OBSERVATIONS ON WOODALL'S CONJECTURE

Discussion notes

Alexander Schrijver[1]

1. Notation and terminology

Let $D = (V, A)$ be a directed graph. For $U \subseteq V$, $\delta^{\text{in}}(U)$ and $\delta^{\text{out}}(U)$ denote the sets of arcs entering U and leaving U, respectively. Moreover, $\delta(U) := \delta^{\text{in}}(U) \cup \delta^{\text{out}}(U)$, $d^{\text{in}}(U) := |\delta^{\text{in}}(U)|$ (the *indegree*), $d^{\text{out}}(U) := |\delta^{\text{out}}(U)|$ (the *outdegree*), and $d(U) := |\delta(U)|$ (the *degree* or *total degree*). If $U = \{u\}$ is a singleton, we replace the argument $\{u\}$ by u. We attach subscript D or A if useful. For $B \subseteq A$, $B^{-1} := \{(u, v) \mid (v, u) \in B\}$.

A *directed cut* is a subset C of A such that $C = \delta^{\text{in}}(U)$ for some subset U of V satisfying $\emptyset \neq U \neq V$ and $\delta^{\text{out}}(U) = \emptyset$. We say that U *determines a directed cut* if U is a subset of V satisfying $\emptyset \neq U \neq V$ and $\delta^{\text{out}}(U) = \emptyset$. Denote by $\sigma(D)$ the minimum size of a directed cut. This is ∞ if D has no directed cut, i.e., if D is strongly connected.

A *directed cut cover* or *dijoin* is a subset B of A intersecting each directed cut. Trivially, B is a directed cut cover if and only if the digraph $(V, A \cup B^{-1})$ is strongly connected. Call a subset B of A *strengthening* if the digraph $(V, (A \setminus B) \cup B^{-1})$ is strongly connected. So each strengthening arc set is a directed cut cover. Call a function $\varphi : A \to [k]$ a *strong coloring* or *strong k-coloring* if $\varphi^{-1}(i)$ is strengthening for each $i \in [k]$.

2. Woodall's conjecture

Woodall [2] conjectures:

Conjecture (Woodall's conjecture). *Let $D = (V, A)$ be a digraph. The maximum number of pairwise disjoint directed cut covers is equal to the minimum size of a directed cut.*

Woodall's conjecture is equivalent to:

Conjecture (Woodall's conjecture). *Let $D = (V, A)$ be a digraph and let $k \geq 2$. Then A can be partitioned into k strengthening sets if and only if each directed cut has size at least k.*

Proof of equivalence. Since each strengthening set is a directed cut cover, necessity in the latter conjecture is direct. To see sufficiency, let each directed cut have size at least k. Add to each arc $a = (u, v)$ two new vertices u_a and v_a, and replace a by arcs (u, u_a), (v_a, v), and $k - 1$ parallel arcs from v_a to u_a. Let D' be the new digraph. Then each directed cut in D' has size at least k. By the first version of Woodall's conjecture, D' contains k disjoint directed cut covers $B'_1, \ldots, B'_k$. For each arc a of D, each B'_i contains precisely one of the

[1] University of Amsterdam and CWI, Amsterdam. The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement n° 339109.

arcs incident with any u_a and one of the arcs incident with v_a. So, if B'_i contains (u, u_a) if and only if it contains (v_a, v). Let B_i be the set of arcs a of D with $(v_a, v) \in B'_i$.

Then each B_i is strengthening. For suppose to the contrary that there is a nonempty proper subset X of V not entered by any arc in $(A \setminus B_i) \cup B_i^{-1}$. So

(1) $$\delta^{\text{out}}(X) = B_i \cap \delta(X).$$

Define

(2) $$X' := X \cup \{u_a \mid a = (u, v) \in A, u \in X\} \cup \{v_a \mid a = (u, v) \in A, u, v \in X\}.$$

Then X' determines a directed cut in D'. Hence B'_i contains an arc a' of D' entering X'. If $a' = (u, u_a)$ for some arc $a = (u, v)$ of D, then $u_a \in X'$ while $u \notin X$, contradicting the definition of X'. If $a' = (v_a, u_a)$ for some arc $a = (u, v)$ of D, then $u_a \in X'$ and $v_a \notin X'$, so $u \in X$ and $v \notin X$, so (u, v) leaves X. Since $a' \in B'_i$, we know $a \notin B_i$. This contradicts (1). If $a' = (v_a, v)$ for some arc $a = (u, v)$ of D, then $v_a \notin X'$ and $v \in X$. So by definition of X', $u \notin X$. So (u, v) enters X. As $(v_a, v) \in B'_i$, we have $a \in B_i$. This again contradicts (1). ∎

3. Decomposition and connection

In this section, we prove a few decomposition results on digraphs that can be useful in proving Woodall's conjecture.

