Gossip in Dynamic Networks

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Abstract

A gossip protocol is a procedure for spreading secrets among a group of agents, using a connection graph. In this talk the problem of designing and analyzing gossip protocols is given a dynamic twist by assuming that when a call is established not only secrets are exchanged but also contact lists, i.e., links in the gossip graph. Thus, each call in the gossip graph changes both the graph and the distribution of secrets. In the talk, we give a full characterization for the class of dynamic gossip graphs where the Learn New Secrets protocol (make a call to an agent if you know the number but not the secret of that agent) is successful.
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This is joint work with Hans van Ditmarsch, Pere Pardo, Rahim Ramezanian and François Schwarzentruber
Overview

- What are Gossip Protocols? Brief History
- Gossip in Totally Connected Networks
- Distributed Protocols
- The Dynamic Turn
- Examples of Gossip Graph Completion
- The Learn New Secrets Protocol
- Results
- Further Questions
An Abstract Perspective on Gossip Spreading

Key notions: gossip graph, gossip secret, gossip call, gossip protocol.

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• What do we assume about the graph?
• What do we assume about the secrets?
• What do we assume about the protocol? In particular: is there a central authority, or is the protocol distributed?
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- Distributed gossip protocols: dynamic turn (this talk) [DvEP+15a, DvEP+15b]
Gossip in totally connected graphs, central authority

Assumptions: graph totally connected, during a call all secrets are exchanged.

Key question: find a minimal sequence of calls to achieve a state where all agents know all secrets. What are the lengths of these minimal sequences?
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**Fact:** In a totally connected graph with $n > 3$ agents, $2n - 4$ calls are sufficient.
Gossip in Totally Connected Graphs: Example

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Now suppose \( e \) is also present. Consider \( ae \), next \( ab; cd; ac; bd \), finally \( ae \). Two extra calls are enough to accommodate one extra agent.
For $n$ agents, $2n - 4$ calls are enough

Let $n > 3$.

Basis: $n = 4$. We have seen that $4 = 8 - 4$ calls are enough.

Induction step: Assume for $n$ agents $2n - 4$ calls are enough. Next, add one extra agent $x$. Start with call from $x$ to $a$, end with call from $a$ to $x$, and all secrets are shared. This shows that $(2n - 4) + 2 = 2(n + 1) - 4$ calls are enough. $\square$
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It is a bit trickier to show that $2n - 4$ calls are needed: see the original [Tij71], or [Hur00] and the references given there.
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Note: in graphs that are weakly but not totally connected, the minimum number of calls to distribute all gossip may be larger than $2n - 4$ [HHL88].
Distributed Protocols

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Possible distributed protocol for gossip spreading:

Search For Secrets

While not every agent knows all secrets, randomly select a pair $xy$ and let $x$ call $y$. 
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- The network is given in distributed fashion: \((x, y) \in N\) iff \(y\) is in the contact list of \(x\) (think of contact lists in smartphones). These contact lists are exchanged (merged) when a call is made.
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- $N \subseteq A^2$ and $S \subseteq A^2$ are relations on $A$.
- $N_{xy}$ expresses that $x$ has a link to $y$ (or: $x$ does know the phone number of $y$).
- $S_{xy}$ expresses that $x$ does know the secret of $y$.
- Alternatively, we can think of $N$ and $S$ as functions in $A \rightarrow \mathcal{P}A$, so that $N_x$ is the set of agents whose numbers are known by $x$, and $S_x$ is the set of agents whose secrets are known by $x$. 
Calls and Their Effects

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A call $xy$ merges the secret lists and the contact lists of $x$ and $y$. Let $G^{xy}$ be the result of this merge in $G$.

If $G = (A, N, S)$ and $x, y \in A$, then $G^{xy} = (A, N', S')$ where

- $N'$ is $N \cup \{(x, y), (y, x)\} \circ N$,
- $S'$ is $S \cup \{(x, y), (y, x)\} \circ S$. 

Calling Sequences, Accessible Secrets
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Possible calling sequence  Define by recursion:

- $\epsilon$ is possible on any $G$,
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Accessible secrets $G = (A, N, S)$ has accessible secrets if $I_A \subseteq S \subseteq N$, where $I_A = \{(a, a) \mid a \in A\}$.

