On the Colin de Verdière graph parameter

Notes for our seminar
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This is an attempt to define the Colin de Verdière graph parameter purely in terms of the nullspace embedding.

1. The Colin de Verdière graph parameter

Colin de Verdière [1]

Let $G = ([n], E)$ be an undirected graph. The corank of a matrix $M$ is the dimension of its nullspace $\ker(M)$.

The Colin de Verdière parameter $\mu(G)$ [1] is defined to be the maximal corank of any symmetric $n \times n$ matrix $M$ with $M_{i,j} < 0$ if $ij \in E$ and $M_{i,j} = 0$ if $i \neq j$ and $ij \notin E$, with precisely one negative eigenvalue and having the Strong Arnold Property:

\begin{equation}
\text{(1)} \quad \text{there is no nonzero real symmetric } n \times n \text{ matrix } X \text{ with } MX = 0 \text{ and } X_{ij} = 0 \text{ whenever } i \text{ and } j \text{ are equal or adjacent.}
\end{equation}

2. The Strong Arnold Property and quadrics

The Strong Arnold Property of $M$ can be formulated in terms of the nullspace embedding defined by $M$. Let $G = ([n], E)$ be an undirected graph and let $M$ be a symmetric $n \times n$ matrix with $M_{i,j} < 0$ if $ij \in E$ and $M_{i,j} = 0$ if $i \neq j$ and $ij \notin E$, with corank $d$, and with precisely one negative eigenvalue. Let $b_1, \ldots, b_d \in \mathbb{R}^n$ be a basis of $\ker(M)$. Define, for each $i \in [n]$, the vector $u_i \in \mathbb{R}^d$ by: $(u_i)_j := (b_j)_i$, for $j = 1, \ldots, d$. So we have $u : [n] \to \mathbb{R}^d$.

Then $u$ is called the nullspace embedding of $G$ defined by $M$. Note that $u$ is unique up to linear transformations of $\mathbb{R}^d$.

The Strong Arnold Property of $M$ is in fact a property only of the graph $G$ and the function $i \mapsto \langle u_i \rangle$. (Throughout, $\langle \ldots \rangle$ denotes the linear space spanned by $\ldots$.) When we have $u : [n] \to \mathbb{R}^d$, define $|G|$ to be the following subset of $\mathbb{R}^d$:

\begin{equation}
|G| := \bigcup\{\langle u_i \rangle \mid i \in [n]\} \cup \bigcup\{\langle u_i, u_j \rangle \mid ij \in E\}.
\end{equation}

A subset $Q$ of $\mathbb{R}^d$ is called a homogeneous quadric if it is the solution set of a nonzero homogeneous quadratic equation. The following was observed in [3]:

**Proposition 1.** $M$ has the Strong Arnold Property if and only $|G|$ is not contained in any homogeneous quadric.

**Proof.** Let $U$ be the $d \times n$ matrix with as columns the vectors $u_i$ for $i \in [n]$.

Suppose that some homogeneous quadric $Q = \{y \mid y^TNy = 0\}$ contains $|G|$, where $N$ is a nonzero symmetric $d \times d$ matrix. Then $X := U^TNU$ is a nonzero symmetric $n \times n$ matrix that contradicts the Strong Arnold Property [1].
Conversely, suppose that $M$ has not the Strong Arnold Property. Let $X$ be a matrix as in (1). As $MX = 0$ and as $X$ is symmetric, we have $X = U^TNU$ for some nonzero symmetric $d \times d$ matrix $N$. Then $Q := \{y \mid y^TNY = 0\}$ is a homogeneous quadric containing $|G|$. \[ \] 3. $M$ exists iff . . .

Having characterized the Strong Arnold Property in terms of the nullspace embedding, we consider in how much the existence of the corresponding matrix $M$ can be expressed in terms of the nullspace embedding $u_1, \ldots, u_n \in \mathbb{R}^d$.

We can assume that $M$ has eigenvalue $-1$ with eigenvector $1$. Indeed, by Brouwer’s fixed point theorem, there exists $x \geq 0$ with $\sum_{i=1}^n x_i = 1$ and $\Delta_1^2 Mx = \lambda x$ for some $\lambda < 0$. So $(\Delta_1 M \Delta_1)1 = \lambda 1$ for some $\lambda < 0$. As $\Delta_1 M \Delta_1$ has precisely one eigenvalue and the same rank as $M$, we can replace $M$ by $\Delta_1 M \Delta_1$. Scaling $M$ then yields eigenvalue $-1$.