If $D = (V, A)$ is a digraph and U is a nonempty proper subset of V, let D/U be the digraph obtained by identifying all vertices in U to one vertex c_U. There is a natural inclusion function $i_{D/U}$ of $A(D/U)$ to $A(D)$. We call the inverse function $(i_{D/U})^{-1} : \mathcal{P}(A(D)) \to \mathcal{P}(A(D/U))$ the corresponding *projection*. Note that if $d(U) \leq 3$, then D is planar if and only if D/U and $D/\overline{U}$ are planar.

We will consider this decomposition also in the reversed order. Let D_1 and D_2 be digraphs and let $v_1 \in V(D_1)$ and $v_2 \in V(D_2)$. We say that digraph $D = (V, A)$ arises by *connecting* D_1 and D_2 at v_1 and v_2 if for some nonempty proper subset U of V, $D_1 = D/U$, $v_1 = c_U$, $D_2 = D/\overline{U}$, and $v_2 = c_{\overline{U}}$. (Here equality is meant up to isomorphism.) If for $a_1 \in \delta_{D_1}(v_1)$ and $a_2 \in \delta_{D_2}(v_2)$ one has $i_{D/U}(a_1) = i_{D/\overline{U}}(a_2)$, we say that a_1 and a_2 are *linked* in this connection.

Theorem 1. *Let $D = (V, A)$ be a digraph and let U be a nonempty proper subset of V with $d^{\text{out}}(U) \leq 1$. Then D is strongly connected if and only if D/U and $D/\overline{U}$ are strongly connected.*

Proof. Let $D_1 := D/\overline{U}$ and $D_2 := D/U$. Necessity is direct, since each directed cut in any D_i yields a directed cut in D. To see sufficiency, suppose X determines a directed cut in D.

If $X \cap U = \emptyset$, then X determines a directed cut in D_2, a contradiction. So $X \cap U \neq \emptyset$. As $X \cap U$ determines no directed cut in D_1, $\delta^{\text{out}}(X \cap U) \neq \emptyset$. Since $\delta^{\text{out}}(X) = \emptyset$, there is an arc from $X \cap U$ to $X \setminus U$.

Similarly, if $X \cup U = V$, then X determines a directed cut in D_1, a contradiction. So $X \cup U \neq V$. As $X \cup U$ determines no directed cut in D_2, $\delta^{\text{out}}(X \cup U) \neq \emptyset$. Since $\delta^{\text{out}}(X) = \emptyset$, there is an arc from $U \setminus X$ to $\overline{X \cup U}$.

Concluding, there at least two arcs from U to $\overline{U}$, contradicting the assumption that $d^{\text{out}}(U) \leq 1$. ▮

Corollary 1a. *Let $D = (V, A)$ be a digraph and let U be a nonempty proper subset of V. Let $B \subseteq A$ such that $d^{\text{out}}_{A'}(U) \leq 1$ where $A' := (A \setminus B) \cup B^{-1}$. Then B is strengthening if and only if the projections of B in D/U and $D/\overline{U}$ are strengthening.*

Proof. Directly from Theorem 1 applied to the digraph (V, A'). ▮

Corollary 1b. *Let $D = (V, A)$ be a weakly connected digraph and let U be a nonempty proper subset of V with $d(U) \leq 3$. Let $\varphi : A \to [k]$. Then φ is a strong coloring for D if and only if $\varphi \circ i_{D/U}$ and $\varphi \circ i_{D/\overline{U}}$ are strong colorings for D/U and $D/\overline{U}$ respectively.*

Proof. Directly from Corollary 1a. ▮

Theorem 2. *Let $D = (V, A)$ be a weakly connected digraph and let U be a nonempty proper subset of V with $d(U) \leq 3$. Then $\sigma(D) = \min\{\sigma(D/U), \sigma(D/\overline{U})\}$.*

Proof. Let $D_1 := D/\overline{U}$ and $D_2 := D/U$. Let $k := \min\{\sigma(D_1), \sigma(D_2)\}$. Clearly, $\sigma(D) \leq k$, since each directed cut in any D_i yields a directed cut in D.

To see the reverse inequality, suppose $\sigma(D) < k$. As D is weakly connected, $\sigma(D) \geq 1$, hence $k \geq 2$. So $\delta^{\text{out}}(X) = \emptyset$ and $d^{\text{in}}(X) < k$ for some nonempty proper subset X of V. Since $d(U) \leq 3$, at least one of X and $\overline{X}$ spans at most one arc in $\delta(U)$. By symmetry, we may assume that X spans at most one arc in $\delta(U)$ — otherwise reverse all arcs and replace X by $\overline{X}$. So there is at most one arc connecting $X \cap U$ and $X \setminus U$. Hence we may assume in addition that no arc leaves $X \cap U$, that is, runs from $X \cap U$ to $X \setminus U$ — otherwise replace U by $\overline{U}$.

If $X \subseteq U$ or $X \subseteq \overline{U}$, $\delta^{\text{in}}(X)$ gives a directed cut of size less than k in D_1 or D_2, contradicting the definition of k. So we know that both $X \cap U$ and $X \setminus U$ are nonempty. Hence $\delta^{\text{in}}(X \cap U)$ is a directed cut.

