by a coin. A typical move of the game is tossing a coin and inspecting the result while keeping it hidden from the other player. Here is a description of what goes on, and what the options of the two players are.

- Players 1 and 2 both stake one euro: Player 1 bets on heads, Player 2 bets on tails.

- Player 1 tosses a coin and observes the outcome (heads or tails), while keeping it concealed from player 2.

- Player 1 *announces* either $\ddagger Head$ or $\ddagger Tail$.

- If 1 announces $\ddagger Tail$, she then simply loses her one euro to player 2 and game ends.

- If 1 announces $\ddagger Head$, she adds one euro to the stake and game continues.

- In response to $\ddagger Head$, 2 either *passes* (gives up) or *challenges* "I don't believe that, you liar") and adds 1 euro to the stake.

- If 2 passes, 1 wins the stake, and the game ends.

- If 2 challenges, and the toss was heads, 1 wins the stake, otherwise 2 wins the stake. The game ends.

Player 1 has two information states: *Heads* and *Tails*, while player 2 has a single information state, for player 2 cannot distinguish the two possible outcomes of the toss. We will give a game theoretic analysis of how player 1 can exploit her 'information advantage' to the utmost, and of how player 2 can react to minimize her losses, on the assumption that the procedure is repeated a large number of times. The following picture gives the extensive game form. The first move is made by Chance; this move gives the outcome of the coin toss. Then player 1 reacts, letting her move depend on the toss outcome. Finally, player 2 decides whether to pass or challenge. This decision does not depend on the coin toss; player 2 cannot distinguish the state where 1 announced $\ddagger Head$ on seeing heads from the state where she is bluffing. In the picture of the extensive game form, this is expressed by a dotted line.

Chance

```
          Heads    Tails
            1              1
        T   │ H              T
      -1,1   2          H      -1,1
          Pass │ Challenge
      1,-1    2,-2         2
                      Pass │ Challenge
                      1,-1     -2,2
```


The leafs of the game tree indicate the payoffs. If the game sequence is *Heads*, ‡*Tail*, the payoffs are −1 euro for player 1 and 1 euro for player 2. The same for the sequence *Tails*, ‡*Tail*. Player 1 gets 1 euro and player 2 gets −1 euro for the sequences *Heads*, ‡*Head*, *Pass*, and *Tail*, ‡*Head*, *Pass* (these are the sequences where 2 gives up). The sequence *Heads*, ‡*Head*, *Challenge* is a win for player 1, with payoff 2 euros, and −2 euros for player 2. The sequence *Tails*, ‡*Head*, *Challenge*, finally, is a win for player 2, with payoff 2 euros, and −2 euros loss for player 1.

Player 1 has four strategies: *(‡Head, ‡Head)* (‡*Head* in case of heads and in case of tails), *(‡Head, ‡Tail)* (‡*Head* in case of heads, ‡*Tail* in case of tails), *(‡Tail, ‡Head)*, and *(‡Tail, ‡Tail)*. Player 2 has two strategies: *Pass* and *Challenge*. To find the strategic game form, one has to take the average of the expected payoffs for the two cases of *heads* and *tails*. E.g., if player 1 plays *(‡Head, ‡Tail)* and player 2 responds with *Challenge*, then in

the long run in $\frac{1}{2}$ of the cases the outcome will be heads, and player 1 wins 2 euros, and in $\frac{1}{2}$ of the cases the outcome will be tails, and player 1 loses 1 euro (for her strategy is just to give up in such cases). Thus, the expected payoff is $\frac{1}{2} \times 2 - \frac{1}{2} \times 1 = \frac{1}{2}$ euro for player 1, and because the game is zero sum, $-\frac{1}{2}$ euro for player 2. The strategic game form is given by:

	Pass	Challenge
$\ddagger Head, \ddagger Head$	1,-1	0,0
$\ddagger Head, \ddagger Tail$	0,0	$\frac{1}{2}, -\frac{1}{2}$
$\ddagger Tail, \ddagger Head$	0,0	$-\frac{3}{2}, \frac{3}{2}$
$\ddagger Tail, \ddagger Tail$	-1,1	-1,1

It is easy to see that there is no pure strategy Nash equilibrium (a Nash equilibrium is a combination of strategies, one for each player, with the property that neither of the players can improve their payoff by unilaterally deviating from her strategy). Clearly, none of the eight strategy pairs has this property.