Fix $G = ([n], E)$, a positive $a \in \mathbb{R}^n$, $U \in \mathbb{R}^{n \times d}$, and $W \in \mathbb{R}^{n \times (n-d-1)}$ such that the matrix $[a, U, W]$ is orthogonal. (So $U$ takes the role of $[u_1, \ldots, u_n]^T$.)

Define $p_i := a_i^{-1} u_i$, $v_i := a_i^{-1} w_i$, and $\beta_i := a_i^2$ for each $i$. Then $\sum_{i=1}^n \beta_i p_i = 0$ and $\sum_{i=1}^n \beta_i = 1$.

**Proposition 2.**

(3) There exists a symmetric matrix $M \in \mathbb{R}^{n \times n}$ of corank $d$, with precisely one negative eigenvalue, with eigenvector $a$, and satisfying $MU = 0$, $M_{i,j} = 0$ if $i \neq j$ and $ij \notin E$, $M_{i,j} < 0$ if $ij \in E$,

if and only if

(4) for all $x_1, \ldots, x_n \in \mathbb{R}^d$ and positive semidefinite $P \in \mathbb{R}^{(n-d-1) \times (n-d-1)}$: if

$$(p_i - p_j)^T(x_i - x_j) + \frac{1}{2}(v_i - v_j)^T P (v_i - v_j) \leq 0$$

for each $ij \in E$, then

$$\sum_{i=1}^n \beta_i (p_i^T x_i + \frac{1}{2} v_i^T P v_i) \leq 0,$$

equality implying that $P = 0$ and $(p_i - p_j)^T(x_i - x_j) = 0$ for all $ij \in E$.

**Proof.** Define

(5) $K := \{ K \in \mathbb{R}^{(n-d-1) \times (n-d-1)} \mid K$ symmetric, $(WKW^T)_{i,j} = a_i a_j$ if $i \neq j$ and $ij \notin E$ and $(WKW^T)_{i,j} < a_i a_j$ if $ij \in E\}$

$= \{ K \in \mathbb{R}^{(n-d-1) \times (n-d-1)} \mid K$ symmetric, tr$(KW^T E_{i,j} W) = a_i a_j$ if $i \neq j$ and $ij \notin E$ and $\text{tr}(KW^TW_{i,j} W) < a_i a_j$ if $ij \in E\}$.

Then (3) is equivalent to: $K$ contains a positive definite matrix $K$.

Indeed, we can assume that $M$ has negative eigenvalue $-1$. Then

(6) \[
\begin{bmatrix}
a^T \\
U^T \\
W^T
\end{bmatrix}
\begin{bmatrix}
M[a, U, W] =
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & K
\end{bmatrix}
\]
for some positive definite $K$. Then

$$
M = [a, U, W] \begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & K
\end{bmatrix}
\begin{bmatrix}
a^T \\
U^T \\
W^T
\end{bmatrix} = -aa^T + WKW^T.
$$

So $M$ as in (3) exists if and only if $K$ contains a positive definite matrix. By convexity, this last is equivalent to: there is no nonzero positive semidefinite matrix $P \in \mathbb{R}^{(n-d-1) \times (n-d-1)}$ such that $\text{tr}(PK) \leq 0$ for all $K \in K$, that is, $\text{tr}(PK) > 0$ for some $K \in K$; equivalently: the following system of linear inequalities has a solution $K$:

$$
\begin{align*}
(8) & \\
& \text{(i)} -\text{tr}(PK) < 0, \\
& \text{(ii)} \text{tr}(KW^T E_{i,j} W) < a_i a_j \text{ for each } i,j \in E, \\
& \text{(iii)} \text{tr}(KW^T E_{i,j} W) = a_i a_j \text{ for each } i,j \notin E \text{ with } i \neq j.
\end{align*}
$$

By Motzkin’s transposition theorem (see Corollary 7.1k in [2]), this is equivalent to: for each nonzero positive semidefinite matrix $P \in \mathbb{R}^{(n-d-1) \times (n-d-1)}$: if $\mu \geq 0$ and $B \in \mathbb{R}^{n \times n}$ is symmetric and satisfies $B_{i,i} = 0$ for all $i$, and $B_{i,j} \geq 0$ if $ij \in E$, and $-\mu P + W^T BW = 0$, then

$$
\begin{align*}
(9) & \\
& \text{(i)} a^T Ba \geq 0, \\
& \text{(ii)} \text{if } a^T Ba = 0, \text{ then } \mu = 0 \text{ and } B_{i,j} = 0 \text{ if } ij \in E.
\end{align*}
$$

