Solving conic optimization problems via self-dual embedding and facial reduction

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Solving conic optimization problems

For convex cone $\mathcal{K} \subseteq \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, consider conic optimization problem:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \in \mathcal{K} \end{array} \tag{1}$$

Let $\theta \in \mathbb{R} \cup \{\pm\infty\}$ denote optimal value, i.e.,

$$\theta := \inf\{c^T x : Ax = b, x \in \mathcal{K}\}.$$

We say (1) is solved if one finds:

- Optimal value θ and point attaining it (if one exists).
- Certificate of optimality, unboundedness, or infeasibility.

Certifying optimality with complementary solutions

- A primal-dual feasible point (x, s, y) of

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \in \mathcal{K} \end{array} \qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & c - A^T y = s \\ & s \in \mathcal{K}^*, \; y \in \mathbb{R}^m, \end{array}$$

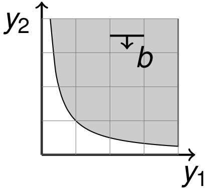
  is called a *complementary solution* if duality gap $s^T x = 0$.

- Given comp. solution, (s, y) certifies x optimal and vice-versa.

- Complementary solutions need not exist, e.g.,

$$\sup\{b^T y : c - A^T y \in \mathcal{K}^*\} \neq \inf\{c^T x : Ax = b, x \in \mathcal{K}\}$$

(Duality gap.)



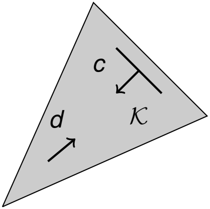
(Optimal value unattained.)

Certifying unboundedness with improving rays

- An *improving ray* is a feas. direction of strictly decreasing cost:

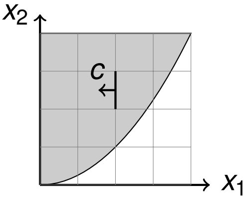
Improving ray d:

$$\begin{aligned} c^T d &< 0, \\ Ad &= 0, \\ d &\in \mathcal{K} \end{aligned}$$



- An improving ray and feasible point certify unboundedness, i.e., that $\inf\{c^T x : Ax = b, x \in \mathcal{K}\} = -\infty$.

- Improving rays need not exist, e.g.,



$Ad = 0$, $d \in \mathcal{K}$ implies
$c^T d \geq 0$

Certifying infeasibility with dual improving rays

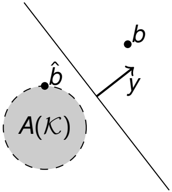
- Set $\{x : Ax = b, x \in \mathcal{K}\}$ empty if hyperplane strictly seperates

$$\{b\}, \quad A(\mathcal{K}) := \{Ax : x \in \mathcal{K}\}$$

- A *dual improving ray* y strictly separates these sets, i.e.,

$$\begin{aligned} y^T b &> 0, \\ -A^T y &\in \mathcal{K}^* \qquad \left(\text{holds iff } y^T(Ax) \leq 0 \;\; \forall x \in \mathcal{K}\right). \end{aligned}$$

- Dual improving rays need not exist when $A(\mathcal{K})$ not closed.



- Closedness of $A(\mathcal{K})$ studied extensively in Pataki '07.

Summary and motivation

In summary, to solve primal problem of

$$\begin{array}{llcll} \text{minimize} & c^T x & \quad & \text{maximize} & b^T y \\ \text{subject to} & Ax = b & & \text{subject to} & c - A^T y = s \\ & x \in \mathcal{K} & & & s \in \mathcal{K}^*, \; y \in \mathbb{R}^m, \end{array}$$

can try to find one of the following objects:

Object	Description
complementary solution	solution and optimality cert.
primal improving ray and feas. point	unboundedness cert.
dual improving ray	infeasibility cert.

This (standard) approach fails when complementary solutions and improving rays don't exist. Motivates different approach...

Overview of approach (P., Friberg, Andersen '15)

If improving rays and complementary solutions don't exist we...

