

Erdős-Szemerédi Sunflower conjecture

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I will give two proofs of the Erdős-Szemerédi Sunflower conjecture: a proof that uses the cap-set problem from Alon et al. [1] and a proof from Naslund and Sawin [5] that uses the slice-rank method. I will also show how Naslund and Sawin [5] apply the same ideas in an attempt towards the (stronger, unproved) Erdős-Rado Sunflower conjecture. A brief note on the connection to algorithms for fast matrix multiplication is included at the end.

1 Recap: the cap-set problem and slice rank

A k -**tensor** is a function $f : X^k \rightarrow \mathbb{F}$ for \mathbb{F} a field and X a finite set. A tensor f has **slice rank one** if it can be written as a product gh where g, h depend on disjoint, non-empty subsets of the variables. For example, we might have $f(x, y, z) = g(x)h(y, z)$ for $k = 3$. The **slice rank** of a tensor f is the minimum number m so that $f = \sum_{i=1}^m f_i$ with f_1, \dots, f_m of slice rank one. The lemma below is a simplification of a lemma of Tao [6].

Lemma 1.1 (Slice rank of diagonal tensors). *If $F : X^k \rightarrow \mathbb{F}$ has the property that $F(x_1, \dots, x_k) \neq 0$ if and only if $x_1 = \dots = x_k$, then F has slice rank $|X|$.*

Definition 1.2. A **cap set** is a subset $A \subseteq \mathbb{F}_3^n$ that does not contain any *line* (three-term arithmetic progression)

$$\{x, x + r, x + 2r\} \text{ for } x \in \mathbb{F}_3^n, r \in \mathbb{F}_3^n \setminus \{0^n\}. \quad \blacktriangle$$

Note that

$$x + y + z = 0^n \iff x = y = z \text{ or } \{x, y, z\} \text{ forms a line,}$$

if lines are defined as above. Hence if A is a cap set, then $x + y + z = 0^n \iff x = y = z$.

Theorem 1.3 (Croot-Lev-Pach-Ellenberg-Gijswijt cap-set bound). *If $A \subseteq \mathbb{F}_3^D$ is a cap set, then $|A| \leq 3C^D$ where $C \leq 2.76$ is a constant.*

Proof sketch. Note that in \mathbb{F}_3 we have $1 - x^2 \neq 0 \iff x = 0$. Hence

$$F : A \times A \times A \rightarrow \mathbb{F}_3 : (x, y, z) \rightarrow \prod_{i=1}^D (1 - (x_i + y_i + z_i)^2)$$

is non-zero if and only if $x + y + z = 0$, which is equivalent to $x = y = z$ (as we saw above). This means F has slice rank $|A|$ by Lemma 1.1. On the other hand, F expands as

$$\sum_{\substack{I, J, K \in \{0, 1, 2\}^D \\ |I| + |J| + |K| \leq 2D}} c_{IJK} x^I y^J z^K = \sum_{\substack{I \in \{0, 1, 2\}^D \\ |I| \leq 2D/3}} x^I f_I(y, z) + \sum_{\substack{J \in \{0, 1, 2\}^D \\ |J| \leq 2D/3}} y^J g_J(x, z) + \sum_{\substack{K \in \{0, 1, 2\}^D \\ |K| \leq 2D/3}} z^K h_K(x, y).$$

This expresses F in terms of tensors of slice rank one, hence this gives an upper bound on $|A|$. See e.g. De Zeeuw [4] for the calculations. □

2 Slice rank applied to sunflower conjectures

Definition 2.1. A k -**sunflower** or Δ -**system** is a set of subsets \mathcal{S} so that $|\mathcal{S}| = k$ and

$$A \cap B = \bigcap_{C \in \mathcal{S}} C \quad \forall A, B \in \mathcal{S} : A \neq B. \quad \blacktriangle$$

A set of subsets \mathcal{A} is called k -**sunflower free** if no k members form a sunflower. We omit k in the case $k = 3$.