Now let's consider the strategy *(‡Tail, ‡Tail)* for 1. This is the strategy of the doomed loser: even when the toss is heads the player still announces $\ddagger Tail$. This is obviously *not* the best thing that 1 can do. *Always* announcing $\ddagger Head$ gives a much better payoff in the long run. In other words, the strategy *(‡Tail,‡Tail)* is strictly dominated by *(‡Head, ‡Head)*. Similar for the strategy of the unconditional liar: *(‡Tail,‡Head)*. It is also strictly dominated by the strategy *(‡Head,‡Head)*. Thus, we are left with:

	Pass	Challenge
$\ddagger Head,\ddagger Head$	1,-1	0,0
$\ddagger Head, \ddagger Tail$	0,0	$\frac{1}{2}, -\frac{1}{2}$

Suppose 1 plays *(‡Head, ‡Head)* with probability p and *(‡Head, ‡Tail)* with probability $1 - p$. Then her expected value is p for her first strategy, and $\frac{1}{2}(1 - p)$ for her second strategy. Any choice of p where the expected payoff for p is different from that for $1 - p$ can be exploited by the other player. Therefore, player 1 should play her first strategy with probability $p = \frac{1}{2}(1-p)$, i.e., $p = \frac{1}{3}$, and her second strategy with probability $1 - p = \frac{2}{3}$. For player 2, we can reason similarly. Suppose 2 plays *Pass* with probability q and *Challenge* with probability $1 - q$. Again, the expected values for q and $1 - q$ should be the same, for otherwise this mixed strategy can be exploited by the other player. The expected value is $-q$ for her first strategy and $-\frac{1}{2}(1 - q)$ for her second strategy. Thus, she should play her first strategy

with probability $q = \frac{1}{2}(1-q)$, i.e., $q = \frac{1}{3}$. Neither player can improve on her payoff by unilateral deviation from these strategies, so the mixed strategy where 1 plays *(‡Head, ‡Head)* in $\frac{1}{3}$ of the cases and 2 plays *Pass* in $\frac{1}{3}$ of the cases is a Nash equilibrium. In other words, the best that player 1 can do is always announcing the truth and raise the stakes when her toss is heads, and lying in one third of the cases when her toss is tails, and 2's best response to this is to Pass in one third of all cases and Challenge two thirds of the time.

The game theoretic analysis yields that lying pays off for player 1, and that player 2, knowing this, may reasonably expect to catch player 1 on a lie in one sixth of all cases. The value of the game is $\frac{1}{3}$ euro, and the solution is $\frac{1}{3}$ (‡*Head*, ‡*Head*), $\frac{2}{3}$ (‡*Head*, ‡*Tail*) as player 1's optimal strategy, and $\frac{1}{3}$ Pass, $\frac{2}{3}$ Challenge as player 2's optimal strategy. It is clear that the honest strategy (‡*Head*, ‡*Tail*) is not the optimal one for player 1: given player 2 plays $\frac{1}{3}$ Pass and $\frac{2}{3}$ Challenge, the expected payoff for player 1 is only $\frac{1}{6}$ if he sticks to the honest strategy. Lying indeed pays off sometimes.