- Find new problem with equal optimal value; specifically, find new cone $\mathcal{C}$:

$$\inf\{c^T x : Ax = b, x \in \mathcal{K}\} = \inf\{c^T x : Ax = b, x \in \mathcal{C}\}$$

- Find improving rays or complementary solutions for new problem; certify equality of optimal values.

To do this, we identify & exploit connection between

- Self-dual embeddings (a conic linear system)
- Facial reduction (a regularization method)

Assuming **oracle access** to embedding solutions, a *complete* algorithm for solving conic problems is obtained.

The self-dual embedding...

...is a conic system introduced by Goldman & Tucker '56:

$$\begin{aligned} Ax - b\tau &= 0, \\ c\tau - A^T y - s &= 0, \\ b^T y - c^T x - \kappa &= 0, \\ (x, s, y, \tau, \kappa) \in \mathcal{K} \times \mathcal{K}^* &\times \mathbb{R}^m \times \mathbb{R}_+ \times \mathbb{R}_+. \end{aligned}$$

All solutions satisfy $\tau\kappa = 0$,

- If $\tau > 0$, $\frac{1}{\tau}(x, s, y)$ is a complementary solution
- If $\kappa > 0$, x and/or y are improving rays
- $\tau = \kappa = 0$ holds for **all** solutions iff **no** improving rays or complementary solutions exist.

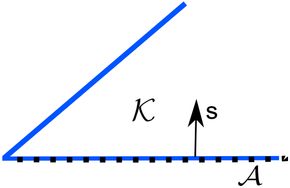
Studied extensively: Ye, Todd, Mizuno; Nesterov; Luo, Sturm, Zhang; Potra, Sheng; de Klerk, Roos, Terlaky.

Basis of solvers: SeDuMi, MOSEK, SCS, SDPT3 4.0.

Facial reduction (Borwein and Wolkowicz '81).

Let $\mathcal{A} := \{x \in \mathbb{R}^n : Ax = b\}$. Facial reduction algorithms find face $\mathcal{F} = \mathcal{K} \cap s^\perp$ containing $\mathcal{A} \cap \mathcal{K}$ by solving

$$\begin{array}{ll} \text{Find} & s \in \mathcal{K}^* \setminus \{0\} \\ \text{subject to} & s^\perp \supseteq \mathcal{A} \end{array}$$



Vector s called a *facial reduction certificate*.

- $\inf\{c^T x : x \in \mathcal{A} \cap \mathcal{K}\} = \inf\{c^T x : x \in \mathcal{A} \cap \mathcal{F}\}$
- A seq. of certs s_i yields $\mathcal{F}_N = \mathcal{K} \cap s_i^\perp \cap \cdots \cap s_N^\perp$ and

$$\begin{array}{lll} \text{minimize} & c^T x & \\ \text{subject to} & Ax = b & x \in \mathcal{F}_N \end{array} \qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & c - A^T y \in \mathcal{F}_N^*. \end{array}$$

Primal satisfies Slater's cond. OR dual has improv. ray.

Self-dual embeddings provide facial reduction certificates (P., Friberg, Andersen '15)

Theorem (P.,F.,A.'15)

If $\tau = \kappa = 0$ *and* (x, s, y, τ, κ) *is in rel. int. of the soln. set of*

$$\begin{aligned} Ax - b\tau &= 0, \\ c\tau - A^T y - s &= 0, \\ b^T y - c^T x - \kappa &= 0, \\ (x, s, y, \tau, \kappa) \in \mathcal{K} \times \mathcal{K}^* &\times \mathbb{R}^m \times \mathbb{R}_+ \times \mathbb{R}_+, \end{aligned}$$

then s *and* x *are optimal facial reduction certificates.*

- I.e., s and x expose smallest faces of $\mathcal{K}$ and $\mathcal{K}^*$ subj. to

$$s^\perp \supseteq \left\{ x \in \mathbb{R}^n : Ax = b \right\} \qquad x^\perp \supseteq \left\{ c - A^T y : y \in \mathbb{R}^m \right\}$$

- Rel. interior restriction follows de Klerk, Roos, Terlaky '98.

