Notes on Binary Spaces

Working text

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Contents

Chapter 1

The Max-Cut Problem

1.1 The Chinese Postman Problem

The following problem was first considered by the Chinese mathematician Mei-Ko Kwan [Kwa62]:

The daily duty of a village postman consists of delivering the mail along every street of the village, starting at the post office and returning to it at the end of his service. The obvious question is to find which route the postman should follow in order to minimize the distance he has to travel.

This problem can be formulated in more mathematical terms as follows:

The Chinese postman problem. *Given a graph $G = (V, E)$ and edge weights $w_e \in \mathbb{Q}_+$ ($e \in E$), find a tour traversing every edge of G at least once and having minimum total weight.*

At this point, let us recall some definitions. A **tour** in a graph G is an alternated sequence $(v_0, e_0, v_1, e_1, \ldots, v_{i-1}, e_{i-1}, v_i, \ldots, e_{k-1}, v_k = v_0)$ of nodes and edges, where the starting node v_0 and the final node v_k are identical and each edge e_i has endnodes v_i and v_{i+1} for $i = 0, 1, \ldots, k-1$. A **postman tour** is a tour that traverses every edge of the graph at least once and a **Eulerian tour** is a tour traversing every edge exactly once. Finally, G is said to be a **Eulerian graph** if it has a Eulerian tour. In this definition, a Eulerian graph is necessarily connected. One can extend the definition, by calling a disconnected graph Eulerian if every connected component is Eulerian. Eulerian graphs have been characterized by Euler in 1736:

THEOREM 1.1.1. *A graph G is Eulerian if and only if every node has an even degree.*

PROOF. It is clear that every node has an even degree if the graph has a Eulerian tour. Conversely, suppose that every node of G has an even degree. We show that G has a Eulerian tour by induction on its number of edges. We can supose that G is connected. Then, $\deg_G(v) \geq 2$ for every node v of G. Hence, G contains a circuit C. By the induction assumption, the graph $H := G \backslash C$, obtained from G by deleting the edges of C, is Eulerian. It is then easy to construct a Eulerian tour of G by piecing together the Eulerian tours in the connected components of H and the circuit C. ∎

A graph which can be decomposed as a union of paiwise edge disjoint circuits is called a **cycle**. It can be easily checked that cycles and Eulerian graphs are the same notions.

PROPOSITION 1.1.2. *A graph is Eulerian if and only if its edge set can be decomposed as a union of pairwise edge disjoint circuits.* ∎

A Eulerian tour in a Eulerian graph can be found, e.g., by applying the following method known as Fleury's method (see, e.g., [Wil72]): Start at any vertex u and traverse the edges in an arbitrary manner, subject only to the following rules: (i) erase edges as they are traversed and isolated nodes as they occur; (ii) at every step use an isthmus (i.e., a cut edge) only if there is no other alternative. This construction always works and produces a Eulerian tour. Other algorithms for finding a Eulerian tour are described in [EJ73].

Let us return to the Chinese postman problem. Clearly, if the graph G is Eulerian, then an optimal postman tour is a Eulerian tour, i.e., it traverses each edge exactly once and its weight is $\sum_{e\in E} w_e$. Suppose now that G is not Eulerian. Let T denote an optimal postman tour. Construct a new graph G' from G by replacing each edge $e \in E$ by a set of as many parallel edges as the number of times e is traversed by the tour T. By construction, the new graph G' is Eulerian. Clearly, the tour T traverses every edge at most twice (for, if an edge e is traversed at least three times, deleting two copies of e in G' still results in a Eulerian graph; a Eulerian tour in this smaller graph would produce a postman tour in G of smaller weight). Therefore, the Chinese postman problem amounts to finding a set of edges of smallest weight whose doubling in G creates a Eulerian graph. Clearly, for $H \subseteq E$, the graph obtained from G by doubling the edges of H is Eulerian if and only if the graph $G\backslash H$ obtained from G by deleting the edges of H is Eulerian. Therefore, an equivalent way of solving the Chinese postman problem consists in solving the following problem:

The maximum weight Eulerian subgraph problem. *Given a graph $G = (V, E)$ and edge weights w_e ($e \in E$), find a subset $F \subseteq E$ for which the graph (V, F) is Eulerian and whose weight $w(F) := \sum_{e\in F} w_e$ is maximum.*

We introduce a few more definitions. Let T be a subset of V of even cardinality. A set $H \subseteq E$ is called a **T-join** if each node $v \in T$ (resp. $v \in V \setminus T$) has an odd degree (resp. an even degree) in the graph (V, H). Note that a T-join always exists. (Indeed, let us suppose that the graph is connected. Then, if $T = \{u_1, v_1, \ldots, u_k, v_k\}$ and P_i is a path joining u_i and v_i for $i = 1, \ldots, k$, the set $P_1 \triangle \ldots \triangle P_k$ is a T-join. If the graph is disconnected, the union of T-joins in its connected components produces a T-join.) One can formulate one more optimization problem:

The minimum weigth T-join problem. *Given a graph $G = (V, E)$, a subset $T \subseteq V$ of even cardinality, and edge weights w_e ($e \in E$), find a T-join H whose weight $w(H)$ is minimum.*

Let

$$T_0 := \{v \in V \mid \deg_G(v) \text{ is odd}\}$$

denote the set of nodes of odd degree in a graph $G = (V, E)$. The full edge set E is obviously a T_0-join. Moreover, the $\emptyset$-joins are precisely the cycles in G. Therefore, for $H \subseteq E$,

$$\begin{aligned} &H \text{ is a cycle, i.e., } (V,H) \text{ is a Eulerian subgraph of } G \\ &\iff H \text{ is a } \emptyset\text{-join} \\ &\iff E \setminus H \text{ is a } T_0\text{-join.} \end{aligned} \tag{1.1.3}$$

Hence, solving the Chinese postman problem is equivalent to finding a maximum weight Eulerian subgraph in G or, equivalently, a minimum weight T_0-join (with nonnegative weights). We will see in the next section that these problems can be reduced to the weighted matching problem and, thus, solved in polynomial time.

1.2 Solving the Minimum Weight T-Join Problem

We present here the algorithm of Edmonds and Johnson [EJ73] for solving the minimum weight T-join problem. This algorithm permits also to solve the maximum weight Eulerian subgraph problem, in view of relation (1.1.3).

A first useful observation is that we can restrict ourselves to nonnegative weights.

LEMMA 1.2.1. *Let $G = (V, E)$ be a graph, $T \subseteq V$ of even cardinality, and $w \in \mathbb{Q}^E$. Set $c_e := |w_e|$ for all $e \in E$, $E_- := \{e \in E \mid w_e < 0\}$, $T_- := \{v \in V \mid v$ is adjacent to an odd number of edges of $E_-\}$, and $T' := T \triangle T_-$. Let H be a subset of E. Then, H is a T-join if and only if $H \triangle E_-$ is a T'-join. Moreover, H is a T-join of minimum weight with respect to the weights w if and only if $H \triangle E_-$ is a minimum T'-join with respect to the weights c.*

PROOF. The first statement is easy to check and the second one follows from the relation: $c(H \triangle E_-) = w(H) - w(E_-)$. ∎

Observe that every T-join admits a decomposition of the form:

$$H = P_1 \cup \ldots \cup P_{\frac{|T|}{2}} \cup C_1 \cup \ldots \cup C_k,$$

where each C_i is a circuit, each P_j is a path joining two nodes of T, and the C_i's and P_j's are pairwise edge disjoint. Hence, a T-join of minimum weight (for nonnegative weights) consists of a union of $\frac{|T|}{2}$ disjoint paths (together with some circuits of zero weight).