As X spans at most one arc of D and no arc leaves X, $d^{\text{out}}(X \setminus U) \leq 1$. Therefore, $d^{\text{in}}(X \setminus U) - d^{\text{out}}(X \setminus U) = d(X \setminus U) - 2d^{\text{out}}(X \setminus U) \geq 0$, since $d(X \setminus U) \geq 2$, as $\sigma(D_2) \geq k \geq 2$. This implies

$$(3) \qquad d^{\text{in}}(X \cap U) = d^{\text{in}}(X) - d^{\text{in}}(X \setminus U) + d^{\text{out}}(X \setminus U) \leq d^{\text{in}}(X) < k.$$

However, as no arc leaves $X \cap U$, $\delta^{\text{in}}(X \cap U)$ is a directed cut, yielding a directed cut in D_1. Hence $d^{\text{in}}(X \cap U) \geq k$, contradicting (3). ▮

4. Reduction

For any k, consider the following conditions on a directed graph D:

(4) D is acyclic and weakly 3-arc-connected, the directed cuts of size k are precisely those determined by the sources and sinks, that each vertex not being a source

or sink has degree 3, and that for each edge a not incident with any sourc or sink there is a cut $\delta(U)$ with $d(U) \leq k$ and $\delta^{\text{in}}(U) = \{a\}$.

We call a digraph satisfying (4) *reduced.* We denote the set of 3-degree vertices not being sources or sinks by V_3.

Theorem 3. *In Woodall's conjecture we can assume that $k \geq 3$ and D is reduced.*

Proof. It was observed by András Frank that Woodall's conjecture is true for $k = 2$ (see Theorem 56.3 in [1]). So we can assume $k \geq 3$. Trivially, if there is a counterexample there is an acyclic counterexample, as we can contract each strong component to one vertex. Choose an acyclic counterexample $D = (V, A)$ minimizing

$$(5) \qquad |V_3| + \sum_{v \in V \setminus V_3} 2d(v) = 4|A| - 5|V_3|,$$

where V_3 is the set of vertices that have degree 3 and are not a source or sink.

We first show:

(6) D is weakly 3-arc-connected.

Suppose to the contrary that $d(U) \leq 2$ for some nonempty proper subset U of V. As $\sigma(D) \geq k \geq 3$, $d^{\text{in}}(U) = 1$ and $d^{\text{out}}(U) = 1$. Let D_1 and D_2 be obtained from $D - U$ and $D - \overline{U}$ respectively by adding an arc from the tail of the arc in $d(U)$ to the head of the arc in $d(U)$. For $i = 1, 2$, $\sigma(D_i) \geq 3$, by Theorem 2. As the sum (5) is smaller for D_i, D_i can be partitioned into k strengthening sets. This implies with Corollary 1a, that D can be partitioned into k strengthening sets.

This shows (6). It implies in particular that each vertex of D has degree at least 3.

We next show that

(7) each directed cut of size k is determined by a sink or by the complement of a source.

Suppose to the contrary that D has a directed cut $\delta^{\text{in}}(U)$ of size k with $\delta^{\text{out}}(U) = \emptyset$ and $2 \leq |U| \leq |V| - 2$. Let D_1 and D_2 be the digraphs obtained by contracting $\overline{U}$ and U, reespectively, to one vertex. Then for each D_i, (5) has decreased. To see this we can assume $i = 2$, by symmetry. As $\delta^{\text{out}}(U) = \emptyset$, U contains a sink s. Choose $t \in U \setminus \{s\}$. Then s and t have a contribution of at least $2k$ and 1, respectively, to the sum (5). In the contracted graph, U becomes a sink of outdegree k. So it has contribution precisely $2k$ to (5). Hence (5) has decreased.

By the minimality of the counterexample, each of D_1 and D_2 can be partitioned into k strengthening sets. As each of the classes of these partitions intersect $\delta^{\text{in}}(U)$ in precisely one arc (since $d^{\text{in}}(U) = k$), we can glue the two partitions together to obtain a k-partition of A. Then by Corollary 1a, each class of the partition is strengthening. This contradicts the assumption that we have a counterexample, thus proving (7).

Next we show:

(8) Let s be a vertex with at least two distinct out-neighbours. Then s is a source

of degree k or s belongs to V_3.

Let u and v be out-neighbours of s with $u \neq v$. Assume that s is not a source of degree k and does not belong to V_3. Let s' be a new vertex, and replace (s,u) and (s,v) by (s,s'), (s',u), (s',v), yielding the new digraph $D' = (V', A')$. Since the number of arcs has increased by 1, while the number of vertices of degree 3 has increased by at least 1, the value of (5) has decreased.