If we modify the game so that player 1 cannot lie anymore, by refusing her the privilege of having a peek at the toss outcome, the game immediately becomes a lot less interesting. In the extensive game form for this version, an extra dotted line indicates that player 1 cannot distinguish the outcome *Heads* from the outcome *Tails*:

Chance

Heads / \ Tails

1

T / | H

-1,1   2   1

Pass / Challenge   H / \ T

1,-1   2,-2   2   -1,1

Pass / \ Challenge

1,-1   -2,2
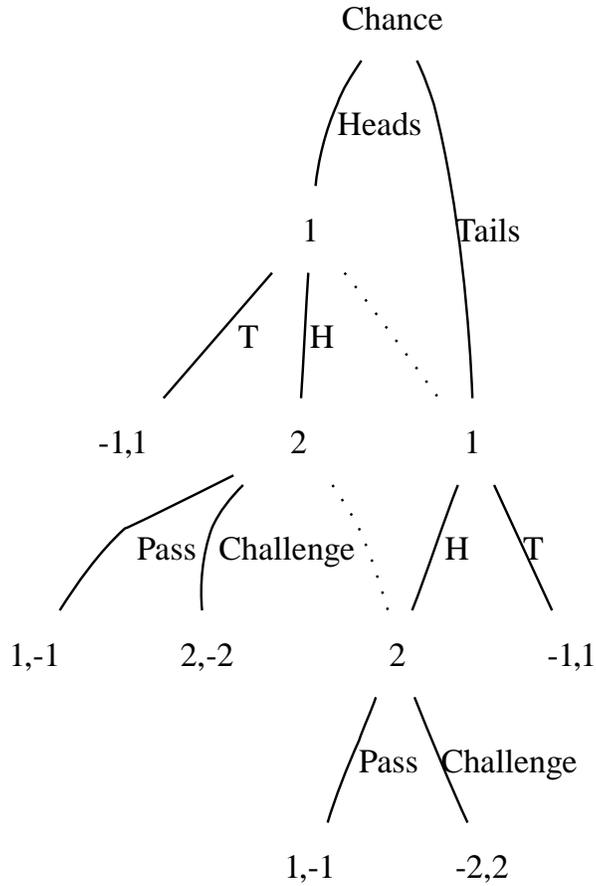
Player 1 has just two strategies left, $\ddagger Head$ and $\ddagger Tail$, and the strategic form of the game becomes:

	Pass	Challenge
$\ddagger Head$	1,-1	0,0
$\ddagger Tail$	-1,1	-1,1

The strategy $\ddagger Tail$ for player 1 is weakly dominated by $\ddagger Head$, so it can be eliminated, and we are left with:

	Pass	Challenge
$\ddagger Head$	1,-1	0,0

The strategy pair *(‡Head, Challenge)* is a Nash equilibrium. The game theoretic analysis predicts that a rational player 1 will always $\ddagger Head$, and

a rational player 2 will always Challenge, and the game becomes a pure zero-sum game of chance. Surely, it is the possibility of lying that makes Liar's Dice an interesting game.

5   Liar's Dice: Epistemic Analysis

One of the authors, Floor Sietsma, has recently extended the epistemic model checker DEMO [Eij07] with factual changes and the possibility to store integer values [Sie07]. The code of this extended model checker is available from `http://student.science.uva.nl/~fsietsma/DEMOFCR/`. We show how the game of Liar's Dice can be modelled using DEMO, and we demonstrate the epistemic models that we get if we trace a particular run of the game.

Fist we will closely examine the different actions that take place in the game and their representations as action models. Let p represent the value of a penny, with 1 signifying heads, and 0 signifying tails. Let agents a and b represent the two players, and let C_1 represent the contents of the purse of player 1 (C for cash), and C_2 that of player 2, with natural number values representing the amounts in euros that each player has in her purse. These natural number registers are available in the new extension of DEMO. Let S_1, S_2 represent the money at stake for each player. Factual change can be thought of as assignment of new values to variables. This is an essential ingredient of the various actions in the game:

Initialisation Both players put one euro at stake, and they both know this. $S_1 := 1, C_1 := C_1 - 1, S_2 := 1, C_2 := C_2 - 1$, together with public announcement of these factual changes.

Heads Factual change of the propositional value of a penny p to 1, with private communication of the result to player 1 ($p = 1$ signifies heads).

Tails Factual change of the propositional value of a penny p to 0, with private communication of the result to player 1. ($p = 0$ signifies tails).