As was already mentioned, the T-joins for $T = \emptyset$ are the cycles.

In the case $|T| = 2$, $T := \{s, t\}$, the (inclusionwise) minimal T-joins are the (s,t)-paths; hence, the minimum weight T-join problem is the problem of finding a shortest path from s to t.

Consider now the case when $T = V$. Then, every perfect matching is a T-join and these are the only T-joins of cardinality $\frac{|V|}{2}$. Hence, the minimum weight perfect matching can be solved as an instance of the minimum weight T-join problem (to see it, modify the weights by adding a large constant).

On the other hand, it turns out that one can solve the general minimum T-join problem (for nonnegative weights) by using a weighted matching algorithm and a shortest path algorithm.

We remind that a set $M \subseteq E$ is called a **matching** in the graph $G = (V, E)$ if every node of G is adjacent to at most one edge of M. Moreover, M is a **perfect matching** if every node of V is adjacent to exactly one edge of M, i.e., if $|M| = \frac{|V|}{2}$.

The Edmonds-Johnson algorithm for the minimum weight T-join problem (with nonnegative weights) [EJ73].
Suppose we are given a graph $G = (V, E)$, a set $T \subseteq V$ of even cardinality, and $w \in \mathbb{Q}_+^E$. The algorithm proceeds as follows:
1. For every $u \neq v \in T$, compute the length $d_w(u, v)$ of a shortest path P_{uv} joining u and v in G, with respect to the weights w.
2. In the complete graph K_T with node set T and with edge weights $d_w(u, v)$ $(u, v \in T)$, find a perfect matching M whose weight $d_w(M)$ is minimum.
3. Output the set $H := \bigcup_{uv \in M} P_{uv}$.

THEOREM 1.2.2. *The set H, given as output of the Edmonds-Johnson algorithm, is a T-join of minimum weight.*

PROOF. We first verify that, if uv, $u'v'$ are two edges of the matching M (defined in step 2), then every edge e which is common to both paths P_{uv} and $P_{u'v'}$ has weight $w_e = 0$. Indeed, if $e \in P_{uv} \cap P_{u'v'}$, then $d_w(u, u') + d_w(v, v') \leq d_w(u, v) + d_w(u', v') - 2w_e$. Hence, the set $M' := M \setminus \{uv, u'v'\} \cup \{uu', vv'\}$ is a perfect matching in K_T with $d_w(M') = d_w(M) - 2w_e$, which implies that $w_e = 0$ by the choice of M.

The set H given as output in step 3 is clearly a T-join. Let H' be a T-join of minimum weight; we can suppose that H' contains no circuit. Then, $H' = P_1 \cup \ldots \cup P_{\frac{|T|}{2}}$, where the P_i's are edge disjoint paths connecting two nodes of T, say u_i and v_i. Then, the set M^* consisting of the edges (u_i, v_i) for $i \leq \frac{|T|}{2}$ is a perfect matching in K_T. We have: $d_w(M^*) = \sum_{i \leq \frac{|T|}{2}} d_w(u_i, v_i) \leq \sum_{i \leq \frac{|T|}{2}} w(P_i) = w(H')$, and $d_w(M) = \sum_{uv \in M} d_w(u, v) = \sum_{uv \in M} w(P_{uv}) = w(\cup_{uv \in M} P_{uv}) = w(H)$. By the choice of M, $d_w(M) \leq d_w(M^*)$, which implies that $w(H) \leq w(H')$, showing that H is a minimum weight T-join. ∎

This algorithm can be implemented with an overall running time in $O(n^3)$, where $n := |V|$. Indeed, step 1 can be performed in $O(n^3)$, e.g., using the Floyd-Warshall algorithm for the all pairs shortest path problem and Edmonds algorithm for the weighted matching problem can also be implemented in $O(n^3)$; see, e.g., [AMO93]. (A recent survey on matching theory can be found in [Ger95].) The minimum weight T-join problem can be solved faster in planar graphs, namely, in time $O(n^{3/2} \log n)$ (see [Gab73, MNS86, Bar90]).

REMARK 1.2.3. Another reduction of the minimum weight T-join problem to the perfect matching problem is proposed by Schrijver [Sch83a]. Construct a new weighted graph (G', w') from the instance (G, T, w) in the following way. For every edge $e = uv$ of G are introduced two new nodes u_e, v_e and edge e is replaced in G' by edge $u_e v_e$. For every node $v \in V$, add all edges $v_e v_f$ for all pairs of edges e, f incident with v. Moreover, if $v \in T$ has an even degree, add a new node v_0 adjacent to all nodes v_e (for $e \in E$ adjacent to v). Do the same for every node $v \in V \setminus T$ of odd degree. Hence, nodes in T correspond in G' to cliques of of odd size and nodes of $V \setminus T$ to cliques of even size. The weights w' are defined as follows: for every edge $e = uv \in E$, the corresponding edge $u_e v_e$ has weight

w_e and all other edges have weight 0. Then, the minimum weight perfect matchings in (G', w') correspond to the minimum weight T-joins in (G, w). Figure 1.2.4 illustrates this construction on an example. ∎

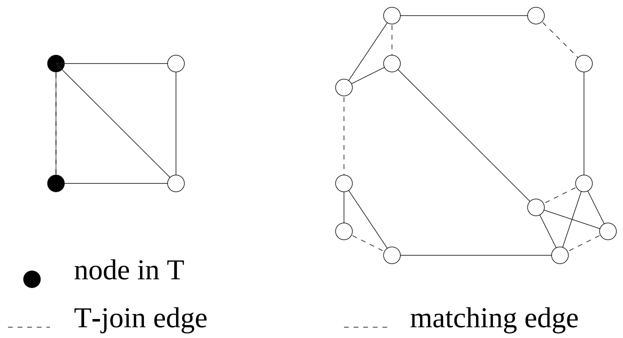


Figure 1.2.4

Hence, one can find a minimum weight T-join (for arbitrary weights) in polynomial time. This suggests that one may also be able to find a linear description of the polyhedra associated with T-joins. This is indeed the case as we will see in Section 1.4. We present in the next section further important results related to T-joins.

1.3 Min-Max Relations for T-Joins

We start with a characterization of T-joins in terms of T-cuts. Given a subset $S \subseteq V$, the cut $\delta_G(S)$ is called a **T-cut**[1] if $|S \cap T|$ is odd.

