The Tutte-Berge formula

A basic result on matchings was found by Tutte [1947]. It characterizes graphs that have a perfect matching. Berge [1958] observed that it implies a min-max formula for the maximum cardinality $\nu(G)$ of a matching in a graph G, the Tutte-Berge formula.

Call a component of a graph *odd* if it has an odd number of vertices. For any graph G, let $o(G)$ denote the number of odd components of G. Then:

Theorem (Tutte-Berge formula). *For each graph $G = (V, E)$,*

$$\nu(G) = \min_{U \subseteq V} \tfrac{1}{2}(|V| + |U| - o(G - U)). \tag{1}$$

Proof. To see $\leq$, we have for each $U \subseteq V$:

$$\nu(G) \leq |U| + \nu(G - U) \leq |U| + \tfrac{1}{2}(|V \setminus U| - o(G - U)) = \tfrac{1}{2}(|V| + |U| - o(G - U)).$$

We prove the reverse inequality by induction on $|V|$, the case $V = \emptyset$ being trivial. We can assume that G is connected, as otherwise we can apply induction to the components of G.

First assume that there exists a vertex v covered by all maximum-size matchings. Then $\nu(G - v) = \nu(G) - 1$, and by induction there exists a subset U' of $V \setminus \{v\}$ with

$$\nu(G - v) = \tfrac{1}{2}(|V \setminus \{v\}| + |U'| - o(G - v - U')).$$

Then $U := U' \cup \{v\}$ gives equality in (1).

So we can assume that there is no such v. In particular, $\nu(G) < \frac{1}{2}|V|$. We show that there exists a matching of size $\frac{1}{2}(|V| - 1)$, which implies the theorem (taking $U := \emptyset$).

Indeed suppose to the contrary that any maximum-size matching M misses at least two distinct vertices u and v. Among all such M, u, v, choose them such that the distance $\text{dist}(u, v)$ of u and v in G is as small as possible.

If $\text{dist}(u, v) = 1$, then u and v are adjacent, and hence we can augment M by uv, contradicting the maximality of $|M|$. So $\text{dist}(u, v) \geq 2$, and hence we can choose an intermediate vertex t on a shortest $u - v$ path. By assumption, there exists a maximum-size matching N missing t. Choose such an N with $|M \cap N|$ maximal.

By the minimality of $\text{dist}(u, v)$, N covers both u and v. Hence, as M and N cover the same number of vertices, there exists a vertex $x \neq t$ covered by M but not by N. Let $x \in e = xy \in M$. Then y is covered by some edge $f \in N$, since otherwise $N \cup \{e\}$ would be a matching larger than N. Replacing N by $(N \setminus \{f\}) \cup \{e\}$ increases the intersection with M, contradicting the choice of N. ∎