Suppose $\sigma(D') < k$. Consider a directed cut $\delta^{\text{in}}(U')$ in D', with U' a nonempty proper subset of V' not left by any arc of V', and with $d^{\text{in}}_{D'}(U') < k$. If U' does not separate s and s', U' would yield a directed cut in D of size less than k, a contradiction. So U' separates s and s', hence, as no arc of D' leaves U', $s \notin U'$ and $s', u, v \in U'$. Therefore, $U := U' \setminus \{s'\}$ determines a directed cut in D of size $d^{\text{in}}_{D'}(U') + 1 \leq k$. Hence, by (7), $|U| = 1$ or $|\overline{U}| = 1$. If $|U| = 1$, then $u = v$ (as $u, v \in U$), contradicting the choice of u and v. If $|\overline{U}| = 1$, then $\overline{U} = \{s\}$, hence s is a source of degree k, contradicting our assumption.

So $\sigma(D') = k$. Hence, since sum (5) has decreased, A' can be partitioned into k strengthening sets. By contracting the new arc (s, s') we obtain a partition for A into k strengthening sets. This contradicts the fact that D is a counterexample, and therefore proves (8).

This implies

(9) all sources and sinks have degree k.

If s is a source of degree at least $k+1$, by (8), all arcs leaving s are parallel. Hence we can delete one of these arcs, not violating the condition that all directed cuts have size at least k. So each source, and by symmetry each sink, has degree k.

Also,

(10) each vertex s not being a source or sink has degree 3.

Otherwise, by (8), all arcs leaving s are parallel, and, by symmetry, all arcs entering s are parallel. Contracting one of these parallel classes to one vertex, we obtain a counterexample with smaller sum (5), a contradiction.

Finally, let $a = (u,v)$ be an edge connecting two vertices in V_3. If $d^{\text{out}}(v) = 2$, then a is the only edge entering v, so for $U := \{v\}$ one has $d(U) = 3$ and $\delta^{\text{in}}(U) = \{a\}$. So we can assume $d^{\text{out}}(v) = 1$ and similarly $d^{\text{in}}(u) = 1$. degree 3, and that for each edge a not incident with any sourc or sink there is a cut $\delta(U)$ with $d(U) = 3$ and $\delta^{\text{in}}(U) = \{a\}$. Let digraph D' arise from D as follows. Remove arc (u,v). The two remaining arcs incident with u are in series, and hence form a directed path, from s to t say. Then replace these two arcs by one arc from s to t. Replace the two arcs incident with v similarly by a path. Thus we obtain the digraph D'. If $\sigma(D') \geq k$, we obtain a counterexample with smaller sum (5). So $\sigma(D') \leq k-1$, and hence a cut in D as required exists. ∎

Theorem 4. *In Woodall's conjecture for $k = 3$ and planar digraphs we can assume that D is reduced and planar.*

Proof. We need to adapt (8) so that the splitting-off construction maintains planarity. Indeed, we can split off arcs that are consecutive in the cyclic order of edges incident with a vertex s.

Let u and v be neighbours of s, where u is an out-neighbour and v is an in-neighbour. Assume that s does not belong to V_3. Let s' be a new vertex, and replace (s,u) and (v,s) by (s',u) and (v,s'), giving digraph D_0. Let D' be the digraph obtained by adding to D_0 the arc (s,s') and let D'' be the digraph obtained by adding to D_0 the arc (s',s). Both for D' and for D'', the number of arcs has increased by 1, while the number of vertices of degree 3 has increased by at least 1.

So the value of (5) has decreased, both for D' and for D''. We have a reduction as before if $\sigma(D') \geq 3$ or $\sigma(D'') \geq 3$, so we may assume $\sigma(D') < 3$ and $\sigma(D'') < 3$.

So there exists a nonempty proper subset U' of $V \cup \{s'\}$ with $\delta^{\text{out}}_{D'}(U') = \emptyset$ and $d^{\text{in}}_{D'}(U') \leq 2$. If U' does not separate s and s', U' would yield a directed cut in D of size less than 3, a contradiction. So U' separates s and s', hence, as no arc of D' leaves U', $s \notin U'$ and $s' \in U'$. So $d^{\text{out}}_{D_0}(U') = 0$ and $d^{\text{in}}_{D_0}(U') \leq 1$. Similarly, there exists a nonempty proper subset U'' of $V \cup \{s'\}$ with $s \in U''$, $s' \notin U''$, $d^{\text{out}}_{D_0}(U'') = 0$ and $d^{\text{in}}_{D_0}(U'') \leq 1$. Then, since $U' \cap U''$ and $U' \cup U''$ do not separate s and s',

(11) $$d_D(U' \cap U'') + d_D(U' \cup U'') = d_{D_0}(U' \cap U'') + d_{D_0}(U' \cup U'') \leq d_{D_0}(U) + d_{D_0}(U'') \leq 2.$$

Since D is 3-connected, this implies $U' \cap U'' = \emptyset$ and $U' \cup U'' = V \cup \{s'\}$. As both U' and U'' determine a directed cut in D_0, it follows that no arc of D_0 connects U' and U''. Therefore, $\delta_D(U'') = \{(s,u),(v,s)\}$, contradicting the 3-connectivity of D. ∎