Announce Player 1 announces either $\ddagger Head$ or $\ddagger Tail$. There are several ways to model this and we will come back to this later.

Pass Player 2 passes and loses, player 1 gets the stakes. $C_1 := C_1 + S_1 + S_2, S_1 := 0, S_2 := 0$.

Challenge Public setting of $C_2 := C_2 - 1, S_2 := S_2 + 1$. followed by public announcement of the value of p. If the outcome is p then $C_1 := C_1 + S_1 + S_2$, otherwise $C_2 := C_2 + S_1 + S_2$ and in any case $S_1 := 0, S_2 := 0$.

We will show how these actions can be defined in Haskell code using DEMO.

```
module Lies
where
import DEMOFCR
```

We first define the cash and stakes of each player as integer registers.

```
c1, c2, s1, s2 :: Reg
c1 = (Rg 1); c2 = (Rg 2)
s1 = (Rg 3); s2 = (Rg 4)
```

This declares four integer registers, and gives them appropriate names. The initial contents of the purses of the two players must also be defined. Let's assume both players have five euros in cash to start with.

```
initCash1, initCash2 :: Integer
initCash1 = 5
initCash2 = 5
```

Initialisation of the game: both players put one euro at state. This is modelled by the following factual change: $S_1 := 1, C_1 := C_1 - 1, S_2 := 1, C_2 := C_2 - 1$. The representation of this in our modelling language is straightforward. We just represent the contents of the registers at startup.

```
initGame :: EM
initGame = (Mo
            [0]
            [(0,([],[(s1,1),(s2,1),
             (c1,(initCash1-1)),(c2,(initCash2-1))]))]
            [a,b]
            [(a,0,0),(b,0,0)]
            [0])
```

Tossing the penny is a factual change of p to 0 or 1. The penny is tossed secretly and before player 1 looks both players don't know the value of the penny. Because of this there are two worlds, one where p is set to 0 and one where p is set to 1, and neither of the two players can distinguish these worlds.

```
toss :: Integer -> FAM
toss c ags = (Mo
              [0,1]
              [(0,(Top,[(P 0,Neg(Top))],[])),
               (1,(Top,[(P 0,Top)],[]))]
              ags
              [(ag,w,w') | w  <- [0,1],
                           w' <- [0,1], ag <- ags]
              [c])
```

Note that the action model has a list that assigns to each world a triple consisting of a precondition, a change to the propositions, and a change to the registers. In world 0, the precondition is $\top$ and the change is to set p to value $\neg\top$, i.e.,$\bot$ (and there is no change to the registers), and in world 1, the precondition is again $\top$ and the change is to set p to value $\top$ (and again, there is no change to the registers).

After the penny is tossed player 1 looks under the cup without showing the penny to player 2. We define a generic function for computing the model of the action where a group of agents looks under the cup. These models consist of two worlds, one where p is true (heads) and one where p is false (tails), the agents in the group can distinguish these two worlds and the other agents cannot.

```
look :: [Agent] -> FAM
look group ags = (Mo
                  [0,1]
                  [(0,(p,[],[])),(1,(Neg(p),[],[]))]
                  ags
                  ([(ag,w,w') | w <- [0,1], w' <- [0,1],
                                ag <- (ags \\ group)] ++
                   [(ag,w,w)  | w <- [0,1], ag <- group])
                  [0,1])
```

In this case, there are no changes to propositions or registers, but world 0 has precondition p, and world 1 has precondition $\neg p$.

Now we define the models of the situation after the penny has been tossed and player 1 has looked at the outcome, distinguishing the two outcomes of the toss:

```
headsg :: EM
headsg = upd (upd initGame (toss 1)) (look [a])

tailsg :: EM
tailsg = upd (upd initGame (toss 0)) (look [a])
```

Before looking at the way to model the announcement of an outcome of the toss by player 1 we will first define the action models for passing and challenging.