Thus, $G$ has accessible secrets iff every agent knows her own secret and moreover, if agent $x$ knows the secret of $y$, $x$ also knows the number of $y$. 
Example

\[ a : \{a\} \rightarrow b : \{b\} \]

\[ c : \{c\} \]
Example

\[ a : \{a\} \rightarrow b : \{b\} \]

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\[ \Rightarrow ab \Rightarrow \]
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\[ a : \{a\} \rightarrow b : \{b\} \]

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\[ a : \{a, b\} \leftrightarrow b : \{a, b\} \]

\[ \rightarrow \rightarrow \]

\[ \rightarrow \rightarrow \]

\[ c : \{c\} \]
Example

\[ a : \{a\} \rightarrow b : \{b\} \]

\[ \Rightarrow \]

\[ c : \{c\} \]

\[ \Rightarrow \]

\[ a : \{a, b\} \leftarrow b : \{a, b\} \]

\[ \Rightarrow \]

\[ c : \{c\} \]

\[ \Rightarrow \]

\[ bc \Rightarrow \]
Example

\[
\begin{align*}
a : \{a\} & \rightarrow b : \{b\} \\
& \quad \quad \quad \downarrow \\
& \quad \quad \quad c : \{c\} \\
\Rightarrow ab \Rightarrow \\
\end{align*}
\]

\[
\begin{align*}
a : \{a, b\} & \leftrightarrow b : \{a, b\} \\
& \quad \quad \quad \downarrow \\
& \quad \quad \quad c : \{c\} \\
\Rightarrow bc \Rightarrow 
\end{align*}
\]
\[ a : \{a, b\} \quad \xleftrightarrow{} \quad b : \{a, b, c\} \]
\[ c : \{a, b, c\} \]
$a : \{a, b\} \iff b : \{a, b, c\}$

$\Rightarrow ac \Rightarrow$
\[\begin{align*}
\text{a} : \{a, b\} & \quad \text{↔} \quad \text{b} : \{a, b, c\} \\
\text{c} : \{a, b, c\} & \quad \\
\Rightarrow & \quad ac \Rightarrow \\
\text{a} : \{a, b, c\} & \quad \text{↔} \quad \text{b} : \{a, b, c\} \\
\text{c} : \{a, b, c\} & \quad 
\end{align*}\]
Example Revisited: Calls in Different Order

\[ a : \{a\} \rightarrow b : \{b\} \]
\[ c : \{c\} \Rightarrow bc \Rightarrow \]
\[ a : \{a\} \rightarrow b : \{b, c\} \]
\[ c : \{b, c\} \]
Example Revisited: Calls in Different Order

\[
\begin{align*}
  a : \{a\} & \rightarrow b : \{b\} \\
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\[ \begin{align*}
  a &: \{a\} \quad b &: \{b\} \quad a &: \{a\} \rightarrow b &: \{b, c\} \\
  c &: \{c\} & \Rightarrow bc & \Rightarrow c &: \{b, c\} \\
  & \Rightarrow ab & \Rightarrow c &: \{b, c\} \\
  & \Rightarrow ac & \Rightarrow c &: \{a, b, c\}
\end{align*} \]
Old Questions, New Answers

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\[ a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow a \]
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Consider a circle with five agents

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\begin{align*}
  a & \rightarrow b \\
  b & \rightarrow c \\
  c & \rightarrow d \\
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\end{align*}
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This needs \(2n - 3 = 7\) calls before everyone knows all secrets [HHL88].
A sequence that works is \(ab; cd; ea; de; ea; ab; bc\) (check at your leisure).
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In our dynamic approach 6 calls are sufficient: \(ab; cd; ea; de; ac; bc\).
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Note: the fifth call \(ac\) in this sequence only works because \(a\) has learned the contact information about \(c\) from \(b\).
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Note: the fifth call \(ac\) in this sequence only works because \(a\) has learned the contact information about \(c\) from \(b\).

This shows that old questions about minimum lengths of calling sequences can receive new answers in this dynamic setting.
**Gossip and Weakly Connected Components**

**Proposition 1** Let $G = (A, N, S)$, and let $\sigma$ be a possible calling sequence for $G$. Then $N^\sigma \subseteq (N \cup N^{-1})^\ast$.

Intuitively, this says that gossip can only spread within the weakly connected components of $G$. 
Gossip and Weakly Connected Components

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A gossip graph $G = (A, N, S)$ is **weakly connected** if for all $x, y \in A$ there is an $N \cup N^{-1}$-path from $x$ to $y$. 
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A gossip graph $G = (A, N, S)$ is **weakly connected** if for all $x, y \in A$ there is an $N \cup N^{-1}$-path from $x$ to $y$.

**Theorem 2** If $\sigma$ is a possible calling sequence for $G = (A, N, S)$, then $G$ is weakly connected iff $G^\sigma$ is weakly connected.

