A proof of Razmyslov's theorem

Notes for our seminar

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Let $d, n \in \mathbb{N}$. Let ρ be the representation $S_n \to \mathrm{End}(\mathbb{C}^d)^{\otimes n}$ given by

$$\rho(\pi)(x_1 \otimes \cdots \otimes x_n) = x_{\pi(1)} \otimes \cdots \otimes x_{\pi(n)} \tag{1}$$

for $\pi \in S_n$ and $x_1, \ldots, x_n \in \mathbb{C}^d$. Note that

$$\mathrm{tr}(\rho(\pi)(X_1 \otimes \ldots \otimes X_n)) = \prod_{\omega \in \Omega_\pi} \mathrm{tr}(\prod_{i=1}^{|\omega|} X_{\pi^i(m_\omega)}) \tag{2}$$

for $\pi \in S_n$ and $X_1, \ldots, X_n \in \mathrm{End}(\mathbb{C}^d)$, where Ω_π is the set of orbits of π and where for any $\omega \in \Omega_\pi$, m_ω is the smallest (equivalently: an arbitrary) element of ω.

For each $\lambda \vdash n$, let K_λ be the isotypical component of $\mathbb{C}S_n$ corresponding to the irreducible representation r_λ. Define

$$L_{\leq d} := \bigoplus_{\substack{\lambda \vdash n \\ \mathrm{height}(\lambda) \leq d}} K_\lambda \quad \text{and} \quad L_{>d} := \bigoplus_{\substack{\lambda \vdash n \\ \mathrm{height}(\lambda) > d}} K_\lambda. \tag{3}$$

Then (cf., e.g., [1]):

$$\mathrm{Ker}(\rho) = L_{>d}. \tag{4}$$

This implies for any two subspaces U, V of $\mathbb{C}S_n$:

$$\rho(U) \subseteq \rho(V) \text{ if and only if } L_{>d} + U \subseteq L_{>d} + V. \tag{5}$$

Let A_n be the alternating subgroup of S_n, and let $A_n^c := S_n \setminus A_n$. Consider the linear function $\varphi : \mathbb{C}S_n \to \mathbb{C}S_n$ determined by

$$\varphi(\pi) := \mathrm{sgn}(\pi)\pi \tag{6}$$

for $\pi \in S_n$. So $\varphi^2 = \mathrm{id}$.

Lemma 1. *For each $\lambda \vdash n$: $\varphi(K_\lambda) = K_{\lambda^*}$.*

Proof. Let Y and Y^* be the Young shapes corresponding to λ and λ^*, respectively, let $T : Y \to Y^*$ be defined by $T(i,j) = (j,i)$ for $(i,j) \in Y$, let $\tilde{\varphi} : S_Y \to S_Y$ be defined by $\tilde{\varphi}(\pi) = \mathrm{sgn}(\pi)\pi$ for $\pi \in S_Y$, let H_Y and V_Y be the subgroups of S_Y of row-stable and column-stable permutations of Y, respectively, and let $h_Y := \sum_{h \in H_Y} h$ and $v_Y := \sum_{v \in V_Y} \mathrm{sgn}(v)v$. Then

$$\begin{aligned} \tilde{\varphi}(v_Y h_Y) &= \sum_{v \in V_Y} \sum_{h \in H_Y} \mathrm{sgn}(v)\tilde{\varphi}(vh) = \sum_{v \in V_Y} \sum_{h \in H_Y} \mathrm{sgn}(v)\mathrm{sgn}(vh)vh = \\ &\sum_{v \in V_Y} \sum_{h \in H_Y} \mathrm{sgn}(h)vh = \sum_{h \in H_{Y^*}} \sum_{v \in V_{Y^*}} \mathrm{sgn}(v)T^{-1}hvT = T^{-1}h_{Y^*}v_{Y^*}T. \end{aligned} \tag{7}$$

Hence for $y, z : [n] \to Y$, one has

(8) $$\varphi(y^{-1}v_Y h_Y z) = \mathrm{sgn}(y^{-1}z)y^{-1}\varphi(v_Y h_Y)z = \mathrm{sgn}(y^{-1}z)y^{-1}T^{-1}h_{Y^*}v_{Y^*}Tz \in K_{\lambda^*}.$$

As K_λ is spanned by all $y^{-1}v_Y h_Y z$ with bijections $y, z : [n] \to Y$, we have the lemma. ∎

Lemma 2. *For each subspace U of $\mathbb{C}S_n$: $U \cap \varphi(U) = (U \cap \mathbb{C}A_n) + (U \cap \mathbb{C}A_n^c)$.*

Proof. If $x \in U \cap \varphi(U)$, then $\varphi(x) \in U$, hence $x = \frac{1}{2}(x+\varphi(x)) + \frac{1}{2}(x-\varphi(x)) \in (U \cap \mathbb{C}A_n) + (U \cap \mathbb{C}A_n^c)$.

Conversely, if $x \in U \cap \mathbb{C}A_n$ then $x = \varphi(x)$, hence $x \in U \cap \varphi(U)$. Similarly, if $x \in U \cap \mathbb{C}A_n^c$ then $x = -\varphi(x)$, hence $x \in U \cap \varphi(U)$. ∎

Theorem 1. $\rho(\mathbb{C}A_n) = \rho(\mathbb{C}S_n) = \rho(\mathbb{C}A_n^c)$ *if and only if* $n > d^2$.

Proof. $\rho(\mathbb{C}A_n) = \rho(\mathbb{C}S_n) = \rho(\mathbb{C}A_n^c) \overset{(5)}{\iff} L_{>d} + \mathbb{C}A_n = \mathbb{C}S_n = L_{>d} + \mathbb{C}A_n^c \iff L_{\le d} \cap \mathbb{C}A_n^c = \{0\} = L_{\le d} \cap \mathbb{C}A_n \overset{\text{Lemma 2}}{\iff} L_{\le d} \cap \varphi(L_{\le d}) = \{0\} \overset{\text{Lemma 1}}{\iff}$ no $\lambda \vdash n$ satisfies $\mathrm{height}(\lambda) \le d$ and $\mathrm{height}(\lambda^*) \le d \iff n > d^2$. ∎

This implies the result of Razmyslov [2]:

Corollary 1a. *If $n > d^2$, then $\mathrm{tr}(X_1 \cdots X_n)$ is a linear combination of products of traces of products of fewer than n of the X_i, where $X_1, \ldots, X_n$ are variable $d \times d$ matrices.*

Note that in fact if $n > d^2$, then $\mathrm{tr}(X_1 \cdots X_n)$ is a linear combination of products of *an even number of* traces of products of the X_i.

By the above, the conclusion of Corollary 1a holds for some fixed n and d if and only if $L_{>d} + U = \mathbb{C}S_n$, where U is the subspace of $\mathbb{C}S_n$ spanned by the permutations with at least two orbits. Note that $L_{>d} + U = \mathbb{C}S_n$ is equivalent to $L_{\le d} \cap V = \{0\}$, where V is the subspace of $\mathbb{C}S_n$ spanned by the permutations with only one orbit.

References

[1] P. Cvitanović, *Group Theory: Birdtracks, Lie's, and Exceptional Groups*, Princeton University Press, Princeton, 2008.

[2] Ju.P. Razmyslov, Trace identities of full matrix algebras over a field of characteristic zero [in Russian], *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya* 38 (1974) 723–756 [English translation: *Mathematics of the USSR. Izvestija* 8 (1974) 727–760].