IV. Stable matchings

1. Stable matchings

Let $G = (V, E)$ be a graph and let for each $v \in V$, $\leq_v$ be a total order on $\delta(v)$. Put $e \preceq f$ if e and f have a vertex v in common with $e \leq_v f$. Call a set M of edges *stable* if for each $e \in E$ there exists an $f \in M$ with $e \preceq f$.

In general, stable matchings need not exist (e.g., generally not for K_3). However, Gale and Shapley [1] showed that if G is bipartite, they do exist:

Theorem 1 (Gale-Shapley theorem)**.** *If G is bipartite, then there exists a stable matching.*

Proof. Let U and W be the colour classes of G. For each edge $e = uw$ with $u \in U$ and $w \in W$, let $\varphi(e)$ be the height of e in $(\delta(w), \leq_w)$. (The *height* of e is the maximum size of a chain with maximum e.) Choose a matching M in G such that for each edge $e = uw$ of G, with $u \in U$ and $w \in W$,

(1) if $f \leq_u e$ for some $f \in M$, then $e \leq_w g$ for some $g \in M$,

and such that $\sum_{e \in M} \varphi(e)$ is as large as possible. (Such a matching exists, since $M = \emptyset$ satisfies (1).) We show that M is stable.

Choose $e = uw \in E$ with $u \in U$ and $w \in W$ and suppose that there is no $e' \in M$ with $e \preceq e'$. Choose e largest in $\leq_u$ with this property. Then by (1) there is no $f \in M$ with $f \leq_u e$; and moreover, there is no $f \in M$ with $e \leq_u f$. Hence u is missed by M.

Since also there is no $g \in M$ with $e \leq_w g$, we can remove any edge in M incident with w and add e to M, so as to obtain a matching satisfying (1) with larger $\sum_{e \in M} \varphi(e)$, a contradiction. ∎

This proof also gives a polynomial-time algorithm to find a stable matching. It was noted by Roth [3] that this algorithm is in fact in use in practice since 1951 in the U.S., to match hospitals and medical students (cf. Roth and Sotomayor [4]).

2. List-edge-colouring

An interesting extension of Kőnig's edge-colouring theorem was shown by Galvin [2], by using the Gale-Shapley theorem on stable matchings (Theorem 1).

Let $G = (V, E)$ be a graph. Then G is *k-list-edge-colourable* if for each choice of finite sets L_e for $e \in E$ with $|L_e| = k$, we can choose $l_e \in L_e$ for $e \in E$ such that $l_e \neq l_f$ if e and f are incident. The smallest k for which G is k-list-edge-colourable is called the *list-edge-colouring number* of G.

Trivially, the list-edge-colouring number of G is at least the edge-colouring number of G, and hence at least the maximum degree $\Delta(G)$ of G. Galvin [2] showed:

Theorem 2. *The list-edge-colouring number of a bipartite graph is equal to its maximum degree.*

Proof. Let $G = (V, E)$ be a bipartite graph, with colour classes U and W, and with maximum degree $k := \Delta(G)$. The theorem follows by applying the following statement to any $\Delta(G)$-edge-colouring $\varphi : E \to \{1, \ldots, \Delta(G)\}$ of G.

(2) Let $\varphi : E \to \mathbb{Z}$ be such that $\varphi(e) \neq \varphi(f)$ if e and f are incident. For each $e = uw \in E$ with $u \in U$ and $w \in W$, let L_e be a finite set satisfying

$$|L_e| > |\{f \in \delta(u) \mid \varphi(f) < \varphi(e)\}| + |\{f \in \delta(w) \mid \varphi(f) > \varphi(e)\}|.$$

Then there exist $l_e \in L_e$ $(e \in E)$ such that $l_e \neq l_f$ if e and f are incident.

So it suffices to prove (2), which is done by induction on $|E|$. Choose $p \in \bigcup L_e$ and let $F := \{e \in E \mid p \in L_e\}$. Define for each $v \in V$ a total order $<_v$ on $\delta_F(v)$ by:

(3)
$$e \leq_v f \iff \varphi(e) \geq \varphi(f), \text{ if } v \in U,$$
$$e \leq_v f \iff \varphi(e) \leq \varphi(f), \text{ if } v \in W,$$

for $e, f \in \delta_F(v)$. By the Gale-Shapley theorem (Theorem 1), F contains a stable matching M. So M is a matching such that for each $e \in F$ there is an $f \in M$ with $e \leq_v f$ for some $v \in e$. Hence for each edge $e = uw \in F \setminus M$, with $u \in U$ and $w \in W$: $\exists f \in M \cap \delta(u) : \varphi(f) < \varphi(e)$ or $\exists f \in M \cap \delta(w) : \varphi(f) > \varphi(e)$. So removing M from E and resetting $L_e := L_e \setminus \{p\}$ for each $e \in F \setminus M$, we can apply induction. ∎

For school scheduling (cf. König's edge-colouring theorem) this theorem can be interpreted as: if we prescribe for each open 'slot' a set of Δ hours, where Δ is the maximum number of open slots over all teachers and all classes, then there exists a feasible schedule.

References

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