Intuition: by proposition 1, gossip cannot create weak connectedness.
Gossip Graph Completion

A gossip graph $G = (A, N, S)$ is complete if it holds for all $x \in A$ that $S_x = A$. That is, a gossip graph is complete if all agents know all secrets.
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Key question 1: find distributed protocols that always create complete graphs.
Precondition: the graph has to be weakly connected to start with, by Theorem 2.
Gossip Graph Completion

A gossip graph $G = (A, N, S)$ is complete if it holds for all $x \in A$ that $S_x = A$. That is, a gossip graph is complete if all agents know all secrets.

**Key question 1:** find distributed protocols that always create complete graphs.

Precondition: the graph has to be weakly connected to start with, by Theorem 2.

**Key question 2:** given some distributed protocol $P$, what is the class of graphs that can be completed by $P$?
Search for Secrets as a Distributed Protocol

Search For Secrets

While not every agent knows all secrets, randomly select a pair $xy$ such that $N_{xy}$ and let $x$ call $y$.

This will complete any weakly connected graph, but it is not efficient.
Learn New Secrets

The following protocol is studied in [AvDGvdH14, AGvdH15] in the context of totally connected graphs.

Learn New Secret Protocol (LNS)

While not every agent is an expert, let an agent $x$ that is not an expert randomly choose an agent $y$ from the list of agents for which $N_{xy}$ but not $S_{xy}$, and perform the call $xy$. 
LNS-permitted and LNS-stuck Sequences

LNS-permitted calling sequences:

- $\epsilon$ is LNS-permitted on any $G$,
- $\sigma; xy$ is LNS-permitted on $G$ iff $\sigma$ is LNS-permitted on $G$ and $xy$ is LNS-permitted on $G^\sigma$. 
LNS-permitted and LNS-stuck Sequences

LNS-permitted calling sequences:

- $\epsilon$ is LNS-permitted on any $G$,
- $\sigma; xy$ is LNS-permitted on $G$ iff $\sigma$ is LNS-permitted on $G$ and $xy$ is LNS-permitted on $G^\sigma$.

A calling sequence $\sigma$ is **LNS-stuck** on $G$ if $\sigma$ is LNS-permitted on $G$, $G^\sigma$ is not complete, and no call is LNS-permitted on $G^\sigma$. 
Example of LNS-stuck Sequence

\[ a : \{ a \} \quad \Rightarrow \quad b : \{ b \} \]

\[ \Rightarrow \quad b : \{ b, c \} \]

\[ c : \{ c \} \quad \Rightarrow \quad b : \{ b, c \} \]

\[ c : \{ b, c \} \]
Example of LNS-stuck Sequence

\[ a : \{a\} \rightarrow b : \{b\} \Rightarrow bc \Rightarrow a : \{a\} \rightarrow b : \{b, c\} \]

\[ c : \{c\} \Rightarrow bc \Rightarrow c : \{b, c\} \]

\[ a : \{a, b, c\} \leftrightarrow b : \{a, b, c\} \rightarrow ab \rightarrow c : \{b, c\} \]
Example of LNS-stuck Sequence

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\[ a : \{a, b, c\} \quad \leftrightarrow \quad b : \{a, b, c\} \quad \Rightarrow \quad ab \quad \Rightarrow \quad c : \{b, c\} \]

\[ \Rightarrow \text{??} \quad \Rightarrow \quad \text{STUCK} \]
Example of LNS-stuck Sequence

Still, the sequence $ab; bc; ca$ is LNS permitted.
Example Graph that Cannot be Completed by LNS

\[ a \rightarrow x \leftarrow e \]

\[ b \rightarrow c \leftarrow d \]
Whoever starts the calling sequence, it cannot be LNS completed.

The reason is that $x$, the spider in the web, never learns enough about the network structure to be able to make a useful call ($x$ only learns contact info of agents whose secret $x$ also learns.

The LNS-permitted sequences are all the permutations of $ax; bx; cx; dx; ex$, and they all get stuck.
Graphs Where LNS is Successful

The LNS protocol is successful on $G$ if either $G$ is complete, or there is an LNS-permitted call $xy$, and after any LNS-permitted call $xy$ the LNS protocol is successful on $G^{xy}$.

It follows that LNS is successful on $G$ iff every sequence of LNS-permitted calls $\sigma$ results in a graph $G^\sigma$ that is complete, or is such that there is an LNS-permitted call, and after any LNS-permitted call $xy$, LNS is successful on $G^{\sigma;xy}$.