Conjecture 2.2 (Erdős-Szemerédi Sunflower conjecture). *Let $k \geq 3$. Then there exists a constant $c_k < 2$ so that for each k -sunflower free set \mathcal{A} of subsets of $\{1, \dots, n\}$ we have $|\mathcal{A}| \leq c_k^n$.*

The cap-set bound gives $c_k \leq 1.938$ (see [5, Theorem 8]). Naslund and Sawin [5] apply the slice rank method directly to the Erdős-Szemerédi Sunflower conjecture.

Theorem 2.3. *If $\mathcal{A} \subseteq \mathcal{P}([n])$ is sunflower-free, then $|\mathcal{A}| \leq 3(n+1)C^n$ for $C = 3/2^{2/3} \leq 1.89$.*

Proof. Identify $\mathcal{P}([n])$ with $\{0, 1\}^n$. Since \mathcal{A} is sunflower-free, any three distinct vectors $x, y, z \in \mathcal{A}$ have $x_i + y_i + z_i = 2$ for some $i \in [n]$. Fix $m \in \{0, \dots, n\}$. Let $\mathcal{A}_m = \{x \in \mathcal{A} \mid \sum_i x_i = m\}$. The function

$$F : \mathcal{A}_m^3 \rightarrow \mathbb{Q} : (x, y, z) \mapsto \prod_{i=1}^n (2 - (x_i + y_i + z_i))$$

takes a non-zero value if and only if $x = y = z$. (If e.g. $x = y \neq z$, then there must be an i for which $x_i = 1 = y_i \neq z_i$, since all vectors have the same size.) By Lemma 1.1, the slice rank of F is $|\mathcal{A}_m|$.

On the other hand, there exist coefficients $c_{I,J,K}$ so that

$$\prod_{i=1}^n (2 - (x_i + y_i + z_i)) = \sum_{\substack{I,J,K \in \{0,1\}^n \\ |I|+|J|+|K| \leq n}} c_{IJK} x^I y^J z^K.$$

By the pigeonhole principle, $|I| + |J| + |K| \leq n$ implies that one of the summands has to be at most $n/3$, which means that we can find functions f_I, g_J, h_K so that

$$\sum_{\substack{I \in \{0,1\}^n \\ |I| \leq n/3}} x^I f_I(y, z) + \sum_{\substack{J \in \{0,1\}^n \\ |J| \leq n/3}} y^J g_J(x, z) + \sum_{\substack{K \in \{0,1\}^n \\ |K| \leq n/3}} z^K h_K(x, y).$$

This expresses F as a sum of $\leq 3 \sum_{k \leq n/3} \binom{n}{k}$ tensors of slice rank one. We conclude

$$|\mathcal{A}| = \sum_{m=0}^n |\mathcal{A}_m| \leq (n+1)3 \sum_{k=0}^{n/3} \binom{n}{k}.$$

To find the precise bound, note that for any $0 < x < 1$ (by the binomial theorem)

$$x^{-n/3}(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^{k-n/3} > \sum_{k=0}^{n/3} \binom{n}{k} x^{k-n/3} > \sum_{k=0}^{n/3} \binom{n}{k}$$

The function $x^{-1/3}(1+x)$ achieves its maximum at $x = 1/2$ with value $3/2^{2/3}$. □

Conjecture 2.4 (Erdős-Rado Sunflower conjecture). *Let $k \geq 3$. Then there exists a constant $C_k > 0$ so that any m -uniform, k -sunflower free family \mathcal{A} has at most C_k^m elements.*

The Erdős-Rado Sunflower conjecture implies the Erdős-Szemerédi Sunflower conjecture. Alon et al. [1, Theorem 2.6] prove that the Erdős-Rado Sunflower conjecture is equivalent to the conjecture below; their proof works in particular for the case $k = 3$.

