1 Borsuk’s conjecture and related questions

Let $\alpha(d)$ denote the minimum number of pieces so that every bounded subset of $\mathbb{R}^d$ can be partitioned into pieces of strictly smaller diameter. Borsuk’s conjecture (1932): $\alpha(d) = d + 1$.

Borsuk-Ulam theorem (Bart L talk): for $S^n \subset \mathbb{R}^{n+1}$ we need exactly $n + 2$ pieces. Borsuk’s conjecture is also shown to be true for $d = 2, 3$, for smooth convex bodies (‘46), centrally-symmetric bodies (‘71) and bodies of revolution (‘95).

David Larman posed two subquestions in the ‘70s:

- Does the conjecture hold for collections of 0-1 vectors (of constant weight)?
- Does the conjecture hold for 2-distance sets (pairwise distances take two values)? If $E$ is a 2-distance set, and need a piece of strictly smaller diameter, than only one of the two distances may occur.

In 1965 Danzer gave finite sets consisting of 0/1 vectors in $\mathbb{R}^d$ that cannot be covered by $1.003^d$ balls of smaller diameter. Kahn-Kalai (1993) answered the first question negatively for $d = 1325$ and $d > 2014$.

Open questions:

- Small values, e.g. only known that $\alpha(4) \leq 9$.
- Last $d$ for which Borsuk’s conjecture is true. Bondarenko (2013): false for $d \geq 65$.
  Shortly after, Thomas Jenrich and Brouwer (Eindhoven) optimized the idea of Bondarenko with computer assistance to $d = 64$.
- Asymptotic behaviour of $\alpha(d)$, e.g. Kahn-Kalai proved $\alpha(d) \geq (1.2)^{\sqrt{d}}$ and an upper-bound of $(\sqrt{3/2} + \epsilon)^d$ is quoted. It is conjectured that there is a $c > 1$ so that $\alpha(d) > c^d$.

Bondarenko associates the vertices of a specific strongly regular graph to the number $\{1, \ldots, 416\}$ and then creates corresponding vertices $\{x_1, \ldots, x_{416}\} \subset S^{64}$ such that for $i \neq j$, $\langle x_i, x_j \rangle = 1/5$ if $i, j$ non-adjacent and $-1/15$ if they are adjacent. Each “piece” then corresponds to a clique, but the largest clique of the graph is of size 5.
2 From Borsuk’s conjecture to intersections of sets

Let \([d] = \{1, \ldots, d\}\) and \([d]^{(k)} = \{A \subseteq [d] \mid |A| = k\}\). We can see each \(A \in [d]^{(k)}\) as a 0/1-vector in \(1_A \in \mathbb{R}^d\) with \((1_A)_i = 1_{i \in A}\). With the usual Euclidean distance, we then find for \(A, B \in [d]^{(k)}\)
\[
\|1_A - 1_B\|^2 = |A \setminus B| + |B \setminus A| = 2(k - |A \cap B|).
\]

**Lemma 1.** Let \(A \subset [4p]^{(2p)}\) for \(p\) prime. If \(|x \cap y| \neq p\) for all \(x, y \in A\), then \(|A| \leq 2\left(\frac{4p}{p-1}\right)\).

**Theorem 1.** For all \(d \geq 2000\), there is a bounded set \(S \subset \mathbb{R}^d\) such that to break \(S\) into pieces of smaller diameter we need \(\geq c\sqrt{d}\) pieces for some \(c > 1\).

- If pieces are small, then we need many pieces.
- We take \(S \subset [d]^{(k)} \subset \mathbb{R}^d\) where each \(A \in S\) corresponds to a set \(x_A \in [4p]^{(2p)}\).
- By (1), \(d(A, B)\) is maximised for \(|x_A \cap x_B| = p\).

**Proof.** Let \(p\) prime. We set \(n = 4p\) and \(d = (\frac{4p}{2})\). For \(x \in [4p]^{(2p)}\), let \(G_x\) denote the complete bipartite graph on vertex classes \(x\) and \(x^c\). Consider
\[
S = \{G_x : x \in [4p]^{(2p)}\}.
\]

By identifying \([d]\) with the edges of the complete graph \(K_{4p}\), we can view \(G_x\) as a subset of \([d]\). Hence it makes sense to look at \(1_{G_x} \in \mathbb{R}^d\) and by (1), the distance between such vectors is maximal if their intersection is minimal. Note that

number of edges common between \(G_x\) and \(G_y = |x \cap y|^2 + |x \cap y^c|^2 = |x \cap y|^2 + (2p - |x \cap y|)^2\)
is minimal for \(|x \cap y| = p\). Hence if \(S' \subset S\) has strictly smaller diameter than \(S\), then \(S' = \{G_x \mid x \in A\}\) has \(|S'| = |A| \leq 2\left(\frac{4p}{p-1}\right)\). The number of pieces we need is hence at least
\[
\frac{|S|}{2\left(\frac{4p}{p-1}\right)} = \frac{(4p)!}{2^p(4p-2)!} = \frac{(4p)!}{2!3!4!} = \frac{(3p+1)\cdots(2p+1)}{(2p)\cdots p} \geq (3/2)^p \geq c\sqrt{d}.
\]

By Bertrand’s postulate, for each \(n \in \mathbb{N}\) there is a prime number \(p\) so that \(n \leq p \leq 2n\). \(\Box\)

3 Frankl-Wilson on modular intersections

**Theorem 2.** Let \(p\) prime, \(A \subset [n]^r\) and \(\lambda_1, \ldots, \lambda_s \in \mathbb{Z}\) for \(s \leq r\) with \(\lambda_i \not\equiv r \mod p\). If for all \(x, y \in A\) with \(x \neq y\)
\[|x \cap y| \equiv \lambda_i \mod p\]
for some \(i \in \{1, \ldots, s\}\), then \(|A| \leq \binom{n}{s}\).

