Tutte-Berge ⇒ Gallai ⇒ Mader

Notes for our seminar
Lex Schrijver

1. The Tutte-Berge formula

For any graph G, let $\nu(G)$ denote the maximum size of a matching in G. Moreover, let $\mathcal{K}(G)$ denote the set of components of G.

Berge [1] derived the following from the characterization of Tutte [5] of the existence of a perfect matching in a graph:

Theorem 1 (Tutte-Berge formula)**.** *Let $G = (V, E)$ be a graph. Then*

$$(1) \qquad \nu(G) = \min_{U \subseteq V} |U| + \sum_{K \in \mathcal{K}(G-U)} \lfloor \tfrac{1}{2}|K| \rfloor.$$

Proof. The maximum is at most the minimum, since for each $U \subseteq V$, each edge of G intersects U or is contained in a component of $G - U$. As U intersects at most $|U|$ disjoint edges, and as any component K contains at most $\lfloor \frac{1}{2}|K| \rfloor$ disjoint edges, we have $\leq$ in (1).

We prove the reverse inequality by induction on $|V|$, the case $V = \emptyset$ being trivial. We can assume that G is connected, as otherwise we can apply induction to the components of G.

First assume that there exists a vertex v covered by all maximum-size matchings. Then $\nu(G - v) = \nu(G) - 1$, and by induction there exists a subset U' of $V \setminus \{v\}$ with

$$(2) \qquad \nu(G - v) = |U'| + \sum_{K \in \mathcal{K}(G-v-U)} \lfloor \tfrac{1}{2}|K| \rfloor.$$

Then $U := U' \cup \{v\}$ gives equality in (1).

So we can assume that there is no such v. We show $2\nu(G) \geq |V| - 1$, which implies $\nu(G) \geq \lceil \frac{1}{2}(|V| - 1) \rceil = \lfloor \frac{1}{2}|V| \rfloor$. Taking $U = \emptyset$ then gives the theorem.

Indeed suppose to the contrary that $2\nu(G) \leq |V| - 2$. So each maximum-size matching M misses at least two distinct vertices u and v. Among all such M, u, v, choose them such that the distance $\text{dist}(u, v)$ of u and v in G is as small as possible.

If $\text{dist}(u, v) = 1$, then u and v are adjacent, and hence we can augment M by uv, contradicting the maximality of $|M|$. So $\text{dist}(u, v) \geq 2$, and hence we can choose an intermediate vertex t on a shortest $u - v$ path. By assumption, there exists a maximum-size matching N missing t.

Consider the component P of the graph $(V, M \cup N)$ containing t. As N misses t, P is a path with end t. As M and N are maximum-size matchings, P contains an equal number of edges in M as in N. Since M misses u and v, P cannot cover both u and v. So by symmetry we can assume that P misses u. Exchanging M and N on P, M becomes a maximum-size matching missing both u and t. Since $\text{dist}(u, t) < \text{dist}(u, v)$, this contradicts the minimality of $\text{dist}(u, v)$. ∎

2. Gallai's theorem

Let $G = (V, E)$ be a graph and let $T \subseteq V$. A path is called a *T-path* if its ends are distinct vertices in T and no internal vertex belongs to T.

Gallai [2] derived the following from the Tutte-Berge formula.

Theorem 2 (Gallai's disjoint T-paths theorem)**.** *Let $G = (V, E)$ be a graph and let $T \subseteq V$. The maximum number of disjoint T-paths is equal to*

$$\min_{U \subseteq V} |U| + \sum_{K \in \mathcal{K}(G-U)} \lfloor \tfrac{1}{2}|K \cap T| \rfloor. \tag{3}$$

Proof. The maximum is at most the minimum, since for each $U \subseteq V$, each T-path intersects U or has both ends in $K \cap T$ for some component K of $G - U$.

To see equality, let μ be equal to the minimum value of (3). Let the graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ arise from G by adding a disjoint copy G' of $G - T$, and making the copy v' of each $v \in V \setminus T$ adjacent to v and to all neighbours of v in G. By the Tutte-Berge formula, $\widetilde{G}$ has a matching M of size $\mu + |V \setminus T|$. To see this, we must prove that for any $\widetilde{U} \subseteq \widetilde{V}$:

$$|\widetilde{U}| + \sum_{\widetilde{K} \in \mathcal{K}(\widetilde{G}-\widetilde{U})} \lfloor \tfrac{1}{2}|\widetilde{K}| \rfloor \geq \mu + |V \setminus T|. \tag{4}$$

Now if for some $v \in V \setminus T$ exactly one of v, v' belongs to $\widetilde{U}$, then we can delete it from $\widetilde{U}$, thereby not increasing the left-hand side of (4).