LEMMA 1.3.1. *A subset $H \subseteq E$ is a T-join if and only if $|H \cap C|$ is odd for every T-cut C.*

PROOF. Let H be a T-join and let $\delta_G(S)$ be a T-cut. Then, $\sum_{v \in S} \deg_H(v) \equiv |H \cap \delta_G(S)| \pmod 2$. On the other hand, $\sum_{v \in S} \deg_H(v) = \sum_{v \in S \cap T} \deg_H(v) + \sum_{v \in S \setminus T} \deg_H(v) \equiv |S \cap T| \equiv 1 \pmod 2$. Hence, $|H \cap \delta_G(S)|$ is odd. Conversely, suppose that $|H \cap C|$ is odd for every T-cut C. Then, $\deg_H(v) = |H \cap \delta_G(\{v\})|$ is odd for every $v \in T$. If $u \in V \setminus T$ and $v \in T$, then $\deg_H(u) = |H \cap \delta_G(u)| \equiv |H \cap \delta_G(\{u, v\})| - |H \cap \delta_G(v)| \pmod 2$ and is, therefore, even. ∎

PROPOSITION 1.3.2. *Let $X \subseteq E$. Then, X meets all T-joins if and only if X contains a T-cut. Moreover, X meets all T-cuts if and only if X contains a T-join.*

PROOF. It suffices to show the first statement. The sufficiency follows from Lemma 1.3.1. Suppose now that X meets all T-joins. Let $G_1, \ldots, G_k$ denote the connected components of the graph $G \backslash X$ (obtained by deleting the edges of X from G). Set $T_i := V(G_i) \cap T$ for

[1] In Section ?? was used the terminology "odd T-cut semimetric" for denoting the same notion.

$i \leq k$. If every set T_i has an even cardinality, let H_i be a T_i-join in G_i. Then, $H_1 \cup \ldots \cup H_k$ is a T-join in G disjoint from X, a contradiction. Hence, we can suppose for instance that $|T_1|$ is odd. Then, $\delta_G(V(G_1))$ is a T-cut contained in X. ∎

In other words, the collection of minimal T-joins and the collection of minimal T-cuts form a pair of blocking clutters[2].

In view of the above result, it is natural to try to relate the minimum size of T-joins with "packings" of T-cuts. Given integral weights $w \in \mathbb{Z}_+^E$, a **w-packing** of T-cuts is a collection $\mathcal{F} = \{C_1, \ldots, C_r\}$ of (not necessarily distinct) T-cuts such that every edge $e \in E$ belongs to at most w_e members of $\mathcal{F}$. Let $\nu(G,T,w)$ denote the maximum number of T-cuts in a w-packing. Let also $\tau(G,T,w)$ denote the minimum weight of a T-join. Hence, if $\mathbb{1}_E$ denotes the all ones vector of $\mathbb{R}^E$, then $\nu(G,T) := \nu(G,T,\mathbb{1}_E)$ denotes the maximum number of disjoint T-cuts and $\tau(G,T) := \tau(G,T,\mathbb{1}_E)$ denotes the minimum cardinality of a T-join. If $k \geq 1$ is an integer, we may also speak of a **k-packing** instead of a $k\mathbb{1}_E$-packing and we set $\nu_k(G,T) := \nu(G,T,k\mathbb{1}_E)$; in particular, $\nu_1(G,T) = \nu(G,T)$. Obviously,

$$\tau(G,T,w) \geq \frac{\nu(G,T,kw)}{k} \geq \nu(G,T,w)$$

for all $w \in \mathbb{Z}_+^E$ and $k \geq 1$ integer. In some cases, equality holds in the above inequalities. The first such result is due to Edmonds and Johnson [EJ73]:

THEOREM 1.3.3. *Let $G = (V,E)$ be a graph and $T \subseteq V$ of even cardinality. Then,*

$$\tau(G,T) = \frac{\nu_2(G,T)}{2}.$$

We will derive Theorem 1.3.3 from Theorem 1.3.5 below. First, let us state some consequences of it. Interestingly, the following result is an extension of Theorem 1.3.3 for arbitrary weights which, in fact, follows from it.

THEOREM 1.3.4. *Let $G = (V,E)$ be a graph, $T \subseteq V$ of even cardinality, and $w \in \mathbb{Z}_+^E$. Then,*

$$\tau(G,T,w) = \frac{\nu(G,T,2w)}{2}.$$

PROOF. Let us construct a new instance (G',T') from (G,T,w) in the following way: An edge $e \in E$ with $w_e \geq 1$ is subdivided by inserting $w_e - 1$ new nodes, thus creating a path P_e of w_e edges in G'; none of these new nodes belongs to T'. An edge $e \in E$ with $w_e = 0$ is contracted; the endnodes u, v of e are replaced by a single node which belongs to T' if and only if exactly one of u, v belongs to T. We show that

$$\tau(G',T') \geq \tau(G,T,w).$$

[2] Clutters and their blockers will be defined in Section 2.3.

For this, let H' be a T'-join in G'. Clearly, H' contains either all or none of the edges of each path P_e (coming from $e \in E$ with $w_e \geq 1$). Hence, H' corresponds in a naturel way to a T-join H in G such that $w(H) = |H'|$, which implies that $|H'| \geq \tau(G, T, w)$. Hence, $\tau(G', T') \geq \tau(G, T, w)$. Using Theorem 1.3.3, we deduce that $\nu_2(G', T') \geq 2\tau(G, T, w)$ and so G' contains a 2-packing of $2\tau(G, T, w)$ T'-cuts. These T'-cuts yield a collection of $2\tau(G, T, w)$ T-cuts in G forming a $2w$-packing. This shows that $\nu(G, T, 2w) \geq 2\tau(G, T, w)$. Hence, equality holds, as the converse inequality always holds. ∎

As an immediate application, we obtain the following min-max relation for the Chinese postman problem. Observe that the T_0-cuts are precisely the cuts of odd cardinality, if T_0 denotes the set of nodes of odd degree in G. Then, the minimum weight of a Chinese postman tour is equal to $w(E) + \frac{q}{2}$, where $q := \nu(G, T_0, 2w)$ denotes the maximum number of odd cardinality cuts covering every edge e at most $2w_e$ times.

The following result was proved by Seymour [Sey81b]. The proof of it that we present below is due to Sebö [Seb87].

THEOREM 1.3.5. [Sey81b] *Let $G = (V, E)$ be a bipartite graph and $T \subseteq V$ of even cardinality. Then,*

$$\tau(G, T) = \nu(G, T).$$

PROOF OF THEOREM 1.3.3. Let $G = (V, E)$ be an arbitrary graph and let G' denote the bipartite graph obtained from G by subdividing each edge e of G as a path of length 2. Clearly, a T-join in G' contains either two or zero of the edges of any such path. Hence, $\tau(G, T) \leq \frac{\tau(G', T)}{2}$. On the other hand, $\nu_2(G, T) \geq \nu(G', T)$. Therefore, using Theorem 1.3.5, $\nu_2(G, T) \geq \nu(G', T) = \tau(G', T) \geq 2\tau(G, T)$. ∎

PROOF OF THEOREM 1.3.5. We start with some observations. Let $G = (V, E)$ be a graph. By a minimum T-join we mean below a minimum cardinality T-join.

(a) *A T-join H is a minimum T-join if and only if $|H \cap C| \leq |C \setminus H|$ for every circuit (or, equivalently, every cycle) C of G.* (Indeed, $H \triangle C$ is again a T-join.)

(b) *Let H_i be a minimum T_i-join, for $i = 1, 2$. Then, $|H_1 \cap C| = |H_2 \cap C|$ for every cycle $C \subseteq H_1 \triangle H_2$.* (This follows from (a).)