Conjecture 2.5. *For every $k \geq 3$, there exists a constant b_k so that for all $D \geq k$ and $A \subseteq (\mathbb{Z}/D\mathbb{Z})^n$ k -sunflower-free we have $|A| \leq b_k^n$.*

A k -sunflower in $(\mathbb{Z}/D\mathbb{Z})^n$ is a set of k elements so that for each coordinate, all k either take the same value or all k take a different value. Naslund and Sawin [5] apply the slice rank method to the conjecture above for the case $k = 3$ and find a better dependence on D than what follows from the cap-set bound.

Theorem 2.6. *Let $D \geq 3$ and $A \subseteq (\mathbb{Z}/D\mathbb{Z})^n$ sunflower-free. Then $|A| \leq (\frac{3}{2^{2/3}}(D-1)^{2/3})^n$.*

Proof. There are D characters $\chi : \mathbb{Z}/D\mathbb{Z} \rightarrow \mathbb{C}^x$, since we have $\chi(0) = 1$ and hence $\chi(1)^D = \chi(D) = 1$. Because

$$\frac{1}{|D|} \sum_{\chi} \chi(a-b) = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases},$$

we find

$$\frac{1}{|D|} \sum_{\chi} \chi(a-b) + \chi(b-c) + \chi(a-c) = \begin{cases} 0 & a, b, c \text{ distinct,} \\ 1 & \text{exactly two of } a, b, c \text{ equal} \\ 3 & a = b = c. \end{cases}$$

This means that

$$T : A \times A \times A \rightarrow \mathbb{C} : (x, y, z) \mapsto \prod_{j=1}^n \left(\frac{1}{|D|} \sum_{\chi} \chi(x_j) \overline{\chi(y_j)} + \chi(y_j) \overline{\chi(z_j)} + \chi(x_j) \overline{\chi(z_j)} - 1 \right)$$

is non-zero if and only if x, y, z form a sunflower (which is impossible) or are equal. This shows $|A|$ equals the slice rank of T (by Lemma 1.1). Again, we can express T as a linear combination of terms

$$\prod_i \chi_i(x_i) \prod_j \psi_j(y_j) \prod_k \xi_k(z_k).$$

Since within each product at most $2n$ character terms are non-trivial (i.e. not equal to the always one character), the pigeonhole principle gives

$$|A| \leq 3 \sum_{k \leq 2n/3} \binom{n}{k} (D-1)^k.$$

(The 3 is from the pigeonhole principle and $D-1$ is the number of non-trivial characters.) Calculations and an amplification argument now give the claimed bound. \square

3 Cap-set bound implies weak sunflower conjecture

Recall that distinct $x, y, z \in \mathbb{F}_3^n$ form a 3-sunflower if for all $i \in [n]$, either $x_i = y_i = z_i$ or x_i, y_i, z_i are distinct, that is, $x_i + y_i + z_i = 0 + 1 + 2 = 0$. This gives the following observation.

Observation 3.1. *A set of vectors \mathcal{A} in \mathbb{F}_3^n is sunflower-free if and only if it is a cap set.*

The cap-set bound hence proves the conjecture below.

Conjecture 3.2 (Cap-set conjecture). *There is an $\epsilon > 0$ and $n_0 \in \mathbb{N}$ so that for $n > n_0$ any set of at least $3^{(1-\epsilon)n}$ vectors in \mathbb{F}_3^n contains a 3-sunflower.*

We will use the following theorem as a black box.

Theorem 3.3 (Theorem 2.4 in [1]). *If the Erdős-Szemerédi Sunflower conjecture does not hold for $k = 3$, then for every $\epsilon > 0$, there exist infinitely many n so that for all $2 \leq c < \frac{1}{\sqrt{\epsilon}}$ there exist sunflower-free families \mathcal{A}_c of n -subsets of $[cn]$ with $|\mathcal{A}_c| \geq \binom{cn}{n}^{1-\epsilon}$.*

Alon et al. [1] use this theorem in their proof that the conjecture below implies the case $k = 3$ of the Erdős-Szemerédi Sunflower conjecture.