When player 2 passes the stakes are added to player 1's cash: $C_2 :=$ $C_2 + S_1 + S_1, S_1 := 0, S_2 := 0$. Player 2 never gets to see the actual value of the penny so there are no changes in the knowledge of the agents about p. The model for this has only one world that indicates the changes in the stakes and cash.

```
pass :: FAM
pass ags = (Mo
            [0]
            [(0,(Top,[],
                    [(s1,(I 0)),
                     (s2,(I 0)),
                     (c1,ASum [Reg c1,Reg s1,Reg s2])]))]
            ags
            [(ag,0,0) | ag <- ags]
            [0])
```

Note that here for the first time we see changes to the registers.

When player 2 decides to challenge player 1, the cup is lifted and both players get to know the value of p. Then the stakes are added to the cash of player 1 in case of heads and player 2 in case of tails, together with one extra euro from the cash of player 2 that player 2 added to the stakes while challenging player 1. So instead of $S2 := S2 + 1, C2 := C2 - 1$ and after that $C1 := C1 + S1 + S2$ in case of heads and $C2 := C2 + S1 + S2$ in case of tails, we use $C1 := C1 + S1 + S2 + 1, C2 := C2 - 1$ in case of heads and $C2 := C2 + S1 + S2$ in case of tails. The action model for this has one world for the case of heads and one world for the case of tails. Both players can distinguish these worlds because the cup was lifted, and the stakes are devided differently in the two worlds.

```
challenge :: FAM
challenge ags =
 Mo
 [0,1]
 [(0,(Neg(p),[],
       [(s1,(I 0)),
        (s2,(I 0)),
        (c2,ASum [Reg c2,Reg s1,Reg s2])])),
   (1,(    p ,[],
       [(s1,(I 0)),
        (s2,(I 0)),
        (c2,ASum [Reg c2,I (-1)]),
        (c1,ASum [Reg c1,Reg s1,Reg s2,I 1])]))]
  ags
  [(ag,w,w) | w <- [0,1], ag <- ags]
  [0,1]
```

When player 1 announces $\ddagger Head$ or $\ddagger Tail$ the stakes change. In case of $\ddagger Head\ C_1 := C_1 - 1, S_1 := S_1 + 1$ and in case of $\ddagger Tail\ C_2 := C_2 + S_1 + S_2, S_1 := 0, S_2 := 0$.

```
announceStakes :: Integer -> FAM
announceStakes 0 ags =
Mo
 [0]
 [(0,(Top,[],[(s1,(I 0)),
   (s2,(I 0)),
   (c2,ASum [Reg c2,Reg s1,Reg s2])]))]
 ags
 [(ag,0,0) | ag <- ags]
 [0]
announceStakes 1 ags =
Mo
 [0]
 [(0,(Top,[],[(s1,ASum [Reg s1,I 1]),
   (c1,ASum [Reg c1,I (-1)])]))]
 ags
 [(ag,0,0) | ag <- ags]
 [0]
```

Now the only thing we have to decide is how we will model the announcement of $\ddagger Head$ or $\ddagger Tail$. Suppose we would use the manipulative update $\ddagger p$ or $\ddagger \neg p$ for this. Then this would mean that the other player believes the claims that are made. However, in a real game of Liar's Dice player 2 knows that player 1 might very well be bluffing and doesn't really believe player 1's claim at all. So to correctly model the game we shouldn't use the manipulative update at all. When player 1 makes an announcement this doesn't even change player 2's knowledge and believes because player 2 doesn't believe player 1.

So instead of the manipulative update we should only use the model for changing the stakes to model the announcement:

```
announce :: Integer -> FAM
announce = announceStakes
```

Now player 2 doesn't know whether p is true but she knows she doesn't know:

```
  bKnows :: Form
  bKnows = Disj [(K b (Neg p)), (K b p)]
```

```
Lies> isTrue (upd tailsg (announce 0)) bKnows
False
Lies> isTrue (upd tailsg (announce 0)) (K b (Neg bKnows))
True
Lies> isTrue (upd headsg (announce 0)) bKnows
False
Lies> isTrue (upd headsg (announce 0)) (K b (Neg bKnows))
True
Lies> isTrue (upd tailsg (announce 1)) bKnows
False
Lies> isTrue (upd tailsg (announce 1)) (K b (Neg bKnows))
True
Lies> isTrue (upd headsg (announce 1)) bKnows
False
Lies> isTrue (upd headsg (announce 1)) (K b (Neg bKnows))
True
```