(c) *Let H be a minimum T-join and let $a \neq b \in V$. Then, there exists a path P joining a and b in G for which $P \triangle H$ is a minimum T'-join, where $T' := T \triangle \{a, b\}$.* Indeed, let H' be a minimum T'-join. Then, $H \triangle H'$ is of the form $P \cup C$, where P is a path from a to b and C is a cycle disjoint from P. Then, $H'' := H' \triangle C = H \triangle P$ is again a minimum T'-join as $|H''| = |H'|$, by (b) above.

We now suppose that G is a bipartite graph. Let $T \subseteq V$ of even cardinality. We show that $\tau(G, T) \leq \nu(G, T)$. The proof is by induction on the size of the graph.

Let $a \neq b \in V$ for which $\tau(G, T \triangle \{a, b\})$ is minimum. Then,

(d) *For every minimum T-join H, $\deg_H(a) = \deg_H(b) = 1$.* Indeed, let H' be a minimum $T\triangle\{a,b\}$-join of the form $H' = P\triangle H$, where P is a path joining a and b; such H' exists from (c) above. Then, $\deg_{H'}(a) = \deg_{H'}(b) = 0$. For, if bc is an edge of H', then $H' \setminus \{bc\}$ would be a $T\triangle\{a,c\}$-join, which contradicts the choice of a, b.

The proof now goes as follows. Let H be a minimum T-join. Let G^* denote the graph obtained from G by contracting all the edges of the set $\delta_G(b)$. Set $B := \{v \in V \mid v \text{ is adjacent to } b\} \cup \{b\}$, let β denote the single node replacing B in G^*, and set

$$T^* := \begin{cases} T \setminus B \cup \{\beta\} & \text{if } |B \cap T| \text{ is odd,} \\ T \setminus B & \text{if } |B \cap T| \text{ is even.} \end{cases}$$

As $\delta_G(b)$ is a T-cut which is disjoint from any T^*-cut, we deduce that $\nu(G,T) \geq \nu(G^*,T^*)+1$. By the induction assumption, $\nu(G^*,T^*) = \tau(G^*,T^*)$. If we can show that $H^* := H \setminus \delta_G(b)$ is a minimum T^*-join in G^*, then $\tau(G^*,T^*) = |H| - 1 = \tau(G,T) - 1$, which implies the desired inequality: $\nu(G,T) \geq \tau(G,T)$.

We now show that H^* is a minimum T^*-join. Clearly, H^* is a T^*-join. If it is not minimum then, by (a), there exists a circuit C in G^* such that $|H^* \cap C| > |C \setminus H^*|$. As G^* is bipartite, this implies that (i) $|H^* \cap C| \geq |C \setminus H^*|$. Then, C corresponds in G to an (x_1,x_2)-path P, where x_1, x_2 are adjacent to b. As $C_0 := P \cup \{bx_1, bx_2\}$ is a circuit in G and H is a minimum T-join, we have that (ii) $|H \cap C_0| \leq |C_0 \setminus H|$. As a consequence, equality holds in both inequalities (i) and (ii), which implies that $bx_1, bx_2 \notin H$. Set $H_0 := H\triangle C_0$. Then, H_0 is a minimum T-join as $|H_0| = |H|$ by equality (ii). But, $\deg_{H_0}(b) \geq 2$, which contradicts (d). ∎

A graph G is called a **Seymour graph** if equality $\tau(G,T) = \nu(G,T)$ holds for all $T \subseteq V$ of even cardinality. Examples of Seymour graphs include: bipartite graphs (by Theorem 1.3.5), series parallel graphs, i.e., with no K_4-minor (see Section 2.3); another class has been found by Gerards [Ger92b]. A characterization of Seymour graphs has been recently given by Ageev, Kostochka and Szigeti [AKS95], solving a conjecture of Sebö.

1.4 T-Join Polyhedra

In this section we present the full linear description of several polyhedra associated with T-joins, namely, of the T-join polytope and of its dominant and, in particular, of the Eulerian subgraph polytope.

Let $G = (V,E)$ be a graph and $T \subseteq V$ of even cardinality. We let $\mathrm{JOIN}^\square(G,T)$ denote the **T-join polytope**, which is defined as the convex hull of the incidence vectors of the T-joins of G, i.e.,

$$\mathrm{JOIN}^\square(G,T) := \mathrm{Conv}(\{\chi^H \mid H \text{ is a } T\text{-join}\}).$$

Moreover, let $\mathrm{JOIN}^\geq(G,T)$ denote the **T-join polyhedron**, which is is defined as the dominant of the T-joins, i.e.,

$$\mathrm{JOIN}^\geq(G,T) := \mathrm{JOIN}^\square(G,T) + \mathbb{R}^E_+ = \{x \in \mathbb{R}^E_+ \mid x \geq z \text{ for some } z \in \mathrm{JOIN}^\square(G,T)\}.$$

We set

$$\mathrm{EUL}^\square(G) := \mathrm{JOIN}^\square(G,\emptyset) = \mathrm{Conv}(\{\chi^H \mid (V,H) \text{ Eulerian subgraph of } G\});$$

$\mathrm{EUL}^{\square}(G)$ is also called the **Eulerian subgraph polytope** of G (recall from (1.1.3) that the $\emptyset$-joins are the Eulerian subgraphs of G). Finally, let $\mathrm{EUL}(G)$ denote the **Eulerian subgraph cone** of G, which is defined as the cone generated by the incidence vectors of the Eulerian subgraphs of G, i.e.,

$$\mathrm{EUL}(G) := \{\sum_H \lambda_H \chi^H \mid \lambda_H \geq 0, (V,H) \text{ is a Eulerian subgraph of } G\}.$$

It turns out that a linear description of any T-join polytope can be easily derived - using the switching operation - from a linear description of the Eulerian subgraph polytope $\mathrm{EUL}^{\square}(G)$. Moreover, the knowledge of the Eulerian subgraph cone $\mathrm{EUL}(G)$ is already sufficient. So, the main two results presented below consist of the description of the T-join polyhedron $\mathrm{JOIN}^{\geq}(G,T)$ and of the Eulerian subgraph cone $\mathrm{EUL}^{\square}(G)$.

1.4.1 The T-Join Polyhedron

In a first step, we present a linear description of the T-join polyhedron; it can be obtained using Theorem 1.3.4 (as shown in Theorem 7.3.6 from [LP86]). By Lemma 1.3.1 every T-join satisfies the inequality $x(C) \geq 1$ for any T-cut C. In fact, these inequalities are sufficient for describing $\mathrm{JOIN}^{\geq}(G,T)$.