A complete classification of relative interior solutions

Let **H** denote solution set of the embedding:

$$\begin{aligned} Ax - b\tau &= 0, \\ c\tau - A^T y - s &= 0, \\ b^T y - c^T x - \kappa &= 0, \\ (x, s, y, \tau, \kappa) \in \mathcal{K} \times \mathcal{K}^* &\times \mathbb{R}^m \times \mathbb{R}_+ \times \mathbb{R}_+. \end{aligned}$$

- If $(x, s, y, \tau, \kappa) \in \operatorname{relint} \mathbf{H}$...

Case	What's obtained?
$\tau > 0$	Complementary solution
$\kappa > 0$	Improving ray(s)
$\tau = \kappa = 0$	Facial reduction certificate(s)

- Conversely, fix $(x, s, y, \tau, \kappa) \in \operatorname{relint} \mathbf{H}$. If complementary solutions exist, $\tau > 0$; if improving rays exist, $\kappa > 0$.

Related converses in de Klerk, Roos, Terlaky '98.

A unified algorithm P., F., A. '15 (Basic version)

Letting $\mathcal{A} := \{x \in \mathbb{R}^n : Ax = b\}$, following solves:

$$\text{min. } c^T x \text{ subj. to } x \in \mathcal{K} \cap \mathcal{A}$$

repeat
  Find relint sol. (x, s, y, τ, κ) of embedding
  if $\tau = \kappa = 0$ **then**
    $\mathcal{K} \leftarrow \mathcal{K} \cap s^{\perp}$
  else
    return complementary solution or improving ray(s)
  end
until algorithm returns;

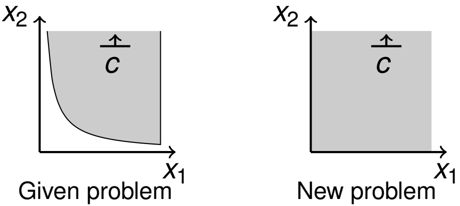
- Letting $\mathcal{K}_0$ and $\mathcal{K}_N$ denote initial and final $\mathcal{K}$,

$$\inf\{c^T x : x \in \mathcal{A} \cap \mathcal{K}_0\} = \inf\{c^T x : x \in \mathcal{A} \cap \mathcal{K}_N\}.$$

- Terminates unless optimal val. *finite but unattained* or $-\infty$ *but no primal improving ray exists*...

Handling unattained finite optimal values

- Goal: if opt. val. unattained for given problem, find new problem with equal opt. val. that's attained.



- Strategy (Abrams '75): Do m facial reduction iterations to given primal P_0, then n iterations to resulting dual. Opts vals. satisfy:

$$\theta_{P_0} = \theta_{P_1} = \cdots = \theta_{P_m} \geq \theta_{D_m} = \theta_{D'_1} = \cdots = \theta_{D'_n} \leq \theta_{P'_n}$$

- If P_m and D'_n satisfy Slater condition, $\theta_{P_0} = \theta_{P'_n}$ and $\theta_{P'_n}$ attained.

A unified algorithm P.,F.,A. '15 (Full version)

Full algorithm (always terminates, new steps in blue):

repeat
  Find relint sol. (x, s, y, τ, κ) to embedding
  if $\tau = \kappa = 0$ **then**
    if $\mathcal{A} \cap \operatorname{relint} \mathcal{K} = \emptyset$ **then**
      $\mathcal{K} \leftarrow \mathcal{K} \cap s^\perp$
    else
      $\mathcal{K} \leftarrow (\mathcal{K}^* \cap x^\perp)^*$
    end
  else
    return complementary solution or improving ray(s)
  end
until algorithm returns;

- Optimal value unchanged, i.e., letting $\mathcal{K}_0$ and $\mathcal{K}_N$ denote initial and final $\mathcal{K}$, and $\mathcal{A} := \{x \in \mathbb{R}^n : Ax = b\}$

$$\inf\{c^T x : x \in \mathcal{A} \cap \mathcal{K}_0\} = \inf\{c^T x : x \in \mathcal{A} \cap \mathcal{K}_N\}.$$

- When $\tau = \kappa = 0$, can show

$$\mathcal{A} \cap \operatorname{relint} \mathcal{K} = \emptyset \Leftrightarrow s \notin (\operatorname{span} \mathcal{K})^\perp.$$

What if we wanted to solve the dual?

