New upper bounds for nonbinary codes based on quadruples

Bart Litjens and Sven Polak

Based on joint work with Lex Schrijver

Korteweg-de Vries Institute for Mathematics
Faculty of Science
University of Amsterdam

June 30th, 2016
Outline of the talk

- Introduction: definitions and notation
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- SDP-bound based on quadruples of code words (I)
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Definitions and notation

Fix $q, n, d \in \mathbb{N}$ with $q \geq 2$. Define $[q] := \{0, \ldots, q - 1\}$. 

Examples
(i) $d_{\min}(\{1112, 2111, 3134\}) = 2$,
(ii) $d_{\min}(\{111, 001, 010, 001, 110\}) = 1$. 

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<table>
<thead>
<tr>
<th>$k$</th>
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<tr>
<td>2</td>
<td>Delsarte (1973)</td>
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<td>3</td>
<td>Schrijver (2005) for $q = 2$ and Gijswijt, Schrijver and Tanaka (2006) for $q \in {3, 4, 5}$</td>
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$$\theta^*(q, n, d) := \max \left\{ \sum_{u, v \in [q]^n} X_{u, v} \mid X \in \mathbb{R}_{\geq 0}^{[q]^n \times [q]^n} \text{ with:} \right. $$

(i) $\text{trace}(X) = 1$,

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Proposition. $A_{q}(n, d) \leq \theta^*(q, n, d)$

Proof. Let $C \subseteq [q]^n$ be a code of minimum distance at least $d$ and maximum size. Define $X$ by $X_{u, v} = 1/|C|$ if $u, v \in C$ and $X_{u, v} = 0$ else. Then $X$ is feasible. Moreover,

$$\sum_{u, v \in [q]^n} X_{u, v} = |C|^2/|C| = A_{q}(n, d),$$

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Reductions

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Reduction of the optimization problem

Let \( G \) be the group of distance preserving permutations of \([q]^n\).

If \( X = (X_{u, v}) \) is an optimum solution, then also \( X_{\pi(u), \pi(v)} \) is optimum for all \( \pi \in G \). Moreover, the feasible region is convex.

\[ \left( \frac{1}{|G|} \right) \sum_{\pi \in G} X_{\pi} \] is a \( G \)-invariant optimum solution. Hence the SDP has at most \( n + 1 \) variables.
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Let $C_k$ be the collection of codes $C \subseteq [q]^n$ with $|C| \leq k$. Given $x : C_2 \rightarrow \mathbb{R}_{\geq 0}$, define the $C_1 \times C_1$-matrix $M_x$ by

$$(M_x)_{C,C'} = x(C \cup C').$$
Delsarte bound – II

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It can be proven that the Delsarte bound equals

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Proposition. \( A_q(n, d) \leq B_q(n, d) \)

Proof. Let \( C \subseteq [q]^n \) be a code of minimum distance at least \( d \)
The quadruple bound

\[ B_q(n, d) := \max \left\{ \sum_{v \in [q]^n} x(\{v\}) \mid x : C_4 \to \mathbb{R}_{\geq 0} \text{ with:} \right. \]

(i) \[ x(\emptyset) = 1, \]
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Proposition. \( A_q(n, d) \leq B_q(n, d) \)

Proof. Let \( C \subseteq [q]^n \) be a code of minimum distance at least \( d \) and maximum size.
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The quadruple bound

\[ B_q(n, d) \coloneqq \max \left\{ \sum_{\nu \in \{q\}^n} x(\{\nu\}) \mid x : C_4 \to \mathbb{R}_{\geq 0} \text{ with:} \right. \]

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\[ \sum_{u \in [q]^n} x(\{u\}) = |C| = A_q(n, d), \]

which yields the proposition.
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\[ \sum_{u \in [q]^n} x(\{u\}) = |C| = A_q(n, d), \]

which yields the proposition. \( \square \)
Suppose that \( q = 2 \) and let \( n, d, w \in \mathbb{N} \).

### Constant weight codes

The **weight** \( \text{wt}(u) \) of a codeword \( u \in \{0, 1\}^n \) is the number of nonzero entries in \( u \).
Intermezzo: constant weight codes

Suppose that \( q = 2 \) and let \( n, d, w \in \mathbb{N} \).