THEOREM 1.4.1. *Let $G = (V,E)$ be a graph and $T \subseteq V$ of even cardinality. Then,*

$$\mathrm{JOIN}^{\geq}(G,T) = \{x \in \mathbb{R}_+^E \mid x(C) \geq 1 \textit{ for all } T\textit{-cuts } C\}.$$

PROOF. Set $P := \{x \in \mathbb{R}_+^E \mid x(C) \geq 1$ for all T-cuts $C\}$. In order to show that $P \subseteq \mathrm{JOIN}^{\geq}(G,T)$, we show that every inequality $w^T x \geq b$ which is valid for $\mathrm{JOIN}^{\geq}(G,T)$ is also valid for P. Let $w^T x \geq b$ be a valid inequality for $\mathrm{JOIN}^{\square}(G,T)$. We can obviously suppose that w, b are integral valued. Morever, $w_e \geq 0$ for all $e \in E$. Hence, $\tau(G,T,w) \geq b$. Let $\{C_1, \ldots, C_r\}$ be a collection of $r := \nu(G,T,2w)$ T-cuts in G forming a $2w$-packing and let $x \in P$. Then,

$$\sum_{e\in E} 2w_e x_e \geq \sum_{e\in E} x_e (\sum_{1\leq i\leq r} \chi_e^{C_i}) = \sum_{1\leq i\leq r} x(C_i) \geq r.$$

This shows that $2w^T x \geq \nu(G,T,2w)$. But, we know from Theorem 1.3.4 that $\nu(G,T,2w) = 2\tau(G,T,w)$. Therefore, $w^T x \geq \tau(G,T,w) \geq b$ for all $x \in P$. This shows that the inequality $w^T x \geq b$ is valid for P. ∎

Let $\mathrm{CUT}^{\geq}(G,T)$ denote the **T-cut polyhedron** of G, which is defined as the dominant of the T-cuts, i.e.,

$$\mathrm{CUT}^{\geq}(G,T) := \{x \in \mathbb{R}_+^E \mid x \geq \sum_C \lambda_C \chi^C \text{ for some } \lambda_C \geq 0, \sum_C \lambda_C = 1, C \;\; T\text{-cut}\}.$$

As the collections of minimal T-cuts and of minimal T-joins form a pair of blocking clutters, we deduce from Theorem 1.4.1 the description of $\mathrm{CUT}^{\geq}(G,T)$ (see (2.3.4) in Section 2.3).

Corollary 1.4.2. *Let $G = (V, E)$ be a graph and $T \subseteq V$ of even cardinality. Then,*

$$\mathrm{CUT}^{\geq}(G,T) = \{x \in \mathbb{R}_+^E \mid x(H) \geq 1 \text{ for all } T\text{-joins } H \text{ in } G\}.$$

▮

Note that the separation problem for the system of inequalities defining $\mathrm{CUT}^{\geq}(G,T)$ can be solved in polynomial time, as it can be formulated as a minimum weight T-join problem. Therefore, one can optimize a linear objective function over $\mathrm{CUT}^{\geq}(G,T)$ in polynomial time, i.e., for nonnegative edge weights, one can find a minimum weight T-cut in a graph in polynomial time. Padberg and Rao [PR82] describe a direct (not using the ellipsoid method) polynomial algorithm for solving the latter problem.

For every $w \in \mathbb{Q}_+^E$ the linear programming problem:

$$\min(w^T x \mid x \in \mathbb{R}_+^E, x(C) \geq 0, \quad \forall C \ T\text{-cut})$$

has an integer optimum solution. Its dual linear program reads:

$$(1.4.3) \qquad \begin{array}{lll} \max & \sum_{C\ T\text{-cut}} y_C & \\ \text{s.t.} & \sum_{C \mid e \in C} y_C \leq w_e & (e \in E) \\ & y_C \geq 0 & (C\ T\text{-cut}). \end{array}$$

The following can be easily derived, using Theorems 1.3.4 and 1.3.5.

Theorem 1.4.4. *For every $w \in \mathbb{Z}_+^E$, the program* (1.4.3) *has a half-integral optimal solution. Moreover, if $w(C)$ is even for all cycles of G, then* (1.4.3) *has an integral optimal solution.* ▮

1.4.2 The Eulerian Subgraph Polytope

We now give a linear description of the Eulerian subgraph cone $\mathrm{EUL}(G)$. Theorem 1.4.5 below was first proved by Seymour [Sey79]. Several other proofs have been given thereafter, in particular, by Aráoz et al. [ACEGK83], by Hoffman and Lee [HL86], by Coullard and Pulleyblank [CP89]. The proof we present here is an adaptation of a proof given by Schrijver [Sch83b] for the linear description of the perfect matching polytope.

Theorem 1.4.5. *Let $G = (V, E)$ be a graph. Its Eulerian subgraph cone* $\mathrm{EUL}(G)$ *is completely described by the inequalities:*

$$(1.4.6) \qquad x_e \geq 0 \ \text{ for } e \in E,$$

$$(1.4.7) \qquad x_e - x(\delta_G(S) \setminus \{e\}) \leq 0 \text{ for } S \subseteq V, e \in \delta_G(S).$$

▮

Proof. Let $Q(G)$ denote the cone in $\mathbb{R}^E$ defined by the inequalities (1.4.6) and (1.4.7). Clearly, the incidence vector of every cycle satisfies the inequalities (1.4.6) and (1.4.7);

hence, $\mathrm{EUL}(G) \subseteq Q(G)$. Suppose that Theorem 1.4.5 does not hold and let G be a graph such that $Q(G) \not\subseteq \mathrm{EUL}(G)$ for which $|V| + |E|$ is minimum. Let x be a vector lying on an extreme ray of $Q(G)$ and such that $x \notin \mathrm{EUL}(G)$. We start with some observations.

(a) $x_e > 0$ *for all edges* $e \in E$ (else, deleting the edge e from G would yield a smaller counterexample).

(b) G *is connected.* For, suppose $G_1, \ldots, G_p$ are the connected components of G. Then, the projection x_i of x on the edge set of G_i belongs to $Q(G_i) = \mathrm{EUL}(G_i)$. Therefore, x_i is a conic combination: $x_i = \sum_{1 \leq j \leq m_i} \alpha_j^i \chi^{C_j^i}$, where all C_j^i $(1 \leq j \leq m_i)$ are cycles in G_i, for $i = 1, \ldots, p$. We can suppose that $\sum_{1 \leq j \leq m_i} \alpha_j^i$ is a constant not depending on $i = 1, \ldots, p$, say equal to α (adding if necessary a multiple of the zero vector to some of the combinations). Then, $\alpha x = \sum_{1 \leq i \leq p} \sum_{1 \leq j_i \leq m_i} \alpha_{j_1}^1 \cdot \ldots \cdot \alpha_{j_p}^p \chi^{C_{j_1}^1 \cup \ldots \cup C_{j_p}^p}$ is a conic combination of cycles of G, which contradicts our assumption that $x \notin \mathrm{EUL}(G)$.

(c) G *has no node of degree 1.* (Else, if v is adjacent to a unique edge e, then $x_e = 0$ by (1.4.7), which contradicts (a) above.)

(d) G *has* $|E| \geq |V| + 2$ *edges.* Indeed, $|E| \geq |V|$ by (c). If $|E| = |V|$ then G is a circuit; hence, by (1.4.7), $x_e = \alpha$ for all $e \in E$ (for some $\alpha \geq 0$), i.e., $x = \alpha \chi^E$ which contradicts the fact that $x \notin \mathrm{EUL}(G)$. If $|E| = |V| + 1$ then all nodes have degree 2 except two nodes of degree 3, or one node of degree 4. Hence, G is one of the following two graphs: Either G consists of a circuit C together with a path joining two nodes of C (and containing no other node of C), or G consists of two circuits sharing exactly one node. In both cases, one finds easily that $x \in \mathrm{EUL}(G)$, yielding a contradiction.