To solve dual

$$\text{max. } b^T y \text{ subj. to } c - A^T y \in \mathcal{K}^*,$$

swap order of facial reduction steps, i.e., replace

if $\{x : Ax = b\} \cap \operatorname{relint} \mathcal{K} = \emptyset$ **then**
  $\mathcal{K} \leftarrow \mathcal{K} \cap s^\perp$
else
  $\mathcal{K} \leftarrow (\mathcal{K}^* \cap x^\perp)^*$
end

with

if $\{c - A^T y : y \in \mathbb{R}^m\} \cap \operatorname{relint} \mathcal{K}^* = \emptyset$ **then**
  $\mathcal{K} \leftarrow (\mathcal{K}^* \cap x^\perp)^*$
else
  $\mathcal{K} \leftarrow \mathcal{K} \cap s^\perp$
end

Now, iterations won't change $\sup\{b^T y : c - A^T y \in \mathcal{K}^*\}$.

Implementation details.

To implement, we need:

- An oracle that produces points in relative interior of embedding solution set
- A way of representing faces, e.g., $\mathcal{K} \cap \boldsymbol{s}^{\perp}$.

For semidefinite programming ($\mathcal{K} = \mathbb{S}^n_+$)

- Faces represented by subspaces of $\mathbb{R}^n$ (Barker & Carlson)
- A central-path-following algorithm can serve as an oracle. Shown in P.F.A. '15 using hammers from Halická, de Klerk, Roos, '02 and de Klerk, Roos, Terlaky '98.

A relative interior oracle for semidefinite programming.

The self-dual embedding of Ye, Todd, Mizuno '94 (strictly feasible; $\theta = 0$ at optimality; p_i, μ new parameters):

$$
\begin{array}{ll}
\text{minimize} & \mu\theta \\
\text{subject to} & \begin{aligned} Ax - b\tau &= p_1\theta \\ -A^T y - s + c\tau &= p_2\theta \\ b^T y - c^T x - \kappa &= p_3\theta \\ p_1^T y + p_2^T x + p_3\tau &= -\mu \end{aligned} \\
\multicolumn{2}{c}{(x, s, y, \tau, \kappa, \theta) \in \mathcal{K} \times \mathcal{K}^* \times \mathbb{R}^m \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}.}
\end{array}
\tag{2}
$$

If (2) an SDP, can find relative interior solution by tracking central path! (de Klerk, Roos, Terlaky; Halická, de Klerk, Roos)

Lemma (P.F.A. '15)

If $(x, s, y, \tau, \kappa, \theta)$ a rel. int. solution of (2), then (x, s, y, τ, κ) a rel. int. solution of simpler embedding.

Representing faces of the PSD cone $\mathbb{S}^n_+$.

Representation due to Barker & Carlson:

- Faces of $\mathbb{S}^n_+$ are sets of form $\{U\hat{X}U^T : \hat{X} \in \mathbb{S}^d_+\}$ where $U \in \mathbb{R}^{n \times d}$ is fixed.

$$X = U \underbrace{\hat{X}}_{\in \mathbb{S}^d_+} U^T$$

- To represent face $\mathbb{S}^n_+ \cap S^\perp$, pick any U such that

$$\text{range } U = \text{null } S.$$

Summary and future work

Summary: Connection between facial reduction and self-dual embeddings affords natural algorithm for solving conic problems; ‘implementable’ for semidefinite programming if we can track central path to its limit point.

Future work:

- How far can we push numerical implementation?
- Formal complexity analysis possible?
- Study embedding solution set (e.g., facial structure) in more detail.

Paper and references

Our Paper (P., Friberg, Andersen, '15):

www.mit.edu/~fperment

or

www.optimization − online.org/