Note that since we did not use the manipulative update to model player 1's announcement (although it is easy to implement in DEMO, of course) the resulting models are still S5-models.

```
Lies> isS5Model (upd headsg (announce 1))
True
Lies> isS5Model (upd headsg (announce 0))
True
Lies> isS5Model (upd tailsg (announce 1))
True
Lies> isS5Model (upd tailsg (announce 0))
True
```

This means that no actual misleading is taking place at all. This is actually very plausible because everyone knows that player 1's announcement might very well be false.

Now we can use these action models to do an epistemic analysis of a game of Liar's Dice. The different possible games are:

1. Player 1 tosses tails and announces ‡$Tail$

2. Player 1 tosses heads and announces ‡$Tail$

3. Player 1 tosses tails and announces ‡$Head$ and player 2 passes

4. Player 1 tosses tails and announces ‡$Head$ and player 2 challenges

5. Player 1 tosses heads and announces ‡$Head$ and player 2 passes

6. Player 1 tosses heads and announces ‡$Head$ and player 2 challenges

The models for these games are:

```
game1, game2, game3, game4, game5, game6 :: EM
game1 = gsm (upd tailsg (announce 0))
game2 = gsm (upd headsg (announce 0))
game3 = gsm (upd (upd tailsg (announce 1)) pass)
game4 = gsm (upd (upd tailsg (announce 1)) challenge)
game5 = gsm (upd (upd headsg (announce 1)) pass)
game6 = gsm (upd (upd headsg (announce 1)) challenge)
```

We will now consider these six different cases in turn.

Game 1 is the game where player 1 tosses tails and admits this.

In this case both players stake one euro and player 2 wins the stakes, so in the end player 1 lost one euro and player 2 won one euro. This can be checked with DEMO:

```
Lies> isTrue game1 (eq (Reg c1) (ASum [I initCash1,I (-1)]))
True
Lies> isTrue game1 (eq (Reg c2) (ASum [I initCash2,I 1]))
True
```

Player 2 doesn't get to know what the value of the penny was:
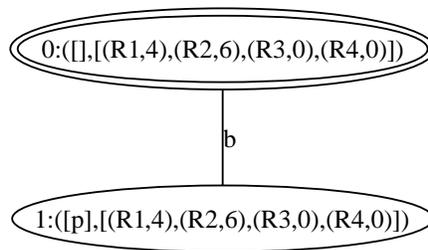
```
Lies> isTrue game1 bKnows
False
```

The model for game 1 is:

```
Lies> game1
Mo
[0,1]
[(0,([],[(R1,4),(R2,6),(R3,0),(R4,0)])),
 (1,([p],[(R1,4),(R2,6),(R3,0),(R4,0)]))]
[a,b]
[(a,0,0),(a,1,1),(b,0,0),(b,0,1),(b,1,0),(b,1,1)]
[0]
```

A picture of this model is below. There are two worlds, one where the toss was heads and one where it was tails. Player 1 can distinguish these worlds, player 2 cannot because player 2 never got to see the penny. In both worlds the cash of player 1 is 4 and that of player 2 is 6 euros, because the devision of the stakes doesn't depend on the value of the penny.