(e) *Suppose that* x *satisfies two inequalities (1.4.7) at equality for the same set* $S := \{v\}$ $(v \in V)$*; then, these two equalities are identical (up to a multiple).* Indeed, suppose these two equalities are: $x_e - x(\delta_G(v) \setminus \{e\}) = 0$ and $x_f - x(\delta_G(v) \setminus \{f\}) = 0$, where e, f are two distinct edges of $\delta_G(v)$. Then, by adding up these two equalities and using (a) above, this implies that $\delta(v) = \{e, f\}$ and the above two equalities are indeed identical (up to the factor -1).

The proof now goes as follows. As x lies on an extreme ray of $Q(G)$, there exists a set of $|E| - 1$ inequalities (1.4.7) that are satisfied at equality by x and are linearly independent. By (e), at most one such inequality is chosen for each set $S = \{v\}$ $(v \in V)$ and, by (d), $|E| \geq |V| + 2$. Therefore, there exist $S \subseteq V$, $f \in \delta_G(S)$ such that $|S|, |V \setminus S| \geq 3$ and

(1.4.8) $$x_f - x(\delta_G(S) \setminus \{f\}) = 0.$$

Let G_1 denote the graph obtained from G by contracting all nodes of S to a single node a_1 (keeping multiple edges) and let x_1 denote the corresponding projection of x on the edge set of G_1. Similarly, G_2 is obtained by contracting all nodes of $V \setminus S$ to a single node a_2 and x_2 is the projection of x on the edge set of G_2. Clearly, $x_i \in Q(G_i)$ for $i = 1, 2$. As G_i is smaller than G, we deduce that $x_i \in \mathrm{EUL}(G_i)$ for $i = 1, 2$. As x can be supposed to be rational valued we can find $k \in \mathbb{Z}_+$ such that kx_1 and kx_2 are integer valued. Then, $x_i = \frac{1}{k}(\sum_{C \in \mathcal{C}_i} \chi^C)$ where $\mathcal{C}_i$ is a collection of (not necessarily distinct) cycles in G_i, for $i = 1, 2$. We can suppose that both families $\mathcal{C}_1$ and $\mathcal{C}_2$ count the same number of members (adding if necessary the empty cycle). We now verify that these decompositions of x_1, x_2 can be combined to yield a decomposition of x as a conic combination of cycles.

For every edge $e \in \delta_G(S)$, let e_i denote the corresponding edge in G_i. As x satisfies the equality (1.4.8), we deduce that x_i satisfies the equality:

$$x_i(f_i) - x_i(\delta_{G_i}(a_i) \setminus \{f_i\}) = 0.$$

Hence, every cycle $C \in \mathcal{C}_i$ entering the decomposition of x_i satisfies:

$$|C \cap \{f_i\}| - |C \cap (\delta_{G_i}(a_i) \setminus \{f_i\})| = 0.$$

This implies that, either $C \cap \delta_G(a_i) = \emptyset$, or $C \cap \delta_G(a_i) = \{f_i, e_i\}$ for some edge $e \in \delta_G(S) \setminus \{f\}$. For $e \in \delta_G(S) \setminus \{f\}$ and for $i = 1, 2$, set

$$\mathcal{C}_i(e) := \{C \in \mathcal{C}_i \mid C \cap \delta_{G_i}(a_i) = \{f_i, e_i\}\}.$$

Set also $\mathcal{C}_i^* := \mathcal{C}_i \setminus \bigcup_{e \neq f, e \in \delta_G(S)} \mathcal{C}_i(e)$ for $i = 1, 2$. Note that

$$|\mathcal{C}_1(e)| = |\mathcal{C}_2(e)| \quad \text{for all } e \in \delta_G(S), e \neq f.$$

Indeed, by construction, $x(e) = x_1(e_1) = x_2(e_2)$ with $x_i(e_i) = |\mathcal{C}_i(e)|$. Finally, note that, for $e \in \delta_G(S)$, $e \neq f$, any $C_1 \in \mathcal{C}_1(e)$ and $C_2 \in \mathcal{C}_2(e)$ can be combined to the cycle

$$C := (C_1 \setminus \{e_1, f_1\}) \cup (C_2 \setminus \{e_2, f_2\}) \cup \{e, f\}$$

of G. Similarly, any $C_1 \in \mathcal{C}_1^*$ and $C_2 \in \mathcal{C}_2^*$ can be combined to the cycle $C := C_1 \cup C_2$ of G. Hence, a decomposition of x as a conic combination of cycles can be constructed by pairing the elements of $\mathcal{C}_1(e)$, $\mathcal{C}_2(e)$ (for $e \in \delta_G(S), e \neq f$) as well as the elements of $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$. This contradicts the assumption that $x \notin \mathrm{EUL}(G)$. ∎

As an application, we can derive a linear description of the Eulerian subgraph polytope $\mathrm{EUL}^\square(G)$. This can be done by using the switching operation, which was described in Section **??**; the next result follows from Theorem 1.4.5, by applying Proposition **??**.

THEOREM 1.4.9. *Let $G = (V, E)$ be a graph. Its Eulerian subgraph polytope $\mathrm{EUL}^\square(G)$ is completely described by the inequalities:*

(1.4.10) $\quad 0 \leq x_e \leq 1$ *for all* $e \in E$,

(1.4.11) $\quad x(F) - x(\delta_G(S) \setminus F) \leq |F| - 1$ *for all* $S \subseteq V, F \subseteq \delta_G(S)$ *with* $|F|$ *odd.* ∎

1.4.3 The T-Join Polytope

We can now give a linear description of the T-join polytope $\mathrm{JOIN}^\square(G, T)$ for arbitrary T; it follows from the description of $\mathrm{EUL}^\square(G)$, using again switching.

THEOREM 1.4.12. *Let $G = (V, E)$ be a graph and $T \subseteq V$ of even cardinality. Then, the T-join polytope $\mathrm{JOIN}^\square(G, T)$ is completely described by the inequalities:*

(1.4.13) $\quad 0 \le x_e \le 1$ *for all* $e \in E$,

(1.4.14) $\quad x(F) - x(\delta_G(S) \setminus F) \le |F| - 1,$ *for* $S \subseteq V, F \subseteq \delta_G(S)$ *with* $|F| + |S \cap T|$ *odd.*

Proof. Let B be a given T-join. Then, the T-joins of G are precisely the sets $B \triangle C$, where C is a cycle of G. Hence, by Corollary **??**, $\mathrm{JOIN}^{\square}(G,T) = r_B(\mathrm{EUL}^{\square}(G))$, where r_B is the switching mapping defined in (**??**). The linear description of $\mathrm{JOIN}^{\square}(G,T)$ now follows from Theorem 1.4.9 and applying Corollary **??**. ▮

In fact, the T-join polytope can be obtained as a projection of the perfect matching polytope. We remind that the **perfect matching polytope** $\mathrm{PMAT}^{\square}(G)$, which is defined as the convex hull of the incidence vectors of the perfect matchings of G, is fully described by the inequalities:

$$(1.4.15) \qquad \begin{cases} x_e \ge 0 & \text{for } e \in E, \\ x(\delta_G(v)) = 1 & \text{for } v \in V, \\ x(\delta_G(S)) \ge 1 & \text{for } S \subseteq V \text{ with } |S| \text{ odd.} \end{cases}$$

This fact was originally proved algorithmically by Edmonds [Edm65]. A short proof for it was given independently by Aráoz et al. [ACEGK83] and Schrijver [Sch83b]. (In fact, as was already mentioned, our proof of Theorem 1.4.12 is an adaptation of their proofs.)