Conjecture 3.4 (Auxiliary conjecture). *There is an $\epsilon_0 > 0$ so that for $D > D_0$ and $n > n_0$ any set of at least $D^{(1-\epsilon_0)n}$ vectors in $[D]^n$ contains a 3-sunflower.*

Theorem 3.5. *The Cap-set conjecture implies the case $k = 3$ of the Erdős-Szemerédi Sunflower conjecture.*

Proof. We first show that the Cap-set conjecture implies the Auxiliary conjecture. Suppose the cap set conjecture holds true for $\epsilon > 0$ and $n_0 \in \mathbb{N}$. Let $D_0 = 3$, $\epsilon_0 = \epsilon$ and let $D > D_0$ and $n > n_0$ be given.

Construct an injective map

$$f : [D] \rightarrow \mathbb{F}_3^d$$

for $d = \log_3(D)$ by mapping each element of $[D]$ to its ternary representation. For each $\mathcal{A} \subseteq [D]^n$, we can construct a set $\mathcal{A}' \subseteq f([D]^n) \subseteq \mathbb{F}_3^{dn}$ by mapping $v = (v_1, \dots, v_n)$ to $v' = (f(v_1), \dots, f(v_n))$. Note that if $v', u', w' \in \mathcal{A}'$ form a sunflower for given $v, u, w \in \mathcal{A}$, then $f(v_i), f(u_i), f(w_i) \in f([D])$ are either all the same or all distinct. Since f is injective, we find that v_i, u_i, w_i are either all the same or all distinct (for all $i \in [n]$). Hence $v, u, w \in \mathcal{A}$ form a sunflower as well. Now note that $|\mathcal{A}'| = |\mathcal{A}|$, so that $|\mathcal{A}| \geq D^{(1-\epsilon)n} = 3^{(1-\epsilon)dn}$ implies that \mathcal{A}' has sunflower by assumption (since $dn \geq n > n_0$). This proves the Auxiliary conjecture.

Suppose we have shown the Auxiliary conjecture for ϵ_0, D_0 and n_0 . Suppose Conjecture 2.2 is false. By Theorem 3.3, we can choose $\epsilon < \min(\epsilon_0/2, 1/D_0^2)$ so that for infinitely many n , for each $2 \leq c \leq D_0$ there exists a sunflower-free family \mathcal{A}_c of at least $\binom{cn}{n}^{1-\epsilon}$ subsets of $[cn]$ of size n . We will now give a family of vectors \mathcal{A} in $[D_0]^{D_0n}$ of size at least $D_0^{(1-\epsilon_0)D_0n}$. This will then have a sunflower by our Auxiliary conjecture, which will correspond to a sunflower in one of the \mathcal{A}_c , which we chose sunflower-free

The vectors in \mathcal{A} have entries in \mathbb{F}_{D_0} and have D_0n coordinates. The family \mathcal{A}_{D_0} consists of subsets of $[D_0n]$ of size n ; for each $A_0 \in \mathcal{A}_{D_0}$, we have a vector $v \in \mathcal{A}$ so that $\{i \in [D_0n] : v_i = 0\} = A_0$. In fact, for each $A_0 \in \mathcal{A}_{D_0}$, $A_1 \in \mathcal{A}_{(D_0-1)}, \dots, A_{(D_0-2)} \in \mathcal{A}_2$ we have a vector $v = v_{A_0, \dots, A_{D_0-2}} \in \mathcal{A}$ for which

$$\{i \in [D_0n] : v_i = 0\} = A_0,$$

and for which similarly, after specifying $(j-1)n$ coordinates using A_1, \dots, A_{j-1} , the set $\{i \in [D_0n] : v_i = j\}$ is determined by the coordinates of A_j . After using each A_j to fill in n coordinates, the remaining n coordinates of each vector are set to $D_0 - 1$. This defines a family \mathcal{A} of size