So we can assume that for each $v \in V \setminus T$, either $v, v' \in \widetilde{U}$ or $v, v' \notin \widetilde{U}$. Define $U := \widetilde{U} \cap V$. Then each component K of $G - U$ is equal to $\widetilde{K} \cap V$ for some component $\widetilde{K}$ of $\widetilde{G} - \widetilde{U}$. Hence

$$|\widetilde{U}| + \sum_{\widetilde{K} \in \mathcal{K}(\widetilde{G}-\widetilde{U})} \lfloor \tfrac{1}{2}|\widetilde{K}| \rfloor = |U| + |V \setminus T| + \sum_{K \in \mathcal{K}(G-U)} \lfloor \tfrac{1}{2}|K \cap T| \rfloor \geq \mu + |V \setminus T|. \tag{5}$$

Thus we have (4).

So $\widetilde{G}$ has a matching M of size $\mu + |V \setminus T|$. Let N be the matching $\{vv' \mid v \in V \setminus T\}$ in $\widetilde{G}$. As $|M| = \mu + |V \setminus T| = \mu + |N|$, the union $M \cup N$ has at least μ components with more edges in M than in N. Each such component is a path connecting two vertices in T. Then contracting the edges in N yields μ disjoint T-paths in G. ∎

3. Mader's theorem

Let $G = (V, E)$ be a graph and let $\mathcal{S}$ be a collection of disjoint nonempty subsets of V. A path in G is called an *$\mathcal{S}$-path* if it connects two different sets in $\mathcal{S}$ and has no internal vertex in any set in $\mathcal{S}$. Denote $T := \bigcup \mathcal{S}$.

Mader [3] showed the following (we follow the proof of [4], deriving Mader's theorem from Gallai's theorem).

Theorem 3 (Mader's disjoint $\mathcal{S}$-paths theorem). *The maximum number of disjoint $\mathcal{S}$-paths is equal to the minimum value of*

$$(6) \qquad |U_0| + \sum_{i=1}^{n} \lfloor \tfrac{1}{2}|B_i| \rfloor,$$

taken over all partitions $U_0, \ldots, U_n$ of V (over all n) such that each $\mathcal{S}$-path intersects U_0 or traverses some edge spanned by some U_i. Here B_i denotes the set of vertices in U_i that belong to T or have a neighbour in $V \setminus (U_0 \cup U_i)$.

Proof. Let μ be the minimum value of (6). Trivially, the maximum number of disjoint $\mathcal{S}$-paths is at most μ, since any $\mathcal{S}$-path disjoint from U_0 and traversing an edge spanned by U_i, traverses at least two vertices in B_i.

To prove the reverse inequality, fix V, and choose a counterexample E, $\mathcal{S}$ minimizing

$$(7) \qquad |E| - |\{\{x, y\} \mid x, y \in V, \exists X, Y \in \mathcal{S} : x \in X, y \in Y, X \neq Y\}|.$$

Then each $X \in \mathcal{S}$ is a stable set of G, since deleting any edge e spanned by X does not change the maximum and minimum value in Mader's theorem (as no $\mathcal{S}$-path traverses e and as deleting e does not change any set B_i), while it decreases (7).

Moreover, $|\mathcal{S}| \geq 2$, since if $|\mathcal{S}| \leq 1$, no $\mathcal{S}$-paths exist, and we can tale $U_0 = \emptyset$ and for the sets $U_1, \ldots, U_n$ all singletons from V.

If $|X| = 1$ for each $X \in \mathcal{S}$, the theorem reduces to Gallai's disjoint T-paths theorem: we can take for U_0 any set U minimizing (3), and for $U_1, \ldots, U_n$ the components of $G - U$.

So $|X| \geq 2$ for some $X \in \mathcal{S}$. Choose $s \in X$. Define

$$(8) \qquad \mathcal{S}' := (\mathcal{S} \setminus \{X\}) \cup \{X \setminus \{s\}, \{s\}\}.$$

Replacing $\mathcal{S}$ by $\mathcal{S}'$ does not decrease the minimum in Mader's theorem (as each $\mathcal{S}$-path is an $\mathcal{S}'$-path and as $\bigcup \mathcal{S}' = T$). But it decreases (7), hence there exists a collection $\mathcal{P}$ of μ disjoint $\mathcal{S}'$-paths.