**Constant weight codes**

The **weight** \( \text{wt}(u) \) of a codeword \( u \in \{0, 1\}^n \) is the number of nonzero entries in \( u \). Let

\[
\mathcal{E} := \{ C \subseteq [q]^n \mid d_{\text{min}}(C) \geq d, \ \text{wt}(u) = w \ \forall \ u \in C \}
\]
Intermezzo: constant weight codes

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The weight $\text{wt}(u)$ of a codeword $u \in \{0, 1\}^n$ is the number of nonzero entries in $u$. Let $\mathcal{E} := \{ C \subseteq [q]^n \mid d_{\text{min}}(C) \geq d, \text{wt}(u) = w \ \forall u \in C \}$ and

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Then $A(n, d, w) \leq B(n, d, w)$, where

$$B(n, d, w) := \max \left\{ \sum_{v \in \{0,1\}^n} x(\{v\}) \mid x : C_4 \rightarrow \mathbb{R}_{\geq 0} \text{ with:} \right. $$

(i) $x(\emptyset) = 1$,

(ii) $x(C) = 0$ if $C \notin E$,

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(i) Schrijver (2005) found upper bounds on constant weight codes for $k = 3$. 
Intermezzo: constant weight codes

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(ii) Possible to find new upper bounds for $k = 4$?
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```
phase.value = pdFEAS
Iteration = 111
mu = 1.0595571803025323e-06
relative gap = 3.3729668079904213e-03
gap = 1.2966860772542390e-02
digits = 2.4719879325070719e+00
objValPrimal = -6.89000228713742179733350691711352e+02
objValDual = -6.86680166557914958473403040732179e+02
p.feas.error = 9.0463901819100310e-08
d.feas.error = 7.0071861627726210e-08
relative eps = 4.9303806576313200e-32
total time = 1440171.900
main loop time = 1439609.910000
**total time = 1440171.900000**
file read time = 550.020000
```

sven@Sven-PC:~/Documents/codesJuni$
Intermezzo: $k = 5$ for binary codes

Suppose that $q = 2$ and let $n, d \in \mathbb{N}$.

SDP-bound on $A_2(n, d)$ based on quintuples, $k = 5$

Let $\mathbf{0} := 0\ldots0$ and let $C'_k$ be the collection of codes $C \subseteq [q]^n$ with $|C| \leq k$ and $\mathbf{0} \in C$. Then $A_2(n, d) \leq Q(n, d)$, where

$$Q(n, d) := \max \left\{ \sum_{v \in [q]^n} x(\{\mathbf{0}, v\}) \mid x : C'_5 \to \mathbb{R}_{\geq 0} \text{ with:} \right. $$

(i) $x(\{\mathbf{0}\}) = 1$,

(ii) $x(C) = 0$ if $d_{\min}(C) < d$,

(iii) $M_x$ is positive semidefinite \}

where $(M_x)_{C, C'} = x(C \cup C')$ for all $x \in C'_3$. 
Intermezzo: $k = 5$ for binary codes

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Intermezzo: \( k = 5 \) for binary codes

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(iii) Possible to prove via semidefinite programming?
(iv) Around 6000 variables, 70 blocks, max block size 400.
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(ii) Proved by a clique search taking two months.

(iii) Possible to prove via semidefinite programming?

(iv) Around 6000 variables, 70 blocks, max block size 400.
Reductions

\[ B_q(n, d) := \max \left\{ \sum_{v \in [q]^n} x(\{v\}) \mid x : C_4 \rightarrow \mathbb{R}_{\geq 0} \text{ with:} \right. \]

(i) \( x(\emptyset) = 1 \),

(ii) \( x(C) = 0 \) if \( d_{\min}(C) < d \),

(iii) \( M_x \) is positive semidefinite \}.

Let \( G = S_n \wr S_n \) be the group of distance preserving permutations of \([q]^n\). If \( x \) is an optimum solution, then also \( x_\pi \) given by

\[ x_\pi(C) := x(\pi \circ C) \]

is optimum for all \( \pi \in G \).

\( \frac{1}{|G|} \sum_{\pi \in G} x_\pi \) is a \( G \)-invariant optimum solution.
Reductions

\[ B_q(n, d) := \max \left\{ \sum_{v \in [q]^n} x(\{v\}) \mid x : C_4 \to \mathbb{R}_{\geq 0} \text{ with:} \right. \]

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\]

Reduction of the optimization problem

- Let \( G = S_q^n \rtimes S_n \) be the group of distance preserving permutations of \([q]^n\).
Reductions