We have seen in Remark 1.2.3 that a minimum weight T-join can be found in a graph G by looking for a minimum weight perfect matching in a suitably defined larger graph G'. Moreover, the T-joins in G are precisely the projections on the edge set of G of the perfect matchings of G'. In fact, it can be checked that the polytope defined by the inequalities (1.4.13) and (1.4.14) is the projection on the edge set of G of the polytope associated with G' and defined by (1.4.15). Therefore, the linear description of the T-join polytope $\mathrm{JOIN}^{\square}(G,T)$ from Theorem 1.4.12 can be alternatively deduced from the above description of the perfect matching polytope $\mathrm{PMAT}^{\square}(G')$.

1.5 The Max-Cut Problem in Planar Graphs

Let $G = (V,E)$ be a graph. Let $\mathrm{MET}^{\square}(G)$ denote the polytope in $\mathbb{R}^E$ defined by the inequalities:

(1.5.1) $\qquad 0 \le x_e \le 1$ for all $e \in E$,

(1.5.2) $\qquad x(F) - x(C \setminus F) \le |F| - 1$ for C cycle of $G, F \subseteq C, |F|$ odd.

Let also $\mathrm{MET}(G)$ denote the cone in $\mathbb{R}^E$ defined by the inequalities:

(1.5.3) $\qquad x_e \ge 0$ for all $e \in E$,

(1.5.4) $\qquad x_e - x(C \setminus \{e\}) \le 0$ for all C cycle of G and $e \in C$.

The polytope $\mathrm{MET}^{\square}(G)$ is called the **semimetric polytope** of G and the cone $\mathrm{MET}(G)$ the **semimetric cone** of G. These two polyhedra have already been introduced in Section **??**; they coincide, in fact, with the projections on the edge set of G of the polyhedra $\mathrm{MET}_n^{\square}$ and MET_n (recall Theorem **??**). Recall that, in the linear description of $\mathrm{MET}^{\square}(G)$, we can restrict ourselves to the inequalities (1.5.1) for which the edge e does not belong to any triangle of G and to the inequalities (1.5.2) for which C is a chordless circuit of G.

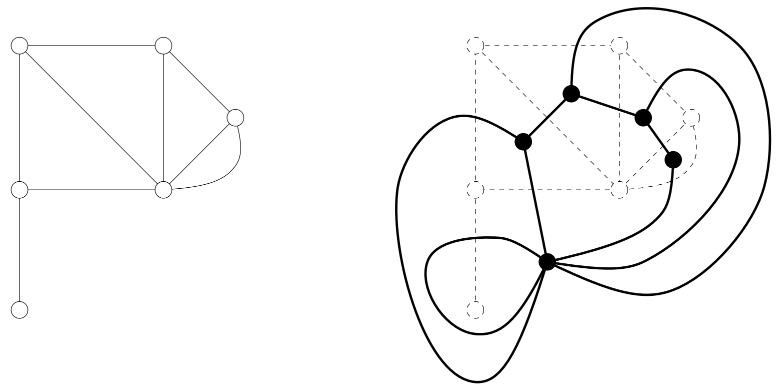
The inequalities (1.5.1) and (1.5.2) are clearly valid for all cuts, i.e.,

$$\mathrm{CUT}(G) \subseteq \mathrm{MET}(G) \text{ and } \mathrm{CUT}^{\square}(G) \subseteq \mathrm{MET}^{\square}(G).$$

We show here that equality holds in the above inclusions for planar graphs and, more generally, for graphs with no K_5-minor and for no other graphs.

A graph G is said to be **planar** if it can be drawn in the plane so that its edges intersect only at their ends. Such an embedding is called a **plane embedding** of G. Suppose that G is a **plane graph**, i.e., a planar embedding of G is already given. As is well known, another graph G^* can be constructed, the **plane dual** of G, in such a way that G^* is also planar and its dual $(G^*)^*$ coincides with G when G is connected.

The plane graph G induces a partition of the plane into a number of connected regions, called the **faces** of G. Then, G^* has one node f^* for each face f of G and one edge e^* for each edge e of G; two vertices f_1^*, f_2^* being connected by the edge e^* if and only if the corresponding faces f_1, f_2 are separated by the edge e in G. To obtain a plane embedding of G^*, simply place each node f^* in the corresponding face f of G and draw each edge e^* in such a way that it crosses the corresponding edge e once (and it does not cross any other edge of G). We illustrate the construction in Figure 1.5.5, where the dual is indicated by thick lines and dots.



plane graph G   plane dual G*

Figure 1.5.5

Observe that a cut in G corresponds precisely to a cycle in the dual G^* and vice versa. Therefore, finding a maximum weight cut in G amounts to finding a maximum weight Eulerian subgraph in G^*. As the latter problem can be solved in polynomial time, the max-cut problem can be solved in polynomial time for planar graphs; this was first proved by Hadlock [Had75], and Orlova and Dorfman [OD75]. An immediate consequence of Theorems 1.4.5 and 1.4.9 is:

THEOREM 1.5.6. *Let G be a planar graph. Then,*

$$\mathrm{CUT}(G) = \mathrm{MET}(G) \text{ and } \mathrm{CUT}^{\square}(G) = \mathrm{MET}^{\square}(G).$$

∎

In order to describe the cut polytope for the class of graphs with no K_5-minor, we need a result on how to compose cut polyhedra.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that the subgraph of G_i induced by the set $V_1 \cap V_2$ is a clique in G_i, for $i = 1, 2$. Then, the graph $G = (V := V_1 \cup V_2, E := E_1 \cup E_2)$ is called the **clique k-sum** of G_1 and G_2, where $k := |V_1 \cap V_2|$.

PROPOSITION 1.5.7. [Bar83b] *Let $G = (V, E)$ be a graph which is constructed as the clique k-sum of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ by identifying a clique K of size $k \leq 3$. Let $\mathrm{CUT}^{\square}(G_i) := \{x \in \mathbb{R}^{E_i} \mid A_i x \leq b_i\}$ be a linear description of the cut polytope of G_i, for $i = 1, 2$. Then, a linear description of the cut polytope $\mathrm{CUT}^{\square}(G)$ of G is obtained by juxtaposing the two linear descriptions: $A_1 x \leq b_1$ and $A_2 x \leq b_2$, after identifying the variables corresponding to the edges of the common clique K.*

PROOF. Set $P := \{x \in \mathbb{R}^E \mid A_1 x \leq b_1, A_2 x \leq b_2\}$. We have to show the inclusion: $P \subseteq \mathrm{CUT}^{\square}(G)$. Let $x \in P$ and let x_i denote the projection of x on the edge set of G_i for $i = 1, 2$. By assumption, $x_i \in \mathrm{CUT}^{\square}(G_i)$ for $i = 1, 2$. Hence, x_i can be decomposed as a convex combination of cuts in G_i, ie.,

$$x_i = \sum_{j \in J^i} \alpha_j^i \delta_{G_i}(S_j^i)$$

where $\alpha_j^i \geq 0, \sum_{j \in J^i} \alpha_j^i = 1, S_j^i \subseteq V(G_i)$ for $j \in J^i$, $i = 1, 2$. We show that these decompositions of x_1 and x_2 can be combined to yield a decomposition of x as a convex combination of cuts in G. This is obvious if $k = 0, 1$, as

$$x = \sum_{j_1 \in J^1,\ j_2 \in J^2} \alpha_{j_1}^1 \alpha_{j_2}^2 \delta_G(S_{j_1}^1 \cup S_{j_2}^2)$$

(assuming in the case $k = 1$ that no set S_j^i contains the common node of $V_1 \cap V_2$).