LNS gossip graph algorithm

Search for an LNS-stuck calling sequence in depth-first fashion, and declare success if no such calling sequence can be found [EG15a].
LNS-maximal Sequences

A calling sequence $\sigma$ for $G$ is LNS-maximal if $\sigma$ is LNS-permitted for $G$, and no calls are LNS-permitted in $G^\sigma$. 
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**Proposition 3** If $\sigma$ is an LNS-maximal calling sequence for $G$, and $G$ has accessible secrets, then $S^\sigma = N^\sigma$. 
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**Proposition 4** If $\sigma$ is an LNS-maximal calling sequence for $G$, and $G$ satisfies $I_A = S \subseteq N$, then $S^\sigma \circ N^* = S^\sigma$. 
Terminal Points, Skin, Sun Graphs

A **terminal point** in $G = (A, N, S)$ is a point $x$ for which $N_x \subseteq \{x\}$. That is, a terminal point is an agent that knows at most her own number.
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Let $s(G)$ be the result of skinning graph $G$, i.e. removing all terminal points from $G$. That is, $s(G) = (B, N', S')$ where

$$B = \{x \in A \mid N_x - \{x\} \neq \emptyset\}, \quad N' = N \cap B^2, \quad S' = S \cap B^2.$$

Note that skinning a graph is not a closure operation: there are graphs with $s(s(G')) \neq s(G')$. 
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Note that skinning a graph is not a closure operation: there are graphs with $s(s(G)) \neq s(G)$.

Call a graph $G = (A, N, S)$ a **sun** if $S = I_A \subseteq N$, $N$ is weakly connected on $G$, and $N$ is strongly connected on $s(G)$. 
Examples of Sun Graphs
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Theorem 5 The LNS protocol is successful for any sun $G$. 
Strongly Connected Components of a Graph

Let $\sim$ be the relation on $G = (A, N, S)$ given by $x \sim y$ iff there is an $N$-path from $x$ to $y$ and there is an $N$-path from $y$ to $x$. Then $\sim$ is an equivalence relation, and a cell in the partition induced by $\sim$ is called a strongly connected component of $G$. Use $[x]_\sim$ for the strongly connected component of $G$ that contains $x$. 
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If \( G = (A, N, S) \) is a gossip graph and \( \sigma \) is a possible calling sequence for \( G \), then we use \([x]_\sim^\sigma\) for the strongly connected component of \( G^\sigma \) that contains \( x \).
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If $G = (A, N, S)$, then the relation $\hat{N}$ on $A$ is defined by means of:

$$\hat{N}xy \text{ iff } [x]_{\sim} \neq [y]_{\sim} \land \exists x' \in [x]_{\sim}\exists y' \in [y]_{\sim} : N x' y'.$$
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Theorem 6 If $G = (A, N, S)$ is a gossip graph with the property that $s(G)$ is connected but not strongly connected, then there are $x, y \in A$ such that

1. $\hat{N}xy$,

2. for all $u \in A$ with $\hat{N}xu$ and $N^*uy$ it holds that $[u]_\sim = [y]_\sim$,

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Theorem 7 For any gossip graph $G = (A, N, S)$ with $S = I_A \subseteq N$ and the property that $s(G)$ is connected but not strongly connected there is an LNS-permitted calling sequence $\sigma$ such that $G^\sigma$ is not complete, but no calls are LNS-permitted in $G^\sigma$. 
**Theorem 6** If \( G = (A, N, S) \) is a gossip graph with the property that \( s(G) \) is connected but not strongly connected, then there are \( x, y \in A \) such that

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**Theorem 7** For any gossip graph \( G = (A, N, S) \) with \( S = I_A \subseteq N \) and the property that \( s(G) \) is connected but not strongly connected there is an LNS-permitted calling sequence \( \sigma \) such that \( G^\sigma \) is not complete, but no calls are LNS-permitted in \( G^\sigma \).

**Theorem 8** For any connected graph \( G = (A, N, S) \) with \( I_A = S \subseteq N \) the following holds: \( s(G) \) is strongly connected iff the LNS protocol is successful for \( G \).
LNS-impossible Graphs

**Question** Characterize the graphs where no CNS sequence is successful.

**Answer** See [DvEP+15a].

Note: the conditions for this are quite strong. Spider in the web above is an example.
Other Protocols: HYN

Help Your Neighbour (HYN) [Her15]
Everyone who ever contacted you becomes your neighbour. Every time you learn new secrets, you incur an obligation to inform them . . .

Theorem 9 HYN completes any weakly connected graph. [Her15]
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- Further features from actual life could be imported. An important one that comes to mind is caller blocking. Which blocking patterns on which graphs can be overcome by which protocols?
- In [EG15b] we try to harness PDL as a logic for dynamic gossip. Details will be given by Malvin in the next talk.
facebook, a great place to spread gossip
References