5. Equivalent strengthenings of Woodall's conjecture

Let D be a reduced digraph and let $v \in V_3$. Call a partition P of $\delta(v)$ *proper* if $d^{\text{in}}(v) = 1$ and $\delta^{\text{in}}(v)$ is not a class of P, or $d^{\text{out}}(v) = 1$ and $\delta^{\text{out}}(v)$ is not a class of P. So if $d^{\text{in}}(v) = 1$, let $\delta^{\text{in}}(v) = \{a\}$ and $\delta^{\text{out}}(v) = \{a_1, a_2\}$; then the proper partitions are $\{\{a, a_1, a_2\}\}$, $\{\{a, a_1\}, \{a_2\}\}$, and $\{\{a, a_2\}, \{a_1\}\}$. Similarly if $d^{\text{in}}(v) = 1$. Call a subset B of A *splitting* if $\delta(v) \not\subseteq B$ for each vertex v. We say that a partition Π of A *extends* a partition P of $\delta(v)$ (for some vertex v) if P is equal to the collection of nonempty intersections of classes of Π with $\delta(v)$.

Theorem 5. *Let $k \geq 3$. Then Woodall's conjecture holds for k if and only if* <u>*at least one*</u> *of the following statements* (i) *and* (ii) *holds:*

(12)
- (i) *if $D = (V, A)$ is a reduced digraph, v is a vertex in V_3, and P is a proper partition of $\delta(v)$, then A has a partition Π into k strengthening sets such that Π extends P;*
- (ii) *if $D = (V, A)$ is a reduced digraph, v is a vertex in V_3, and P is a proper partition of $\delta(v)$ with $|P| = 2$, then A has a partition Π into k splitting strengthening sets such that Π extends P.*

For $k = 3$, this equivalence is maintained if all digraphs are restricted to planar digraphs.

Proof. I. Sufficiency is direct, since Woodall's conjecture is known to be true for reduced digraphs with $V_3 = \emptyset$, that is, for k-regular bipartite graphs with all edges oriented from one colour class to the other.

II. To see necessity, we first show that the negations of (12)(i) and (12)(ii) together imply the negation of

(13) if $D = (V, A)$ is a reduced digraph, v is a vertex in V_3, and P is a proper partition of $\delta(v)$ with $|P| = 2$, then A has a partition Π into k strengthening sets such that Π extends P.

Let $D = D_1$, $v = v_1$, $P = P_1$ be a counterexample to (12)(i), and let $D = D_2$, $v = v_2$, $P = P_2$ be a counterexample to (12)(ii). If $|P| = 2$ we have a counterexample to (13), so we can assume that $|P| = 1$. Now for each vertex $u \in V_3(D_2) \setminus \{v_2\}$:

(14) if $d^{\text{in}}_{D_2}(u) = d^{\text{out}}_{D_1}(v)$, connect D_2 and a copy of D_1 at u and v; if $d^{\text{in}}_{D_2}(u) = d^{\text{in}}_{D_1}(v)$, connect D_2 and a copy of $(D_1)^{-1}$ at u and v.

The final digraph forms with v_2 and P_2 a counterexample to (13).

III. We next show the following with respect to the graph D_{12} of in Figure 1. Note that $\sigma(D_{12}) = k$.

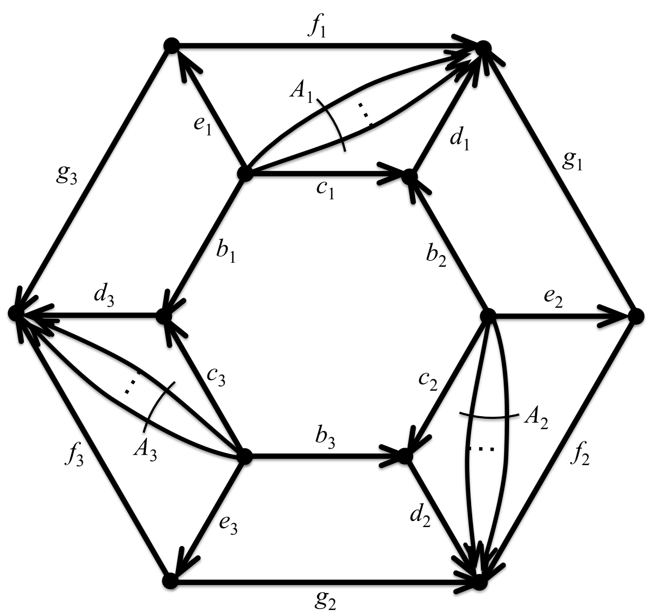

Figure 1

The digraph D_{12}, where $|A_i| = k - 3$ for each $i = 1, 2, 3$

(15) There is no strong k-coloring φ for D_{12} such that $\varphi(c_i) = \varphi(d_i)$ and $\varphi(e_i) = \varphi(f_i)$ for $i = 1, 2, 3$.

