Survey of Problems, Questions, and Conjectures

We here collect unsolved problems, questions, and conjectures mentioned in this book. For terminology and background, we refer to the pages indicated.

1 (page 41). Is NP $\neq$ P?

2 (page 42). Is P = NP $\cap$ co-NP?

3 (page 65). The Hirsch conjecture: A $d$-dimensional polytope with $m$ facets has diameter at most $m - d$.

4 (page 161). Is there an $O(nm)$-time algorithm for finding a maximum flow?

5 (page 232). Berge [1982b] posed the following conjecture generalizing the Gallai-Milgram theorem. Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}^+$. Then for each path collection $P$ partitioning $V$ and minimizing

$$\sum_{P \in P} \min\{|VP|, k\},$$

there exist disjoint stable sets $C_1, \ldots, C_k$ in $D$ such that each $P \in P$ intersects $\min\{|VP|, k\}$ of them. This was proved by Saks [1986] for acyclic graphs.

6 (page 403). The following open problem was mentioned by Fulkerson [1971b]: Let $A$ and $B$ be families of subsets of a set $S$ and let $w \in \mathbb{Z}^S$. What is the maximum number $k$ of common transversals $T_1, \ldots, T_k$ of $A$ and $B$ such that

$$\chi^{T_1} + \cdots + \chi^{T_k} \leq w?$$

7 (page 459). Can the weighted matching problem be formulated as a linear programming problem of size bounded by a polynomial in the size of the graph, by extending the set of variables? That is, is the matching polytope of a graph $G = (V, E)$ equal to the projection of some polytope $\{x \mid Ax \leq b\}$ with $A$ and $b$ having size bounded by a polynomial in $|V| + |E|$?

8 (pages 472, 646). The 5-flow conjecture of Tutte [1954a]:

(3) (?) each bridgeless graph has a nowhere-zero 5-flow. (?)
(A nowhere-zero $k$-flow is a flow over $\mathbb{Z}_k$ in some orientation of the graph, taking value 0 nowhere.)

9 (pages 472,498,645,1426). The 4-flow conjecture of Tutte [1966]:

(4) (?) each bridgeless graph without Petersen graph minor has a nowhere-zero 4-flow. (?)

This implies the four-colour theorem. For cubic graphs, (4) was proved by Robertson, Seymour, and Thomas [1997], Sanders, Seymour, and Thomas [2000], and Sanders and Thomas [2000].

Seymour [1981c] showed that the 4-flow conjecture is equivalent to the following more general conjecture, also due to Tutte [1966]:

(5) (?) each bridgeless matroid without $F^*_7$, $M^*(K_5)$, or $M(P_{10})$ minor has a nowhere-zero flow over $\mathbb{GF}(4)$. (?)

Here $P_{10}$ denotes the Petersen graph.


(6) (?) each 4-edge-connected graph has a nowhere-zero 3-flow. (?)

11 (page 473). The weak 3-flow conjecture of Jaeger [1988]:

(7) (?) there exists a number $k$ such that each $k$-edge-connected graph has a nowhere-zero 3-flow. (?)

12 (page 473). The following circular flow conjecture of Jaeger [1984] generalizes both the 3-flow and the 5-flow conjecture:

(8) (?) for each $k \geq 1$, any $4k$-connected graph has an orientation such that in each vertex, the indegree and the outdegree differ by an integer multiple of $2k + 1$. (?)

13 (pages 475,645). The generalized Fulkerson conjecture of Seymour [1979a]:

(9) (?) $\lceil \chi''(G) \rceil = \lceil \frac{1}{2} \chi'(G_2) \rceil$ (?)

for each graph $G$. (Here $\chi''(G)$ denotes the fractional edge-colouring number of $G$, and $G_2$ the graph obtained from $G$ by replacing each edge by two parallel edges.) This is equivalent to the conjecture that

(10) (?) for each $k$-graph $G$ there exists a family of $2k$ perfect matchings, covering each edge precisely twice. (?)

(A $k$-graph is a $k$-regular graph $G = (V, E)$ with $|\delta(U)| \geq k$ for each odd-size subset $U$ of $V$.)

(A nowhere-zero $k$-flow is a flow over $\mathbb{Z}_k$ in some orientation of the graph, taking value 0 nowhere.)
14 (pages 476,645). Fulkerson [1971a] asked if in each bridgeless cubic graph there exist 6 perfect matchings, covering each edge precisely twice (the Fulkerson conjecture). It is a special case of Seymour’s generalized Fulkerson conjecture.