\[ B_q(n, d) := \max \left\{ \sum_{v \in [q]^n} \chi(\{v\}) \mid \chi : C_4 \to \mathbb{R}_{\geq 0} \text{ with:} \right. \]

(i) \( \chi(\emptyset) = 1 \),

(ii) \( \chi(C) = 0 \) if \( d_{\min}(C) < d \),

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Reduction of the optimization problem

- Let \( G = S_q^n \times S_n \) be the group of distance preserving permutations of \([q]^n\).

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- \( \frac{1}{|G|} \sum_{\pi \in G} x^\pi \) is a \( G \)-invariant optimum solution.
Let $\Pi$ denote the collection of set partitions of $\{1, 2, 3, 4\}$ in at most $q$ parts.
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Then $\Pi$ is in bijection with the orbits of the natural action of $S_q$ on $[q]^4$ via the map

$$P \mapsto S_q \cdot (a_1, a_2, a_3, a_4),$$

with $a_i = a_j$ if and only if $i$ and $j$ are in the same class in $P$. 

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with $a_i = a_j$ if and only if $i$ and $j$ are in the same class in $P$.

For example, if we assume that $q \geq 3$, then

$$\{\{1, 3\}, \{2\}, \{4\}\} \mapsto S_q \cdot (0, 1, 0, 2).$$
Let $\Omega$ be the set of $G$-orbits of $C_4$. 

$\Rightarrow |\Omega|$ bounded by a polynomial in $n$. 

Bart Litjens and Sven Polak
Semidefinite code bounds
5th SDP-day, June 30th, 2016
Let $\Omega$ be the set of $G$-orbits of $C_4$.

The map $([q]^n)^4 \rightarrow C_4$, $(v_1, v_2, v_3, v_4) \mapsto \{v_1, v_2, v_3, v_4\}$ gives a surjection $\{\text{degree } n \text{ monomials in } |\Pi| \text{ variables}\} \twoheadrightarrow \Omega \setminus \{\emptyset\}$.
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G-orbits of $C_4$ – II

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For example, writing $\{\{1, 2\}, \{3\}, \{4\}\}$ as $12, 3, 4$, letting $n = 4$ and $q \geq 3$ then

$$x_{1234}x_{12,3,4}^2 \mapsto G \cdot \{0000, 0000, 0001, 0112\} = G \cdot \{0000, 0001, 0112\}.$$
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$$= G \cdot \{0000, 0001, 0112\}.$$ 

$$\Rightarrow |\Omega| \text{ bounded by a polynomial in } n.$$
Reductions

- Replace variable $x(C)$ in the matrix $M_x$, with $C \in C_4$, by $y(w)$, with $w \in \Omega$ the orbit containing $C$. 

Get a matrix $M_y$ that is invariant under the action of $G$ on rows and columns.

$$M_y \in \text{End}_G(R_{C_2}).$$

Using representation theory of the isometry group, one can construct a matrix $U$, independent of $y$, such that

Theorem (Maschke's theorem + Schur's lemma)

$\text{End}_G(R_{C_2}) \cong - \rightarrow \bigoplus_i R_{m_i \times m_i}$ (as linear spaces), via $A \mapsto U^t A U$.

Moreover, $A$ is positive semidefinite if and only if each of the blocks of $U^t A U$ is.
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$$\implies M_y \in \text{End}_G(\mathbb{R}^{c_2}).$$
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The blocks

- Blocks parametrized by quadruples of Young shapes of certain bound heights.
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- Given a block (a quadruple of Young shapes), the size is determined by the number of semistandard Young tableaux, i.e., fillings of the shapes.

- The coefficients can be computed in time polynomial in $n$. 
### Table: New upper bounds on $A_q(n, d)$

<table>
<thead>
<tr>
<th>$q$</th>
<th>$n$</th>
<th>$d$</th>
<th>Best lower bound known</th>
<th>New upper bound</th>
<th>Best upper bound previously known</th>
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<tr>
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<td>489</td>
<td>545</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>5</td>
<td>53</td>
<td>87</td>
<td>108</td>
</tr>
</tbody>
</table>
Fix $n_2, n_3, d \in \mathbb{Z}_{\geq 0}$.

- A mixed binary/ternary code is a subset of $[2]^{n_2}[3]^{n_3}$.
Mixed binary/ternary codes