Suppose such a coloring φ exists. Then, for each $i = 1, 2, 3$, $\varphi(A_i) \cup \{\varphi(c_i), \varphi(e_i)\}$ is equal to $\varphi(A_i) \cup \{\varphi(d_i), \varphi(f_i)\}$. As this set has size $k - 1$ it follows that $\varphi(b_i) = \varphi(g_i)$.

and $\varphi(A_i \cup \{c_i, f_i\}) = [k] \setminus \{\varphi(b_i)\}$. Since $A_i \cup \{c_i, f_i, b_{i+2}, g_{i+1}\}$ (taking indices mod 3) is a directed cut, $\varphi(b_i)$ must belong to $\{\varphi(b_{i+2}), \varphi(g_{i+1})\} = \{\varphi(b_{i+2}), \varphi(b_{i+1})\}$. As this holds for all $i = 1, 2, 3$, we know that $\varphi(b_1) = \varphi(b_2) = \varphi(b_3) := t$. Then $\varphi(A_i \cup \{d_i, e_i\}) = [k] \setminus \{t\}$ for all $i = 1, 2, 3$. This implies that $\varphi^{-1}(t)$ does not intersect the set of edges from the inner hexagon to the outer hexagon, a contradiction.

IV. By part II of this proof, we can assume that we have a counterexample D, v, P for (13). By symmetry, we can assume that v has indegree 2 and outdegree 1 in D. Let $P = \{\{a, a'\}, \{a''\}\}$, where $\delta^{\text{out}}_D(v) = \{a'\}$ and $\delta^{\text{in}}_D(v) = \{a, a''\}$. So each strong k-coloring of D satisfies $\varphi(a') = \varphi(a'')$. Again we take repeated connections, now of D_{12} with copies of D or D^{-1}.

If $u \in V_3(D_{12})$ has indegree 1 and outdegree 2, then $\delta^{\text{in}}_{D_{12}}(u) = \{e_i\}$ and $\delta^{\text{out}}_{D_{12}}(u) = \{f_i, g_{i+2}\}$ for some i. Then connect D_{12} and a copy of D at u and v such that f_i and a'' are linked, e_i and a' are linked, and f_i and a are linked.

If $u \in V_3(D_{12})$ has outdegree 1 and indegree 2, then $\delta^{\text{out}}_{D_{12}}(u) = \{d_i\}$ and $\delta^{\text{in}}_{D_{12}}(u) = \{c_i, b_{i+1}\}$ for some i. Then connect D_{12} and a copy of D^{-1} at u and v such that c_i and $(a'')^{-1}$ are linked, d_i and $(a')^{-1}$ are linked, and b_{i+1} and a^{-1} are linked.

The final digraph H is a counterexample to Woodall's conjecture, by (15). ∎

As in part IV of this proof one shows:

Theorem 6. *For each $k \geq 3$,* (12)(ii) *is equivalent to the following weakened form of it:*

(16) *if $D = (V, A)$ is a reduced digraph, then A can be partitioned into k splitting strengthening sets.*

This equivalence is maintained if all digraphs are restricted to planar digraphs.

Proof. If D, v, P is a counterexample to (12)(ii), the construction with D_{12} in part IV of the proof of Theorem 5 gives a counterexample to (16). ∎

6. Is the following lemma true?

The following lemma, if true, would imply the equivalence of Woodall's conjecture and statement (16) (???).

Lemma 1. True?? *Let $G = (V, E)$ be an undirected graph. Let $\mathcal{C}$ be a collection of nonempty proper subsets of V such that*

(17) *for all $U, W \in \mathcal{C}$: if $U \cap W \neq \emptyset$ then $U \cap W \in \mathcal{C}$; if $U \cup W \neq V$ then $U \cup W \in \mathcal{C}$,*

and such that $d_E(U) \geq 2$ for each $U \in \mathcal{C}$. Then E has an orientation A such that $d^{\text{in}}_A(U) \geq 1$ and $d^{\text{out}}_A(U) \geq 1$ for each $U \in \mathcal{C}$.

Proof. (**Attempt!**) Choose a counterexample with $|V| + |E|$ smallest. Let $\preceq$ be the pre-order of V given by

(18) $u \preceq v \iff \forall U \in \mathcal{C} : v \in U \Rightarrow u \in U.$

Then $\preceq$ is a partial order, as we can contract strong components of $\preceq$. So $\mathcal{C} \cup \{\emptyset, V\}$ is equal to the collection of down-ideals in $(V, \preceq)$. Moreover, G has no circuit C, as otherwise we can choose an orientation of C and then restrict $\mathcal{C}$ to those $U \in \mathcal{C}$ not splitting C.

We show:

(19) there is no $U \in \mathcal{C}$ with $2 \leq |U| \leq |V| - 2$ such that $d_E(U) = 2$.