15 (page 476). Berge [1979a] conjectures that the edges of any bridgeless cubic graph can be covered by 5 perfect matchings. (This would follow from the Fulkerson conjecture.)

16 (page 476). Gol’dberg [1973] and Seymour [1979a] conjecture that for each (not necessarily simple) graph $G$ one has

\[ \chi'(G) \leq \max\{\Delta(G) + 1, |\chi''(G)|\}. \]

An equivalent conjecture was stated by Andersen [1977].

17 (page 476). Seymour [1981c] conjectures the following generalization of the four-colour theorem:

\[ \chi'(G) \leq \max\{\Delta(G) + 1, |\chi''(G)|\}. \]

For $k = 3$, this is equivalent to the four-colour theorem. For $k = 4$ and $k = 5$, it was derived from the case $k = 3$ by Guenin [2002b].

18 (pages 476,644). Lovász [1987] conjectures more generally:

\[ \chi'(G) \leq \max\{\Delta(G) + 1, |\chi''(G)|\}. \]

This is equivalent to stating that the incidence vectors of perfect matchings in a graph without Petersen graph minor, form a Hilbert base.

19 (page 481). The following question was asked by Vizing [1968]: Is there a simple planar graph of maximum degree 6 and with edge-colouring number 7?

20 (page 481). Vizing [1965a] asked if a minimum edge-colouring of a graph can be obtained from an arbitrary edge-colouring by iteratively swapping colours on a colour-alternating path or circuit and deleting empty colours.

21 (page 482). Vizing [1976] conjectures that the list-edge-colouring number of any graph is equal to its edge-colouring number.

(The list-edge-colouring number $\chi'(G)$ of a graph $G = (V,E)$ is the minimum number $k$ such that for each choice of sets $L_e$ for $e \in E$ with $|L_e| = k$, one can select $l_e \in L_e$ for $e \in E$ such that for any two incident edges $e, f$ one has $l_e \neq l_f$.)

22 (page 482). Behzad [1965] and Vizing [1968] conjecture that the total colouring number of a simple graph $G$ is at most $\Delta(G) + 2$. (The total colouring...
number of a graph $G = (V, E)$ is a colouring of $V \cup E$ such that each colour consists of a stable set and a matching, vertex-disjoint.)

23 (page 482). More generally, Vizing [1968] conjectures that the total colouring number of a graph $G$ is at most $\Delta(G) + \mu(G) + 1$, where $\mu(G)$ is the maximum edge multiplicity of $G$.

24 (pages 497, 645). Seymour [1979b] conjectures that each even integer vector in the circuit cone of a graph is a nonnegative integer combination of incidence vectors of circuits.

25 (pages 497, 645, 1427). A special case of this is the circuit double cover conjecture (asked by Szekeres [1973] and conjectured by Seymour [1979b]): each bridgeless graph has circuits such that each edge is covered by precisely two of them.

Janss and Tarsi [1989] proved that the circuit double cover conjecture is equivalent to a generalization to matroids:

\[(14) \quad (?) \text{ each bridgeless binary matroid without } F_7 \text{ minor has a circuit double cover.} \ (?) \]

26 (page 509). Is the system of $T$-join constraints totally dual quarter-integral?

27 (page 517). L. Lovász asked for the complexity of the following problem: given a graph $G = (V, E)$, vertices $s, t \in V$, and a length function $l : E \to \mathbb{Q}$ such that each circuit has nonnegative length, find a shortest odd $s-t$ path.

28 (page 545). What is the complexity of deciding if a given graph has a 2-factor without circuits of length at most 4?

29 (page 545). What is the complexity of finding a maximum-weight 2-factor without circuits of length at most 3?

30 (page 646). Tarsi [1986] mentioned the following strengthening of the circuit double cover conjecture:

\[(15) \quad (?) \text{ in each bridgeless graph there exists a family of at most 5 cycles covering each edge precisely twice.} \ (?) \]

31 (page 657). Is the dual of any algebraic matroid again algebraic?

32 (page 892). A special case of a question asked by A. Frank (cf. Schrijver [1979b], Frank [1995]) amounts to the following:

\[(16) \quad (?) \text{ Let } G = (V, E) \text{ be an undirected graph and let } s \in V. \text{ Suppose that for each vertex } t \neq s, \text{ there exist } k \text{ internally vertex-disjoint } s-t \text{ paths. Then } G \text{ has } k \text{ spanning trees such that for each vertex} \]

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$t \neq s$, the $s-t$ paths in these trees are internally vertex-disjoint.

