Seminar combinatorics (Borsuk's conjecture)

Carla Groenland

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1 Borsuk's conjecture and related questions

Let $\alpha(d)$ denote the minimum number of pieces so that every bounded subset of $\mathbb{R}^d$ can be partitioned into pieces of strictly smaller diameter. Borsuk's conjecture (1932): $\alpha(d) = d + 1$.

Borsuk-Ulam theorem (Bart L talk): for $S^n \subset \mathbb{R}^{n+1}$ we need exactly $n+2$ pieces. Borsuk's conjecture is also shown to be true for $d = 2, 3$, for smooth convex bodies ('46), centrally-symmetric bodies ('71) and bodies of revolution ('95).

David Larman posed two subquestions in the '70s:

- Does the conjecture hold for collections of 0-1 vectors (of constant weight)?
- Does the conjecture hold for 2-distance sets (pairwise distances take two values)? If E is a 2-distance set, and need a piece of strictly smaller diameter, than only one of the two distances may occur.

In 1965 Danzer gave finite sets consisting of 0/1 vectors in $\mathbb{R}^d$ that cannot be covered by 1.003^d balls of smaller diameter. Kahn-Kalai (1993) answered the first question negatively for $d = 1325$ and $d > 2014$.

Open questions:

- Small values, e.g. only known that $\alpha(4) \leq 9$.
- Last d for which Borsuk's conjecture is true. Bondarenko (2013): false for $d \geq 65$. Shortly after, Thomas Jenrich and Brouwer (Eindhoven) optimized the idea of Bondarenko with computer assistance to $d = 64$.
- Asymptotic behaviour of $\alpha(d)$, e.g. Kahn-Kalai proved $\alpha(d) \geq (1.2)^{\sqrt{d}}$ and an upper-bound of $(\sqrt{3/2} + \epsilon)^d$ is quoted. It is conjectured that there is a $c > 1$ so that $\alpha(d) > c^d$.

Bondarenko associates the vertices of a specific strongly regular graph to the number $\{1, \ldots, 416\}$ and then creates corresponding vertices $\{x_1, \ldots, x_{416}\} \subset S^{64}$ such that for $i \neq j$, $\langle x_i, x_j \rangle = 1/5$ if i, j non-adjacent and $-1/15$ if they are adjacent. Each "piece" then corresponds to a clique, but the largest clique of the graph is of size 5.

2 From Borsuk's conjecture to intersections of sets

Let $[d] = \{1, \dots, d\}$ and $[d]^{(k)} = \{A \subseteq [d] \mid |A| = k\}$. We can see each $A \in [d]^{(k)}$ as a 0/1-vector in $1_A \in \mathbb{R}^d$ with $(1_A)_i = 1_{i \in A}$. With the usual Euclidean distance, we then find for $A, B \in [d]^{(k)}$

$$\|1_A - 1_B\|^2 = |A \setminus B| + |B \setminus A| = 2(k - |A \cap B|). \tag{1}$$

Lemma 1. *Let $\mathcal{A} \subset [4p]^{(2p)}$ for p prime. If $|x \cap y| \neq p$ for all $x, y \in \mathcal{A}$, then $|\mathcal{A}| \leq 2\binom{4p}{p-1}$.*

Theorem 1. *For all $d \geq 2000$, there is a bounded set $S \subset \mathbb{R}^d$ such that to break S into pieces of smaller diameter we need $\geq c^{\sqrt{d}}$ pieces for some $c > 1$.*

- If pieces are small, then we need many pieces.
- We take $S \subset [d]^{(k)} \subset \mathbb{R}^d$ where each $A \in S$ corresponds to a set $x_A \in [4p]^{(2p)}$.
- By (1), $d(A, B)$ is maximised for $|x_A \cap x_B| = p$.

Proof. Let p prime. We set $n = 4p$ and $d = \binom{4p}{2}$. For $x \in [4p]^{(2p)}$, let G_x denote the complete bipartite graph on vertex classes x and x^c. Consider

$$S = \{G_x : x \in [4p]^{(2p)}\}.$$

By identifying $[d]$ with the edges of the complete graph K_{4p}, we can view G_x as a subset of $[d]$. Hence it makes sense to look at $1_{G_x} \in \mathbb{R}^d$ and by (1), the distance between such vectors is maximal if their intersection is minimal. Note that

$$\text{number of edges common between } G_x \text{ and } G_y = |x \cap y|^2 + |x \cap y^c|^2 = |x \cap y|^2 + (2p - |x \cap y|)^2$$

is minimal for $|x \cap y| = p$. Hence if $S' \subset S$ has strictly smaller diameter than S, then $S' = \{G_x \mid x \in A\}$ has $|S| = |A| \leq 2\binom{4p}{p-1}$. The number of pieces we need is hence at least

$$\frac{|S|}{2\binom{4p}{p-1}} = \frac{\binom{4p}{2p}}{4\binom{4p}{p-1}} = \frac{(4p)!(p-1)!(3p+1)!}{(4p)!(2p)!(2p)!} = \frac{(3p+1)\cdots(2p+1)}{(2p)\cdots p} \geq (3/2)^p \geq c'^{\sqrt{d}}.$$

By Bertrand's postulate, for each $n \in \mathbb{N}$ there is a prime number p so that $n \leq p \leq 2n$. □

3 Frankl-Wilson on modular intersections

Theorem 2. *Let p prime, $\mathcal{A} \subset [n]^r$ and $\lambda_1, \dots, \lambda_s \in \mathbb{Z}$ for $s \leq r$ with $\lambda_i \not\equiv r \mod p$. If for all $x, y \in \mathcal{A}$ with $x \neq y$*

$$|x \cap y| \equiv \lambda_i \mod p$$

for some $i \in \{1, \dots, s\}$, then $|\mathcal{A}| \leq \binom{n}{s}$.