Game 2 is the game where player 1 falsely announces $\ddagger Head$. Just like in game 1, player 1 loses one euro and player 2 wins one euro, and player 2 doesn't get to know the value of the penny.

```
Lies> isTrue game2 (eq (Reg c1) (ASum [I initCash1,I (-1)]))
True
Lies> isTrue game2 (eq (Reg c2) (ASum [I initCash2,I 1]))
True
Lies> isTrue game2 bKnows
False
```

The model for this game is almost the same as for game 1: the difference is that now the world where p is true is actual instead of the world where p is false.
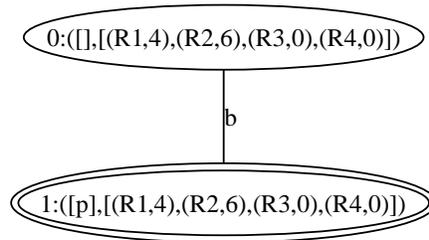
```
Lies> game2
Mo
```

```
[0,1]
[(0,([],[(R1,4),(R2,6),(R3,0),(R4,0)])),
 (1,([p],[(R1,4),(R2,6),(R3,0),(R4,0)]))]
[a,b]
[(a,0,0),(a,1,1),(b,0,0),(b,0,1),(b,1,0),(b,1,1)]
[1]
```

The picture of this model is:

The third game is the case where player 1 tosses tails but falsely announces ‡*Head* and player 2 passes. In this case player 1 stakes two euros and player 2 stakes one euro, and player 1 gets to keep the stakes, so the final payoff is that player 1 wins one euro and player 2 loses one euro:

```
Lies> isTrue game3 (eq (Reg c1) (ASum [I initCash1,I 1]))
True
Lies> isTrue game3 (eq (Reg c2) (ASum [I initCash1,I (-1)]))
True
```

Player 2 passes, so the cup is never lifted and player 2 doesn't know the value of the penny:

```
Lies> isTrue game3 bKnows
False
```

The model for this game is:

```
Lies> game3
Mo
[0,1]
[(0,([],[(R1,6),(R2,4),(R3,0),(R4,0)])),
 (1,([p],[(R1,6),(R2,4),(R3,0),(R4,0)]))]
[a,b]
[(a,0,0),(a,1,1),(b,0,0),(b,0,1),(b,1,0),(b,1,1)]
[0]
```

This model has the same two worlds as the models for game 1 and 2 except for the changes in the player's cash.

In the fourth game, player 1 tosses tails but falsely announces $\ddagger Head$ and player 2 challenges player 1. This means that both player stake one extra euro and then the cup is lifted and player 2 gets the stakes.

In this case player 2 does know the value of the penny:
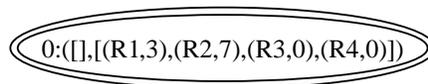
```
Lies> isTrue game4 bKnows
True
```

The payoffs are -2 euros for player 1 and 2 euros for player 2:

```
Lies> isTrue game4 (eq (Reg c1) (ASum [I initCash1,I (-2)]))
True
Lies> isTrue game4 (eq (Reg c2) (ASum [I initCash2,I 2]))
True
```

The model for this game is:

```
Lies> game4
Mo
[0]
[(0,([],[(R1,3),(R2,7),(R3,0),(R4,0)]))]
[a,b]
[(a,0,0),(b,0,0)]
[0]
```

This model has only one world because none of the players consider any other world possible, because both players know the values of the penny. In this world p is false (because the toss was tails), player 1's cash is 3 euros and player 2's cash is 7 euros. A picture of this model is below.

The fifth game is the game where player 1 tosses heads and truthfully announces this and player 2 passes. In this case the cup isn't lifted so player 2 doesn't know the value of the penny again:

```
Lies> isTrue game5 bKnows
False
```

The payoffs are 1 for player 1 and -1 for player 2:

```
Lies> isTrue game5 (eq (Reg c1) (ASum [I initCash1,I 1]))
True
Lies> isTrue game5 (eq (Reg c2) (ASum [I initCash2,I (-1)]))
True
```

The model for game 5 has two worlds again because player 2 doesn't know the value of the penny.

```
Lies> game5
Mo
[0,1]
[(0,([],[(R1,6),(R2,4),(R3,0),(R4,0)])),
 (1,([p],[(R1,6),(R2,4),(R3,0),(R4,0)]))]
[a,b]
[(a,0,0),(a,1,1),(b,0,0),(b,0,1),(b,1,0),(b,1,1)]
[1]
```