(The spanning trees need not be edge-disjoint — otherwise $G = K_3$ would form a counterexample.) For $k = 2$, (16) was proved by Itai and Rodeh [1984, 1988], and for $k = 3$ by Cheriyan and Maheshwari [1988] and Zehavi and Itai [1989].

33 (page 962). Can a maximum number of disjoint directed cut covers in a directed graph be found in polynomial time?


(17) (?) In a digraph, the minimum size of a directed cut is equal to the maximum number of disjoint directed cut covers. (?)

35 (page 985). Let $G = (V, E)$ be a complete undirected graph, and consider the system

$$
0 \leq x_e \leq 1 \text{ for each edge } e, \\
x(\delta(v)) = 2 \text{ for each vertex } v, \\
x(\delta(U)) \geq 2 \text{ for each } U \subseteq V \text{ with } \emptyset \neq U \neq V.
$$

Let $l : E \to \mathbb{R}_+$ be a length function. Is the minimum length of a Hamiltonian circuit at most $1/2$ times the minimum value of $l^T x$ over (18)?

36 (page 990). Padberg and Grötschel [1985] conjecture that the diameter of the symmetric traveling salesman polytope of a complete graph is at most 2.

37 (page 1076). Frank [1994a] conjectures:

(19) (?) Let $D = (V, A)$ be a simple acyclic directed graph. Then the minimum size of a $k$-vertex-connector for $D$ is equal to the maximum of $\sum_{v \in V} \max\{0, k - \deg^\text{in}(v)\}$ and $\sum_{v \in V} \max\{0, k - \deg^\text{out}(v)\}$. (?)

(A $k$-vertex-connector for $D$ is a set of (new) arcs whose addition to $D$ makes it $k$-vertex-connected.)

38 (page 1087). Hadwiger’s conjecture (Hadwiger [1943]): If $\chi(G) \geq k$, then $G$ contains $K_k$ as a minor.

Hadwiger’s conjecture is trivial for $k = 1, 2, 3$, was shown by Hadwiger [1943] for $k = 4$ (also by Dirac [1952]), is equivalent to the four-colour theorem for $k = 5$ (by a theorem of Wagner [1937a]), and was derived from the four-colour theorem for $k = 6$ by Robertson, Seymour, and Thomas [1993]. For $k \geq 7$, the conjecture is unsettled.

39 (page 1099). Chvátal [1973a] asked if for each fixed $t$, the stable set problem for graphs for which the stable set polytope arises from $P(G)$ by at most
$t$ rounds of cutting planes, is polynomial-time solvable. Here $P(G)$ is the polytope determined by the nonnegativity and clique inequalities.

40 (page 1099). Chvátal [1975b] conjectures that there is no polynomial $p(n)$ such that for each graph $G$ with $n$ vertices we can obtain the inequality $x(V) \leq \alpha(G)$ from the system defining $Q(G)$ by adding at most $p(n)$ cutting planes. Here $Q(G)$ is the polytope determined by the nonnegativity and edge inequalities. (This conjecture would be implied by $\text{NP} \neq \text{co-NP}$.)

41 (page 1105). Gyárfás [1987] conjectures that there exists a function $g : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that for each graph $G$ without odd holes.

42 (page 1107). Can perfection of a graph be tested in polynomial time?

43 (page 1131). Berge [1982a] conjectures the following. A directed graph $D = (V, A)$ is called $\alpha$-diperfect if for every induced subgraph $D' = (V', A')$ and each maximum-size stable set $S$ in $D'$ there is a partition of $V'$ into directed paths each intersecting $S$ in exactly one vertex. Then for each directed graph $D$:

\begin{equation}
\text{(20) } D \text{ is } \alpha\text{-diperfect if and only if } D \text{ has no induced subgraph } C \text{ whose underlying undirected graph is a chordless odd circuit of length } \geq 5, \text{ say with vertices } v_1, \ldots, v_{2k+1} \text{ (in order) such that each of } v_1, v_2, v_3, v_4, v_6, v_8, \ldots, v_{2k} \text{ is a source or a sink. } \text{(?)}
\end{equation}

44 (page 1170). Is $\vartheta(C_n) = \Theta(C_n)$ for each odd $n$?

45 (page 1170). Can Haemers’ bound $\eta(G)$ on the Shannon capacity of a graph $G$ be computed in polynomial time?

46 (page 1187). Is every $t$-perfect graph strongly $t$-perfect?

