On the Shannon capacity of sums and products of graphs

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$\Theta(G)$ is the Shannon capacity of a graph G.
$\alpha(G)$ is the stable set number of a graph G.
GH is the strong product of graphs G and H.
$G+H$ is the disjoint union of graphs G and H.

Proposition 1. $\Theta(GH) > \Theta(G)\Theta(H)$ *if and only if* $\Theta(G+H) > \Theta(G) + \Theta(H)$.

Proof. First assume $\Theta(GH) > \Theta(G)\Theta(H)$. Then (using $\Theta(X+Y) \geq \Theta(X) + \Theta(Y)$):

(1) $$\Theta(G+H)^2 = \Theta((G+H)^2) = \Theta(G^2 + 2GH + H^2) \geq \Theta(G^2) + 2\Theta(GH) + \Theta(H^2) = \Theta(G)^2 + 2\Theta(GH) + \Theta(H)^2 > \Theta(G)^2 + 2\Theta(G)\Theta(H) + \Theta(H)^2 = (\Theta(G) + \Theta(H))^2.$$

Second assume $\Theta(GH) \leq \Theta(G)\Theta(H)$. Then for all i, j (using $\Theta(X)\Theta(Y)\Theta(Z) \leq \Theta(XYZ)$):

(2) $$\Theta(G^iH^j)\Theta(G)^j\Theta(H)^i = \Theta(G^iH^j)\Theta(G^j)\Theta(H^i) \leq \Theta((GH)^{i+j}) = \Theta(GH)^{i+j} \leq \Theta(G)^{i+j}\Theta(H)^{i+j}.$$

So $\Theta(G^iH^j) \leq \Theta(G)^i\Theta(H)^j$. Hence for each k (using $\alpha(X+Y) = \alpha(X) + \alpha(Y)$):

(3) $$\alpha((G+H)^k) = \alpha(\sum_{i=0}^{k} \binom{k}{i} G^iH^{k-i}) = \sum_{i=0}^{k} \binom{k}{i} \alpha(G^iH^{k-i}) \leq \sum_{i=0}^{k} \binom{k}{i} \Theta(G^iH^{k-i}) \leq \sum_{i=0}^{k} \binom{k}{i} \Theta(G)^i\Theta(H)^{k-i} = (\Theta(G) + \Theta(H))^k.$$

Taking k-th roots and $k \to \infty$ gives $\Theta(G+H) \leq \Theta(G) + \Theta(H)$. ∎