In game 6 player 1 tosses heads and truthfully announces this and player 2 challenges player 1. In this case both players add one extra euro to the stakes, the cup is lifted and player 1 gets to keep the stakes. The model for this has one world where p is true, player 1 has 7 euros and player 2 has 3 euros.

```
Lies> game6
Mo
[1]
[(1,([p],[(R1,7),(R2,3),(R3,0),(R4,0)]))]
[a,b]
[(a,1,1),(b,1,1)]
[1]
```

In this case player 2 knows the value of the penny and the payoffs are 2 euros for player 1 and -2 euros for player 2:

```
Lies> isTrue game6 bKnows
True
Lies> isTrue game6 (eq (Reg c1) (ASum [I initCash1,I 2]))
True
Lies> isTrue game6 (eq (Reg c2) (ASum [I initCash2,I (-2)]))
True
```

According to Augustine, the defining characteristic of a lie is the intent to mislead someone into believing something that is not true. Our analysis shows that this is not what happens in dice games like Liar's Dice. All updates in the game turn out to be S5 updates, and the players are never duped into believing a falsehood.

6   Conclusion

There are still two discrepancies in the paper that we have to address. The first one is between our treatment of lying in public discourse and our treatment of lying in games. As we have seen, lying in public discourse can lead to KD45 models, which illustrates the fact that genuine misleading takes place. We argued that the players in a game like Liar's Dice are never actually misled, so in a sense no real lying takes place here at all. Still, lying in public discourse and lying in games seem to be connected somehow.

The second discrepancy is between the game theoretical analysis of lying in games in terms of mixed strategies that use probabilities, and the logical analysis in terms of truth values. To see that these perspectives do still not quite match, consider the game situation where player 1 tosses the penny, observes the result, and announces 'heads'. In our logical analysis this does *not* lead to the false belief of player 2 that the coin has landed heads; it does not lead to a belief change at all. But the game theoretical analysis reveals that a rational agent would have formed a belief about the probability that the claim is true. So it seems that the logical analysis is still too crude.

This defect could be remedied by using probabilistic beliefs and probabilistic updates, in the style of [BGK06], which would allow us to express the probability of actions in the game. With these, we can model the fact that the game theoretical analysis in terms of mixed strategies is common knowledge. For if this is the case, it is common knowledge that if the toss is tail, then player 1 will announce 'heads' with probability $\frac{1}{3}$ and 'tails' with probability $\frac{2}{3}$. Interestingly, this is also relevant for the first discrepancy. For why are the players not duped into believing falsehoods, in the game of Liar's Poker? Because they look further than a single run of the game, and they know that as the game gets repeated they can adhere to mixed strategies. Therefore, an analysis in terms of manipulative probabilistic updates might work for both lying in public discourse and lying in games.

But there is need here for further work. Even if we switch to a probabilistic version of the logic of communication and change, we have to attach probabilities to the update actions that we start with. This leaves open the problem of

how to use logic to *derive* the correct Nash equilibria in the first place. In future work we will explore the possibility of letting agents find such solutions by iterative playing of the game and updating their probabilities until a fixpoint representing an equilibrium is reached.

Other areas of future work are the connection of the logic of lying with belief revision and the modelling of agency. Believing a lie might have the consequence that an initial true belief is given up in favour of a false one. This will only happen, however, if the original true belief is held weakly enough to be replaced by the lie. In modelling a lie as a publicly announced falsehood that is believed by the audience we have left out the liar. To get the liar back into the picture, one has to analyse the preconditions for a lie, in terms of the epistemics of the input model. For agent i to be the originator of a lie, i has to know ϕ and announce $\neg\phi$, so $K_i\phi$ is a precondition of the lying action. This issue will be taken up in future work.

Finally, we must mention the fact that in philosophy and logic there is a long standing interest in liar paradoxes. Now it seems that our language is not powerful enough to express such paradoxes. What happens if we add a mechanism for self reference to dynamic epistemic logic? Does this immediately lead to either incompleteness or inconsistency? What is the simplest possible way of expressing liar paradoxes in (an extension of) dynamic epistemic logic, and what happens as a result?

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