47 (page 1195). $T$-perfection is closed under taking induced subgraphs and under contracting all edges in $\delta(v)$ where $v$ is a vertex not contained in a triangle. What are the minimally non-$t$-perfect graphs under this operation?

48 (page 1242). For any $k$, let $f(k)$ be the smallest number such that in any $f(k)$-connected undirected graph, for any choice of distinct vertices $s_1, t_1, \ldots, s_k, t_k$ there exist vertex-disjoint $s_1 - t_1, \ldots, s_k - t_k$ paths. Thomassen [1980] conjectures that $f(k) = 2k + 2$ for $k \geq 2$.

49 (page 1242). For any $k$, let $g(k)$ be the smallest number such that in any $g(k)$-edge-connected undirected graph, for any choice of vertices $s_1, t_1, \ldots, s_k, t_k$ there exist edge-disjoint $s_1 - t_1, \ldots, s_k - t_k$ paths. Thomassen [1980] conjectures that $g(k) = k$ if $k$ is odd and $g(k) = k + 1$ if $k$ is even.
50 (page 1243). What is the complexity of the $k$ arc-disjoint paths problem in directed planar graphs, for any fixed $k \geq 2$? This is even unknown for $k = 2$, also if we restrict ourselves to two opposite nets.

51 (page 1274). Karzanov [1991] conjectures that if the nets in a multflow problem form two disjoint triangles and if the capacities and demands are integer and satisfy the Euler condition, then the existence of a fractional multflow implies the existence of a half-integer multflow.

52 (page 1274). The previous conjecture implies that for each graph $H = (T, R)$ without three disjoint edges, there is an integer $k$ such that for each graph $G = (V, E)$ with $V \supseteq T$ and any $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$, if there is a feasible multflow, then there exists a $\frac{1}{k}$-integer multflow.

53 (page 1276). Okamura [1998] conjectures the following. Let $G = (V, E)$ be an $l$-edge-connected graph (for some $l$). Let $H = (T, R)$ be a ‘demand’ graph, with $T \subseteq V$, such that $d_R(U) \leq l$ for each $U \subseteq V$. Then the edge-disjoint paths problem has a half-integer solution.

54 (page 1293). Is each Mader matroid a gammoid?

55 (page 1294). Is each Mader matroid linear?

56 (page 1299). Is the undirected edge-disjoint paths problem for planar graphs polynomial-time solvable if all terminals are on the outer boundary? Is it NP-complete?

57 (page 1310). Is the integer multflow problem polynomial-time solvable if the graph and the nets form a planar graph such that the nets are spanned by a fixed number of faces?

58 (page 1310). Pfeiffer [1990] raised the question if the edge-disjoint paths problem has a half-integer solution if the graph $G + H$ (the union of the supply graph and the demand graph) is embeddable in the torus and there exists a quarter-integer solution.

59 (page 1320). Let $G = (V, E)$ be a planar bipartite graph and let $q$ be a vertex on the outer boundary. Do there exist disjoint cuts $C_1, \ldots, C_p$ such that any pair $s, t$ of vertices with $s$ and $t$ on the outer boundary, or with $s = q$, is separated by dist$_G(s, t)$ cuts?

60 (page 1345). Fu and Goddyn [1999] asked: Is the class of graphs for which the incidence vectors of cuts form a Hilbert base, closed under taking minors?

61 (page 1382). Füredi, Kahn, and Seymour [1993] conjecture that for each hypergraph $H = (V, \mathcal{E})$ and each $w : \mathcal{E} \to \mathbb{R}_+$, there exists a matching $\mathcal{M} \subseteq \mathcal{E}$ such that
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\[ \sum_{F \in \mathcal{M}} (|F| - 1 + \frac{1}{|F|})w(F) \geq \nu^*_w(H), \]

where \( \nu^*_w(H) \) is the maximum weight \( w^Ty \) of a fractional matching \( y : \mathcal{E} \to \mathbb{R}_+ \).

62 (pages 1387, 1408). Seymour [1981a] conjectures:

\[ \text{(22) (?) a binary hypergraph is ideal if and only if it has no } \mathcal{O}(K_5), \]
\[ b(\mathcal{O}(K_5)), \text{ or } F_7 \text{ minor. (?) } \]

63 (page 1392). Seymour [1990b] asked the following. Suppose that \( H = (V, \mathcal{E}) \) is a hypergraph without \( J_n \) minor \((n \geq 3)\). Let \( l, w : V \to \mathbb{Z}_+ \) be such that

\[ \tau(H^w) \cdot \tau(b(H)^l) > l^Tw. \]

Is there a minor \( H' \) of \( H \) and \( l', w' : VH' \to \{0, 1\} \) such that

\[ \tau((H')^{w'}) \cdot \tau(b(H')^{l'}) > l'^Tw'. \]

and such that \( \tau((H')^{w'}) \leq \tau(H^w) \) and \( \tau(b(H')^{l'}) \leq \tau(b(H)^l) \)?