Theorem implies the Lemma: let \(p\) prime and \(A \subset [4p]^{2p}\) be given so that \(|x \cap y| \neq p\) for all \(x, y \in A\). Let \(\lambda_i = i\) for \(i \in \{1, \ldots, p-1\}\), then \(\lambda_i \not\equiv r \mod p\). Note that \(|x \cap y| \equiv 0 \mod p\) for \(x \neq y\) can only happen if \(|x \cap y| \in \{0, p\}\). The intersection cannot be \(p\) by assumption, and \(x \cap y = \emptyset\) if and only if \(x = y^c\). Halving \(A\) if necessary, we may hence apply the theorem.
The proof of the theorem relies on the linear algebra method: we associate each \( x \in [n]^r \) with a vector \( v_x \) in a vector space of dimension (at most) \( \binom{n}{s} \). By proving that the \( v_x \) for \( x \in \mathcal{A} \) are linear independent, we may then conclude

\[
|\mathcal{A}| = |\{v_x : x \in \mathcal{A}\}| \leq \binom{n}{s},
\]

Another observation that is applied, is that the polynomial

\[
(t - \lambda_1) \cdots (t - \lambda_s)
\]

evaluates to 0 mod \( p \) for \( t = |x \cap y| \) for \( x, y \in \mathcal{A} \) if and only if \( x \neq y \).

**Proof.** Let \( M(i, j) \) denote the \( \binom{n}{i} \times \binom{n}{j} \)-matrix with components

\[
M(i, j)_{xy} = 1_{x \subseteq y}
\]

for \( x \in [n]^{(i)}, y \in [n]^{(j)} \). Let \( V \) be the vector space spanned by the rows of \( M(s, r) \) over \( \mathbb{R} \). We have \( \binom{n}{s} \) rows, so the dimension of \( V \) is at most \( \binom{n}{s} \). Let \( i \in \{0, \ldots, s-1\} \) be given and note that

\[
(M(i, s)M(s, r))_{xy} = \sum_{z \in [n]^s} 1_{x \subseteq z} 1_{z \subseteq y} = M(i, r)_{xy} \binom{r-i}{s-i}.
\]

Premultiplying by a matrix corresponds to taking row operations, so that \( M(i, r) = CM(i, s)M(s, r) \) (for some \( C \in \mathbb{R}^s \)) also has all rows in \( V \). For the same reason, \( M(i) = M(i, r)^T M(i, r) \) has all rows in \( V \). For \( x, y \in [n]^{(r)} \),

\[
(M(i, r)^T M(i, r))_{xy} = \sum_{z \in [n]^i} 1_{x \subseteq z} 1_{z \subseteq y} = \binom{|x \cap y|}{i}.
\]

Recall \( \{\binom{i}{t} : i \in \{0, \ldots, s\}\} \) forms a basis for the polynomials of degree \( \leq s \) over the integers, so we can write the integer polynomial

\[
(t - \lambda_1) \cdots (t - \lambda_s) = \sum_{i=0}^{s} a_i \binom{t}{i}
\]

for certain \( a_i \in \mathbb{Z} \). Let \( M = \sum_{i=0}^{s} a_i M(i) \), then \( M \) has all rows in \( V \) and

\[
M_{xy} = \sum_{i=0}^{s} a_i M(i)_{xy} = \sum_{i=0}^{s} a_i \binom{|x \cap y|}{i} = (|x \cap y| - \lambda_1) \cdots (|x \cap y| - \lambda_s)
\]

is equivalent to zero mod \( p \) for \( x, y \in \mathcal{A} \) if and only if \( x \neq y \). Hence the submatrix corresponding to \( \mathcal{A} \) has linear independent rows over \( \mathbb{Z}_p \), hence over \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) and we may conclude \( |\mathcal{A}| \leq \binom{n}{s} \).

In the paper of Frankl-Wilson, they already note that their theorem implies \( \chi(\mathbb{R}^d) \) has an exponential lower bound (points must get different colours if their distance is exactly 1; let this distance correspond to intersection size \( p \) so that colour classes forbid this intersection size and have to be small). Another corollary is a lower bound for Ramsey numbers \( R(t, t) \): suppose we 2-colour the edges of \( G \) with \( V(G) = [p^3](p^2-1) \) with \( xy \in E(G) \) if and only if \( |x \cap y| \mod p = -1 \). If we have a clique of size \( t \), then only \( p-1, 2p-1, \ldots, p^2-p-1 \) are allowed as intersection sizes; if we have an independent set, then the modular FW applies for \( s = p-1 \). We find \( \chi(\mathbb{R}^n) = \Omega\left(\frac{27}{16}n^{1/8}\right) \) and \( R(t) > t^{\log_2(t)/\log_2 \log_2(t)} \).