For any $n \in \mathbb{N}$, let $[n] := \{1, \ldots, n\}$. Let $n, k, t \in \mathbb{N}$. Call a function $\varphi : [n]^t \to [n]^k$ an embedding if $\varphi$ is injective and for each $i \in [k]$ either $\varphi(x)_i$ is constant over $x \in [n]^t$ or there exists $j \in [t]$ such that $\varphi(x)_i = x_j$ for all $x \in [n]^t$. Call a function $\chi$ on $[n]^t$ reducible if $\chi(x) = \chi(y)$ whenever $y$ arises from $x$ by replacing each value $n$ by $n - 1$.

Shelah [2] showed:

**Theorem** (Shelah’s Claim). For $r, n, t \in \mathbb{N}$ with $n \geq 2$ there exists $k = k(r, n, t)$ such that for each $\chi : [n]^k \to [r]$ there exists an embedding $\varphi : [n]^t \to [n]^k$ with $\chi \circ \varphi$ reducible.

**Proof.** By induction on $t$, with $k(r, n, 0) = 0$ being trivial. If $t \geq 1$, define

\[(1) \quad k(r, n, t) := k(r^n, n, t - 1) + r^{n^k(r^n, n, t - 1)}.\]

Set $k := k(n, r, t)$, $\ell := k(r^n, n, t - 1)$, and $m := r^{n^k(r^n, n, t - 1)}$, and choose $\chi : [n]^k \to [r]$.

For each $i = 0, \ldots, m$, let $w_i$ be the word $(n - 1)^m i n^i$ in $[n]^m$, and define $\chi_i : [n]^\ell \to [r]$ by $\chi_i(w) := \chi(wu_i)$ for $w \in [n]^\ell$. Since $m + 1 > m = r^n\ell$, there exist $i < j \in \{0, \ldots, m\}$ such that $\chi_i = \chi_j$. Define the embedding $\psi : [n]^1 \to [n]^m$ by $\psi(a) = (n - 1)^m a^i i n^i$ for $a \in [n]$. So $\chi(\psi(w(n - 1))) = \chi(\psi(w(n)))$ for all $w \in [n]^\ell$.

Next define $\chi' : [n]^\ell \to [r]^n$ by

\[(2) \quad \chi'(w) := (\chi(\psi(1)), \ldots, \chi(\psi(n)))\]

for $w \in [n]^\ell$. By the induction hypothesis and by definition of $\ell$, there exists an embedding $\varphi' : [n]^\ell \to [n]^k$ with $\chi' \circ \varphi'$ reducible. Then $\varphi := \varphi' \times \psi$ is an embedding $[n]^t \times [n]^1 \to [n]^\ell \times [n]^m$ with $\chi \circ \varphi$ reducible.

This implies the theorem of Hales and Jewett [1]:

**Corollary** (Hales-Jewett theorem). For each $r, n \in \mathbb{N}$ there exists $m = m(r, n)$ such that for each $\chi : [n]^m \to [r]$ there exists an embedding $\varphi : [n]^1 \to [n]^m$ with $\chi \circ \varphi$ constant.

**Proof.** By induction on $n$, the case $n = 1$ being trivial. Assume $n \geq 2$. Let $t := m(r, n - 1)$ and $m := m(r, n) := k(r, n, t)$. By Shelah’s Claim, there exists an embedding $\varphi : [n]^t \to [n]^m$ with $\chi \circ \varphi$ reducible. By the induction hypothesis, there exists an embedding $\psi : [n - 1]^1 \to [n - 1]^1$ with $\chi \circ \varphi \circ \psi$ constant. We can extend $\psi$ to an embedding $\psi' : [n]^1 \to [n]^t$. Then $\varphi \circ \psi'$ is an embedding $[n]^1 \to [n]^m$ with $\chi \circ \varphi \circ \psi'$ constant.

**References**