$$|\mathcal{A}_{D_0}| \cdots |\mathcal{A}_2| \geq \left(\binom{D_0n}{n} \binom{(D_0-1)n}{n} \cdots \binom{2n}{n} \right)^{1-\epsilon} = \binom{D_0n}{n, n, \dots, n}^{1-\epsilon} \geq D_0^{D_0n(1-o_n(1))(1-\epsilon_0/2)}$$

using Stirlings approximation formula. This means that for n sufficiently large, we can find such a family \mathcal{A} of size at least $D_0^{(1-\epsilon_0)D_0n}$ as desired.

Finally, suppose $u, v, w \in \mathcal{A}$ form a sunflower. Since \mathcal{A}_{D_0} is sunflower-free, they must have the same set of zeros. Inductively (from D_0 to 2), \mathcal{A}_j being sunflower-free implies that u, v, w take the value $D_0 - j$ on the same coordinates. This implies $u = v = w$, a contradiction. \square

4 Matrix multiplication

Definition 4.1. An Abelian group G with at least two elements and a subset S of G satisfy the **no three disjoint equivoluminous subsets** property if no three non-empty disjoint subsets $T_1, T_2, T_3 \subseteq S$ have the same sum in G . \blacktriangle

Coppersmith and Winograd show that if there exists a sequence of pairs (G_n, S_n) with the no three disjoint equivoluminous subset property so that $\frac{\log(|G_n|)}{|S_n|} \rightarrow 0$, that then fast matrix multiplication is possible (that is, for each $\epsilon > 0$, there is an $O(n^{2+\epsilon})$ time algorithm).

Theorem 4.2. *If the Erdős-Szemerédi Sunflower conjecture holds for $k = 3$ and ϵ_0 , then $\frac{\log(G)}{|S|} \geq \epsilon_0$ for all (G, S) with the no three disjoint equivoluminous subset property.*

Proof. Given S , consider all its $2^{|S|}$ subsets. For each $T \subseteq S$, denote $\sigma(T) = \sum_{g \in T} g$. By the pigeonhole principle, there exists a $g \in G$ so that $|\{T : \sigma(T) = g\}| \geq \frac{2^{|S|}}{|G|}$. If $\frac{\log(G)}{|S|} < \epsilon_0$, then $|G| < 2^{|S|\epsilon_0} = 2^{|S|2^{-|S|+\epsilon_0|S|}}$, so that $2^{|S|}/|G| > 2^{(1-\epsilon_0)|S|}$. Hence we can find a sunflower T'_1, T'_2, T'_3 so that $\sigma(T'_i) = g$ for $i \in \{1, 2, 3\}$. The sets $T_i := T'_i \setminus (T'_1 \cap T'_2 \cap T'_3)$ are disjoint and also have the same sum. This violates the no three disjoint equivoluminous subsets property. \square

They moreover give a “multicolored version” of the Cap-set conjecture and prove that this implies negative results for techniques of Cohn et al. [3].

Conjecture 4.3 (Multicolored sunflower conjecture). *There exists an $\epsilon > 0$ so that for $n > n_0$ every $\mathcal{A} \subseteq \mathbb{F}_3^n \times \mathbb{F}_3^n \times \mathbb{F}_3^n$ of at least $3^{(1-\epsilon)n}$ ordered sunflowers (i.e. triples (x, y, z) so that $\{x, y, z\}$ forms a sunflower in \mathbb{F}_3^n) contains a multicolored sunflower (triples (x_i, y_i, z_i) for $i = 1, 2, 3$ so that x_1, y_2, z_3 form a sunflower).*

Blasiak et al. [2] check that the proof of Ellenberg and Gijswijt works in the multicolored case, disproving the strong USP conjecture. This implies that the method of Cohn et al. [3] cannot work in Abelian groups such as \mathbb{F}_p (although it might still work for e.g. non-Abelian groups).

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