For assume such a U exists. Let $G' := G/\overline{U}$ and $G'' := G/U$. Define

(20) $$\mathcal{C}' := \{X \in \mathcal{C} \mid X \subseteq U\} \cup \{(X \cap U) \cup \{c_{\overline{U}}\} \mid X \in \mathcal{C}, \overline{U} \subseteq X\},$$
$$\mathcal{C}'' := \{X \in \mathcal{C} \mid X \subseteq \overline{U}\} \cup \{(X \cap \overline{U}) \cup \{c_U\} \mid X \in \mathcal{C}, U \subseteq X\},$$

where, as before, c_U and $c_{\overline{U}}$ are the vertices obtained by contracting U and $\overline{U}$, respectively, to one vertex. Then $\mathcal{C}'$ and $\mathcal{C}'$ are closed under union and intersection, and $d_{E'}(X) \geq 2$ for each $X \in \mathcal{C}'$ and $d_{E''}(X) \geq 2$ for each $X \in \mathcal{C}''$. So by induction, G' and G'' have orientations A' and A'' such that $d^{\text{in}}_{A'}(X) \geq 1$ and $d^{\text{out}}_{A'}(X) \geq 1$ for each $X \in \mathcal{C}'$ and $d^{\text{in}}_{A''}(X) \geq 1$ and $d^{\text{out}}_{A''}(X) \geq 1$ for each $X \in \mathcal{C}''$, and such that at each vertex the corresponding edges are oriented in series. As the edges in $\delta_E(U)$ are oriented in opposite directions with respect to U, we can assume that in A' and A'' the orientations coincide. Hence there is an orientation A of E that on the edges of G' coincides with A' and on the edges of G'' coincide with A''. We show that $d^{\text{in}}(X) \geq 1$ and $d^{\text{out}}(X) \geq 1$ for each $X \in \mathcal{C}$.

This is direct if $X \subseteq U$, or $U \subseteq X$, or $X \subseteq \overline{U}$, or $\overline{U} \subseteq X$. So assume to the contrary that each of $U \cap X$, $U \setminus X$, $\overline{U} \cap X$, and $\overline{U} \setminus X$ is nonempty. Then

(21) $$d^{\text{in}}(U \cap X) + d^{\text{in}}(U \cup X) \leq d^{\text{in}}(U) + d^{\text{in}}(X) \leq 1,$$

implying that $d^{\text{in}}(U \cap X) = 0$ or $d^{\text{in}}(U \cup X) = 0$, a contradiction. One similarly shows that $d^{\text{out}}(X) \geq 1$. This proves (19).

Let $V^{\max}$ and $V^{\min}$ be the sets of maximal and minimal elements of V with respect to $\preceq$. Let $V_2^{\max}$ and $V_2^{\min}$ be the sets of vertices of degree 2 in $V^{\max}$ and $V^{\min}$, respectively.

(22) Each edge e of G is incident with a vertex in $V_2^{\max} \cup V_2^{\min}$.

Consider $G - e$. As our counterexample is minimal, there exists $U \in \mathcal{C}$ with $d_{G-e}(U) \leq 1$. Then U splits e and $d_G(U) = 2$. Let $e = \{u, v\}$ with $u \in U$ and $v \notin U$. Hence by (19), $|U| = 1$ or $|U| = |V| - 1$. So $U = \{u\}$ or $U = V \setminus \{v\}$. Hence $u \in V^{\min}$ or $v \in V^{\max}$, proving (22).

(23) Let $u \prec w \prec v$ and $e = \{u, v\} \in E$. Then $u \in V_2^{\min}$ and $v \in V_2^{\max}$.

Let $G' = (V, E')$ be the graph obtained from G by replacing $e = \{u, v\}$ by $e' := \{u, w\}$. Suppose that $d_{G'}(U) \geq 2$ for all $U \in \mathcal{C}$. Then E can be oriented to A' so that $d^{\text{in}}_{A'}(U) \geq 1$ and $d^{\text{out}}_{A'}(U) \geq 1$ for all $U \in \mathcal{C}$. By symmetry, we can assume that e' is oriented as (u, w). Let A be the orientation of E that is equal to A' on $E \setminus \{e\}$, and orients e as (u, v).

Then $d^{\text{in}}_A(U) \geq 1$ for each $U \in \mathcal{C}$. Otherwise, if $d^{\text{in}}(A)(U) = 0$, then (u, w) enters U. So $u \notin U$ while $w \in U$, contradicting the fact that $u \preceq w$.

Moreover, $d^{\text{out}}_A(U) \geq 1$ for each $U \in \mathcal{C}$. Otherwise, if $d^{\text{out}}(A)(U) = 0$, then (u, v) does not leave U. So $v \in U$ and $w \notin U$, contradicting the fact that $w \preceq v$.

So $d_{G'}(U) \leq 1$ for some $U \in \mathcal{C}$. Then $e \in \delta_G(U)$ and $e' \notin \delta_{G'}(U)$. So U splits u and v and does not split u and w. So $u, w \in U$ and $v \notin U$. Moreover, $d_G(U) = 2$. So $U = V \setminus \{v\}$, hence $v \in V^{\max}$ and $d_G(v) = 2$. Similarly, $u \in V^{\min}$ and $d_u = 2$. This proves (23).