Necessarily, there is a path $P_0 \in \mathcal{P}$ connecting s with another vertex in X (otherwise $\mathcal{P}$ forms μ disjoint $\mathcal{S}$-paths). Then all other paths in $\mathcal{P}$ are $\mathcal{S}$-paths. Let u be an internal vertex of P_0 (u exists, since X is a stable set). Define

$$(9) \qquad \mathcal{S}'' := (\mathcal{S} \setminus \{X\}) \cup \{X \cup \{u\}\}.$$

Replacing $\mathcal{S}$ by $\mathcal{S}''$ does not decrease the minimum in Mader's theorem (as each $\mathcal{S}$-path is an $\mathcal{S}''$-path and as $\bigcup \mathcal{S}'' \supseteq T$). But it decreases (7), hence there exists a collection $\mathcal{Q}$ of μ disjoint $\mathcal{S}''$-paths. Choose $\mathcal{Q}$ such that $\mathcal{Q}$ uses a minimal number of edges not used by $\mathcal{P}$.

Necessarily, u is an end of some path $Q_0 \in \mathcal{Q}$ (otherwise $\mathcal{Q}$ forms μ disjoint $\mathcal{S}$-paths). Then all other paths in $\mathcal{Q}$ are $\mathcal{S}$-paths. As $|\mathcal{P}| = |\mathcal{Q}|$ and as u is not an end of any path in $\mathcal{P}$, there exists an end r of some path $P \in \mathcal{P}$ that is not an end of any path in $\mathcal{Q}$.

Then P intersects some path in $\mathcal{Q}$ (otherwise $(\mathcal{Q} \setminus \{Q_0\}) \cup \{P\}$ would form μ disjoint $\mathcal{S}$-paths). So when following P starting from r, there is a first vertex w that is on some path in $\mathcal{Q}$, say on $Q \in \mathcal{Q}$.

Let t' and t'' be the ends of Q, and let Q' and Q'' be the $w - t'$ and $w - t''$ subpaths of Q (possibly of length 0). Let P' be the $r - w$ part of P, and let Y be the set in $\mathcal{S}''$ containing r. Then

(10) $\quad t'' \notin Y$ implies $EQ' \subseteq EP$; similarly: $t' \notin Y$ implies $EQ'' \subseteq EP$.

Indeed, if $t'' \notin Y$ and $EQ' \not\subseteq EP$, we can replace part Q' of Q by P', to obtain a collection $\mathcal{Q}'$ of μ disjoint $\mathcal{S}''$-paths with a fewer number of edges not used by $\mathcal{P}$. This contradicts our minimality assumption. So we have the first statement in (10), and by symmetry also the second.

Since Q is an $\mathcal{S}''$-path, at least one of t', t'' does not belong to Y. By symmetry we can assume that $t'' \notin Y$. So by (10), $EQ' \subseteq EP$.

If $P \neq P_0$, then $\bigcup \mathcal{S}''$ intersects P only in the ends of P. So $EQ' \subseteq EP$ implies that t' is the other end of P (than r). As $r \in Y$, we know $t' \notin Y$. So by (10), $EQ'' \subseteq EP$, hence also t'' is the other end of P. So $t'' = t'$, a contradiction.

So $P = P_0$. As Y contains r and as both ends of P_0 belong to X, we know $Y = X \cup \{u\}$. Moreover, w must be on the $r - u$ part of P_0 (since u is covered by Q_0 and since w is the first vertex from r on P_0 covered by $\mathcal{Q}$). So $t' = u$, and hence, as t' is an end of Q, we know $Q = Q_0$. Also, Q' is equal to the $w - u$ part of P. As $u \in Y$, we know $t'' \notin Y$, so the path $P'Q''$ is an $\mathcal{S}$-path. So replacing $Q_0 = Q'Q''$ by $P'Q''$ gives μ disjoint $\mathcal{S}$-paths, as required. ∎

References

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[4] A. Schrijver, A short proof of Mader's $\mathcal{S}$-paths theorem, *Journal of Combinatorial Theory, Series B* 82 (2001) 319–321.

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