Since the conditions are homogeneous, we can assume $\mu = 0$ or $\mu = 1$. So the existence of $M$ is equivalent to: for each symmetric $B \in \mathbb{R}^{n \times n}$ with $B_{i,i} = 0$ for all $i$, and $B_{i,j} \geq 0$ if $ij \in E$:

$$
\begin{align*}
(10) & \\
& \text{(i)} \text{ if } W^T BW = 0, \text{ then } a^T Ba \geq 0, \\
& \text{(ii)} \text{ if } W^T BW = 0 \text{ and } a^T Ba = 0, \text{ then } B_{i,j} = 0 \text{ for all } ij \in E, \\
& \text{(iii)} \text{ if } W^T BW \text{ is nonzero and positive semidefinite, then } a^T Ba > 0.
\end{align*}
$$

Conditions (10)(i) and (ii) (i.e., the case $\mu = 0$) are in fact equivalent to: $K \neq \emptyset$. That is, there is no $K \in K$ such that $\text{tr}(PK) < 0$ for all $K \in K$, and such that $Ma = -a$ and $MU = 0$. (So no condition on the other eigenvalues.)

For any symmetric $B \in \mathbb{R}^{n \times n}$ there exist unique $y \in \mathbb{R}^n$, $Z \in \mathbb{R}^{n \times d}$, and a symmetric $P \in \mathbb{R}^{(n-d-1) \times (n-d-1)}$ with

$$
B = [a, U]\begin{bmatrix}
y^T \\
Z^T
\end{bmatrix} + [y, Z]\begin{bmatrix}
a^T \\
U^T
\end{bmatrix} + WPW^T.
$$

Note $P = W^T BW$.

If $B_{i,i} = 0$ we can eliminate $y_i$: since

$$
a_i y_i + u_i^T z_i + a_i y_i + z_i^T u_i + w_i^T P w_i = B_{i,i} = 0,
$$

we have
Therefore, for all $i, j$:

\[ B_{i,j} = a_i y_j + u_i^T z_j + a_j y_i + z_i^T u_j + w_i^T P w_j = a_i (-a_i^{-1}(u_i^T z_i + \frac{1}{2} w_i^T P w_i)) + u_i^T z_j + a_j (-a_i^{-1}(u_i^T z_i + \frac{1}{2} w_i^T P w_i)) + z_i^T u_j + w_i^T P w_j = a_i a_j (-p_j^T x_j + p_i^T x_j - p_i^T x_i + x_i^T p_j - \frac{1}{2} (v_i - v_j)^T P (v_i - v_j)) = -a_i a_j ((p_i - p_j)^T (x_i - x_j) + \frac{1}{2} (v_i - v_j)^T P (v_i - v_j)). \]

where $x_i := a_i^{-1} z_i$ for each $i$. Then, with [11] and [13], since $a^T U = 0$ and $a^T W = 0$:

\[ a^T B a = a^T a y^T a + a^T a y a^T a = -\sum_{i=1}^{n} (2 u_i^T z_i + w_i^T P w_i) = -\sum_{i=1}^{n} \beta_i (2 p_i^T x_i + v_i^T P v_i). \]

Therefore, the condition: for each symmetric $B \in \mathbb{R}^{n \times n}$ with $B_{i,i} = 0$ for all $i$, and $B_{i,j} \geq 0$ if $ij \in E$ [10] holds, is equivalent to [4].

Set $P = Q^T Q$ for some matrix $Q \in \mathbb{R}^{s \times (n-1)}$ for some $s$. Consider $p_i(t) := (p_i + t x_i, \sqrt{t} Q v_i) \in \mathbb{R}^{d+s}$ for $t \in \mathbb{R}$, for each $i$.

If $P = 0$, then

\[ x_i = \frac{d}{dt} p_i(t) \big|_{t=0}. \]

Hence

\[ (p_i - p_j)^T (x_i - x_j) = \frac{1}{2} \frac{d}{dt} |p_i(t) - p_j(t)|^2 \big|_{t=0} \]

and

\[ \sum_{i=1}^{n} \beta_i p_i^T x_i = \frac{1}{2} \frac{d}{dt} \sum_{i=1}^{n} \beta_i |p_i(t)|^2 \big|_{t=0}. \]

References