Fix $n_2, n_3, d \in \mathbb{Z}_{\geq 0}$.

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- A mixed binary/ternary code is a subset of \([2]^{n_2} [3]^{n_3}\).

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**Definition**

\[
N(n_2, n_3, d) := \max\{|C| \mid C \subseteq [2]^{n_2} [3]^{n_3}, \ d_{\min}(C) \geq d\}.
\]
Motivation: football pools

Source: http://www.uefa.com/uefaeuro/draws/
Motivation: the (extended) football pool problem

Fix $0 \leq e \leq n_2 + n_3$. Suppose $n_3$ games are played with possible outcome win/draw/loss and $n_2$ games with possible outcome win/loss.
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How many forms need to be filled in to make sure that, whatever the outcome, there is at least one form with $e$ good answers?
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**Covering problem**

How many forms need to be filled in to make sure that, whatever the outcome, there is at least one form with $e$ good answers?

**Packing problem**

How many forms can be filled in such that, whatever the outcome, there are no two or more forms with more than $e$ good answers?
Motivation: the (extended) football pool problem

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**Packing problem**

How many forms can be filled in such that, whatever the outcome, there are no two or more forms with more than $e$ good answers?

$\implies$ amounts to determining $N(n_2, n_3, d)$ with $d = 2e + 1$. 
Bounds on $N(n_2, n_3, d)$

- Lower bounds: all but one best known lower bounds found on a Spanish forum about football pools.

Upper bounds: Delsarte linear programming bound cannot be applied directly anymore.

Problem: set of mixed binary/ternary words in general does not form an association scheme with respect to the Hamming distance.

Solution: it has a product scheme structure.

$\Rightarrow$ Linear programming bound with $\leq \left( \left( n_2 + n_3 + 1 \right) \left( n_2 + n_3 + 2 \right) \right)^2$ constraints (Brouwer, Hämäläinen, Östergård and Sloane, 1998).
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Let $0 := 0 \ldots 0$ and let $C'_3$ be the collection of codes $C \subseteq [2]^{n_2}[3]^{n_3}$ with $|C| \leq 3$ and $0 \in C$. Then $N(n_2, n_3, d) \leq N_3(n_2, n_3, d)$, where

$$N_3(n_2, n_3, d) := \max \left\{ \sum_{v \in [2]^{n_2}[3]^{n_3}} x(\{0, v\}) \mid x : C'_3 \rightarrow \mathbb{R}_{\geq 0} \text{ with:} \right. \left. \begin{array}{l}
(i) \quad x(\{0\}) = 1, \\
(ii) \quad x(C) = 0 \text{ if } d_{\min}(C) < d, \\
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where $(M_x)_{C,C'} = x(C \cup C')$ for all $C, C' \in C'_2$. 
Reductions and results

- Symmetry reduction using the group \((S_2^{n_2} \rtimes S_{n_2}) \times (S_3^{n_3} \rtimes S_{n_3})\).

Results (L., 2016)

In total 135 improved upper bounds were found: 131 from the SDP with \(k = 3\), one new bound from the SDP with \(k = 4\) and three implicit improvements.
Reductions and results

- Symmetry reduction using the group \((S_{n_2}^2 \times S_{n_2}) \times (S_{n_3}^3 \times S_{n_3})\).

- Blocks and coefficients of the constraint matrices can be expressed in terms of the blocks and coefficients for the corresponding programs with \(n_2 = 0\) and \(n_3 = 0\) respectively.

Results (L., 2016)

In total 135 improved upper bounds were found: 131 from the SDP with \(k = 3\), one new bound from the SDP with \(k = 4\) and three implicit improvements.
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In total 135 improved upper bounds were found: 131 from the SDP with \(k = 3\), one new bound from the SDP with \(k = 4\) and three implicit improvements.
A selection of the results

Table: A part of the table with best known bounds on $N(n_2, n_3, 4)$. The improved bounds are boldface.

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