Here, for each \( n \geq 3 \): \( J_n := \) the hypergraph with vertex set \( \{1, \ldots, n\} \) and edges \( \{2, \ldots, n\}, \{1, 2\}, \ldots, \{1, n\} \).

64 (page 1392). Seymour [1990b] also asked the following. Let \( H = (V, \mathcal{E}) \) be a nonideal hypergraph. Is the minimum of \( \tau(H') \) over all parallelizations and minors \( H' \) of \( H \) with \( \tau^*(H') < \tau(H') \) attained by a minor of \( H' \)?

65 (page 1395). Cornuéjols and Novick [1994] conjecture that there are only finitely many minimally nonideal hypergraphs \( H \) with \( r_{min}(H) > 2 \) and \( \tau(H) > 2 \).

66 (page 1396). Ding [1993] asked whether there exists a number \( t \) such that each minimally nonideal hypergraph \( H \) satisfies \( r_{min}(H) \leq t \) or \( \tau(H) \leq t \).

(The above conjecture of Cornuéjols and Novick [1994] implies a positive answer to this question.)

67 (page 1396). Ding [1993] conjectures that for each fixed \( k \geq 2 \), each minor-minimal hypergraph \( H \) with \( \tau_k(H) < k \cdot \tau(H) \), contains some \( J_n \) minor \((n \geq 3)\) or satisfies the regularity conditions of Lehman’s theorems (Theorem 78.4 and 78.5).

68 (page 1401). Conforti and Cornuéjols [1993] conjecture:

\[ \text{(25) (?) a hypergraph is Mengerian if and only if it is packing. (?) } \]


\[ \text{(26) (?) each minimally nonideal hypergraph } H \text{ with } r_{min}(H) \tau(H) = |VH| + 1 \text{ is minimally nonpacking. (?) } \]

71 (page 1404). Seymour [1981a] conjectures that $T_{30}$ is the unique minor-minimal binary ideal hypergraph $H$ with the property $\nu_2(H) < 2\tau(H)$.

Here the hypergraph $T_{30}$ arises as follows. Replace each edge of the Petersen graph by a path of length 2, making the graph $G$. Let $T := VG \setminus \{v\}$, where $v$ is an arbitrary vertex of $v$ of degree 3. Let $E$ be the collection of $T$-joins. Then $T_{30} := (EG, E)$.

72 (page 1405). P.D. Seymour (personal communication 1975) conjectures that for each ideal hypergraph $H$ there exists an integer $k$ such that $\nu_k(H) = k \cdot \tau(H)$ and such that $k = 2^i$ for some $i$. He also asks if $k = 4$ would do in all cases.

73 (page 1405). Seymour [1979a] conjectures that for each ideal hypergraph $H$, the g.c.d. of those $k$ with $\nu_k(H) = k \cdot \tau(H)$ is equal to 1 or 2.

74 (page 1409). Is the following true for binary hypergraphs $H$:

$$\nu(H^w) = \tau(H^w) \text{ for each } w : V \rightarrow \mathbb{Z}_+ \text{ with } w(B) \text{ even for all } B \in b(H) \iff \frac{1}{2} \nu_2(H^w) = \tau(H^w) \text{ for each } w : V \rightarrow \mathbb{Z}_+ \iff H \text{ has no } \mathcal{O}(K_5), b(\mathcal{O}(K_5)), F_7, \text{ or } T_{15} \text{ minor}. (?)$$

Here $T_{15}$ is the hypergraph of $VP_{10}$-joins in the Petersen graph $P_{10}$.

75 (page 1421). Seymour [1981a] conjectures that for any binary matroid $M$:

$$M \text{ is 1-cycling } \iff M \text{ is 1-flowing } \iff M \text{ has no } AG(3,2), T_{11}, \text{ or } T_{11}^1 \text{ minor}. (?)$$

Here $T_{11}$ is the binary matroid represented by the 11 vectors in $\{0,1\}^5$ with precisely 3 or 5 ones. Moreover, $AG(3,2)$ is the matroid with 8 elements obtained from the 3-dimensional affine geometry over $GF(2)$. 