(24) No two vertices in $V_2^{\min}$ are adjacent.

For suppose $u, v \in V^{\min}$, $d_G(u) = d_G(v) = 2$, and $e := \{u, v\} \in E$. Applying induction to $\mathcal{C}' := \{U \in \mathcal{C} \mid U \text{ does not split } \{u, v\}\}$, we obtain an orientation A' of $E \setminus \{e\}$ such that $d^{\text{in}}_{A'}(U) \geq 1$ and $d^{\text{out}}_{A'}(U) \geq 1$ for all $U \in \mathcal{C}'$. Since $\{u, v\}) \in \mathcal{C}'$ and $d_G(\{u, v\}) = 2$, we can assume that the edge in $\delta_G(\{u, v\})$ is oriented towards u, and the edge in $\delta_G(\{u, v\})$ incodent with v is oriented away from v. Now orient e as (u, v), yielding A. Then $d^{\text{in}}_A(U) \geq 1$ and $d^{\text{out}}_A(U) \geq 1$ for each $U \in \mathcal{C}$. If not, by symmetry we can assume that $d^{\text{in}}_A(U) = 0$. So $U \notin \mathcal{C}'$, thatis, U splits u and v; and $u \in U$, $v \notin U$. Now $d_{A'}(U \cup \{v\}) \geq 1$. So A' contains an arc entering $U \cup \{v\}$, but not entering U, hence entering v. However, A' contains no arc entering v. This proves (24).

. . . . ∎

7. $k = 3$ and treelike digraphs

Call a digraph D *treelike* if it is reduced and there is no arc (u, v) with $u, v \in V_3$ and $d^{\text{out}}(u) = 2$ and $d^{\text{in}}(v) = 2$. We can consider them as follows. A *birooted tree* is an oriented tree T having an edge $a = (u, v)$ such that $T - a$ consists of a rooted tree with root v, and an antirooted tree with antiroot u (that is, if we reverse all orientations we obtain a rooted tree with root u). A *birooted forest* is the disjoint union of birooted trees. The sources and sinks then are the vertices of degree 1. Call a birooted tree *binary* if all degrees are 3 or 1.

Then a treelike digraph arises from a binary birooted forest by partitioning the sinks into k-tuples and identifying each such k-tuple to one sink, and similarly partitioning the sources into k-tuples and identifying each such k-tuple to one source.

A digraph $D = (V, A)$ is *weakly internally* 4*-edge-connected* if $d(U) \geq 4$ for each subset U of V with $2 \leq |U| \leq |V| - 2$.

Theorem 7. *For $k = 3$, each of* (12)(i) *and* (12)(ii) *is equivalent to its restriction to weakly internally* 4*-edge-connected treelike digraphs. This equivalence is maintained if all digraphs are restricted to planar digraphs.*

Proof. Consider any of the two statements (12)(i) and (12)(ii). Suppose the statement holds for weakly internally 4-edge-connected treelike digraphs, and suppose it does not hold for general reduced digraphs $D = (V, A)$. Choose a counterexample D, v, P with $|A|$ smallest.

Then D is weakly internally 4-edge-connected. For suppose that there is a subset U of V with $2 \leq |U| \leq |V| - 2$ and $d(U) = 3$. We can assume that v belongs to U. By the minimality, $A(D/\overline{U})$ has a partition Π into k strengthening sets such that Π extends P and such that Π on $\delta(U)$ satisfies the condition in the statement. Let P' be the partition induced by Π on $\delta(U)$. Then, again by he minimality, $A(D/U)$ has a partition Π' into k strengthening sets such that Π' extends P'. Π and Π' together partition $A(D)$ into k strengthening sets. This contradicts our assumption.

So D is weakly internally 4-edge-connected. Now suppose there exists an arc (v,u) with $v,u \in V_3$ and $d^{\text{out}}(u) = 2$ and $d^{\text{in}}(v) = 2$. As D is reduced, (v,u) belongs to a cut $\delta(U)$ with $\delta^{\text{in}}(U) = \{(v,u)\}$ and $d^{\text{out}}(U) \leq 2$. Then $u \in U$ and $v \notin U$. As $d^{\text{in}}(u) = 2$, $U \neq \{u\}$. Similarly, as $d^{\text{out}}(v) = 2$, $U \neq V \setminus \{v\}$. So U determines a cut of size at most 3, while $2 \leq |U| \leq |V| - 2$, a contradiction. ∎

Theorem 8. *For $k = 3$, Woodall's conjecture holds if and only if at least one of the following holds:*

(25) (i) *if $D = (V, A)$ is a cubic treelike digraph and v is a vertex in V_3, A has a partition into k strengthening sets that is not splitting at v;*

(ii) *if $D = (V, A)$ is a cubic treelike digraph, then A has a partition into k strengthening sets that is splitting at each vertex in V_3.*

This equivalence is maintained if all digraphs are restricted to planar digraphs.

References

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