Theorem implies the Lemma: let p prime and $\mathcal{A} \subset [4p]^{2p}$ be given so that $|x \cap y| \neq p$ for all $x, y \in \mathcal{A}$. Let $\lambda_i = i$ for $i \in \{1, \dots, p-1\}$, then $\lambda_i \not\equiv r \mod p$. Note that $|x \cap y| \equiv 0 \mod p$ for $x \neq y$ can only happen if $|x \cap y| \in \{0, p\}$. The intersection cannot be p by assumption, and $x \cap y = \emptyset$ if and only if $x = y^c$. Halving $\mathcal{A}$ if necessary, we may hence apply the theorem.

The proof of the theorem relies on the *linear algebra method*: we associate each $x \in [n]^r$ with a vector v_x in a vector space of dimension (at most) $\binom{n}{s}$. By proving that the v_x for $x \in \mathcal{A}$ are linear independent, we may then conclude

$$|\mathcal{A}| = |\{v_x : x \in \mathcal{A}\}| \leq \binom{n}{s}.$$

Another observation that is applied, is that the polynomial

$$(t - \lambda_1) \cdots (t - \lambda_s)$$

evaluates to 0 mod p for $t = |x \cap y|$ for $x, y \in \mathcal{A}$ if and only if $x \neq y$.

Proof. Let $M(i,j)$ denote the $\binom{n}{i} \times \binom{n}{j}$-matrix with components

$$M(i,j)_{xy} = 1_{x \subseteq y}$$

for $x \in [n]^{(i)}, y \in [n]^{(j)}$. Let V be the vector space spanned by the rows of $M(s,r)$ over $\mathbb{R}$. We have $\binom{n}{s}$ rows, so the dimension of V is at most $\binom{n}{s}$. Let $i \in \{0, \ldots, s-1\}$ be given and note that

$$(M(i,s)M(s,r))_{xy} = \sum_{z \in [n]^s} 1_{x \subseteq z} 1_{z \subseteq y} = M(i,r)_{xy} \binom{r-i}{s-i}.$$

Premultiplying by a matrix corresponds to taking row operations, so that $M(i,r) = CM(i,s)M(s,r)$ (for some $C \in \mathbb{R}^*$) also has all rows in V. For the same reason, $M(i) = M(i,r)^T M(i,r)$ has all rows in V. For $x, y \in [n]^{(r)}$,

$$(M(i,r)^T M(i,r))_{xy} = \sum_{z \in [n]^i} 1_{z \subseteq x} 1_{z \subseteq y} = \binom{|x \cap y|}{i}.$$

Recall $\{\binom{t}{i} : i \in \{0, \ldots, s\}\}$ forms a basis for the polynomials of degree $\leq s$ over the integers, so we can write the integer polynomial

$$(t - \lambda_1) \cdots (t \ldots \lambda_s) = \sum_{i=0}^{s} a_i \binom{t}{i}$$

for certain $a_i \in \mathbb{Z}$. Let $M = \sum_{i=0}^{s} a_i M(i)$, then M has all rows in V and

$$M_{xy} = \sum_{i=0}^{s} a_i M(i)_{xy} = \sum_{i=0}^{s} a_i \binom{|x \cap y|}{i} = (|x \cap y| - \lambda_1) \cdots (x \cap y - \lambda_s)$$

is equivalent to zero mod p for $x, y \in \mathcal{A}$ if and only if $x \neq 0$. Hence the submatrix corresponding to $\mathcal{A}$ has linear independent rows over $\mathbb{Z}_p$, hence over $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and we may conclude $|\mathcal{A}| \leq \binom{n}{s}$. $\square$

In the paper of Frankl-Wilson, they already note that their theorem implies $\chi(\mathbb{R}^d)$ has an exponential lower bound (points must get different colours if their distance is exactly 1; let this distance correspond to intersection size p so that colour classes forbid this intersection size and have to be small). Another corollary is a lower bound for Ramsey numbers $R(t,t)$: suppose we 2-colour the edges of G with $V(G) = [p^3]^{(p^2-1)}$ with $xy \in E(G)$ if and only if $|x \cap y| \mod p = -1$. If we have a clique of size t, then only $p-1, 2p-1, \ldots, p^2-p-1$ are allowed as intersection sizes; if we have an independent set, then the modular FW applies for $s = p-1$. We find $\chi(\mathbb{R}^n) = \Omega(\frac{27}{16}^{n/8})$ and $R(t) > t^{c \log_2(t)/\log_2 \log_2(t)}$.