We now consider the case $k = 3$ (we will omit the case $k = 2$ which is similar, but easier). Suppose that the common clique is $K := \{v_1, v_2, v_3\}$ with edges $e_1 := v_2 v_3, e_2 := v_1 v_3, e_3 := v_1 v_2$. Note that every cut in G_i contains either zero or two edges of the circuit (e_1, e_2, e_3). Hence, $J^i := J_0^i \cup J_{12}^i \cup J_{13}^i \cup J_{23}^i$, where $j \in J_0^i$ if $e_1, e_2, e_3 \notin \delta_{G_i}(S_j^i)$, $j \in J_{12}^i$ if $e_1, e_2 \in \delta_{G_i}(S_j^i)$, and similarly for J_{13}^i and J_{23}^i. Moreover, we can suppose that $v_1, v_2, v_3 \notin S_j^i$ for $j \in J_0^i$, $S_j^i \cap \{v_1, v_2, v_3\} = \{v_3\}$ for $j \in J_{12}^i$, $S_j^i \cap \{v_1, v_2, v_3\} = \{v_2\}$ for $j \in J_{13}^i$, and $S_j^i \cap \{v_1, v_2, v_3\} = \{v_1\}$ for $j \in J_{23}^i$. Set

$$\lambda_0^i := \sum_{j \in J_0^i} \alpha_j^i,\ \lambda_{12}^i := \sum_{j \in J_{12}^i} \alpha_j^i,\ \lambda_{13}^i := \sum_{j \in J_{13}^i} \alpha_j^i,\ \lambda_{23}^i := \sum_{j \in J_{23}^i} \alpha_j^i.$$

Then, $\lambda_0^i + \lambda_{12}^i + \lambda_{13}^i + \lambda_{23}^i = 1$ and, as $x_i(e_j) = x(e_j)$ for $j = 1, 2, 3,\ i = 1, 2$,

$$\lambda_{12}^i = \frac{x(e_1) + x(e_2) - x(e_3)}{2}, \lambda_{13}^i = \frac{x(e_1) + x(e_3) - x(e_2)}{2}, \lambda_{23}^i = \frac{x(e_2) + x(e_3) - x(e_1)}{2}$$

do not depend on $i = 1, 2$. The decompositions of x_1 and x_2 can then be combined as:

$$\begin{aligned} x = \ & \frac{1}{\lambda_0^i} \sum_{j_1 \in J_0^1, j_2 \in J_0^2} \alpha_{j_1}^1 \alpha_{j_2}^2 \delta(S_{j_1}^1 \cup S_{j_2}^2) + \frac{1}{\lambda_{12}^i} \sum_{j_1 \in J_{12}^1, j_2 \in J_{12}^2} \alpha_{j_1}^1 \alpha_{j_2}^2 \delta(S_{j_1}^1 \cup S_{j_2}^2) \\ & + \frac{1}{\lambda_{13}^i} \sum_{j_1 \in J_{13}^1, j_2 \in J_{13}^2} \alpha_{j_1}^1 \alpha_{j_2}^2 \delta(S_{j_1}^1 \cup S_{j_2}^2) + \frac{1}{\lambda_{23}^i} \sum_{j_1 \in J_{23}^1, j_2 \in J_{23}^2} \alpha_{j_1}^1 \alpha_{j_2}^2 \delta(S_{j_1}^1 \cup S_{j_2}^2). \end{aligned}$$

This shows that $x \in \mathrm{CUT}^\square(G)$. ∎

The following decomposition result for graphs with no K_5-minor was proved by Wagner [Wag64, Wag70]:

Theorem 1.5.8. *Let G be a connected graph with no K_5-minor. Then, there exists a graph G' containing G as a subgraph and such that G' can be obtained by means of clique k-sums ($k \leq 3$) from planar graphs and copies of the graph V_8 shown in Figure* 1.5.9. ∎

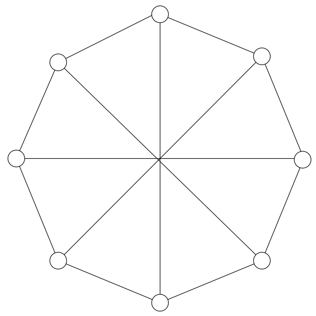


Figure 1.5.9: The graph V_8

Theorem 1.5.10. *Let G be a graph. Then,* $\mathrm{CUT}(G) = \mathrm{MET}(G)$ *or, equivalently,* $\mathrm{CUT}^\square(G) = \mathrm{MET}^\square(G)$ *if and only if G has no K_5-minor.*

Proof. Let $\mathcal{G}$ denote the class consisting of the graphs G for which $\mathrm{CUT}(G) = \mathrm{MET}(G)$. Clearly, $G \in \mathcal{G}$ if and only if $\mathrm{CUT}^\square(G) = \mathrm{MET}^\square(G)$ (because of switching; recall Proposition **??**). We first show that the class $\mathcal{G}$ is closed under taking minors. Suppose $G \in \mathcal{G}$ and let $e := uv$ be an edge of G. We verify that the graph $G \backslash e$ obtained by deleting the edge e belongs to $\mathcal{G}$. For this let $x \in \mathrm{MET}(G \backslash e)$. Then, x can be extended to a vector $y \in \mathrm{MET}(G)$ (as $\mathrm{MET}(G \backslash e)$ is the projection of MET_n on $\mathbb{R}^{E \backslash \{e\}}$). Then, $y \in \mathrm{CUT}(G)$, which implies that $x \in \mathrm{CUT}(G \backslash e)$. We now verify that the graph G/e obtained by contracting the edge e belongs to $\mathcal{G}$. For this let $x \in \mathrm{MET}(G/e)$. Then, x can be extended to $y \in \mathrm{MET}(G)$ simply by setting $y_e := 0$. Hence, $y \in \mathrm{CUT}(G)$ which implies that $x \in \mathrm{CUT}(G/e)$.

Note that $K_5 \notin \mathcal{G}$. For this, consider the vector $x \in \mathbb{R}^{E_5}$ defined by $x_{12} = x_{13} = x_{23} = x_{45} = 2$ and $x_{ij} = 1$ for $i = 1, 2, 3$ and $j = 4, 5$. Then, $x \in \mathrm{MET}(K_5) \setminus \mathrm{CUT}(K_5)$.

Therefore, if a graph G belongs to the class $\mathcal{G}$, then G has no K_5-minor. Conversely, suppose that G has no K_5-minor. Then, G belongs to $\mathcal{G}$, by applying Theorems 1.5.6 and 1.5.8, Proposition 1.5.7, and the fact ([BG82]) that $V_8 \in \mathcal{G}$. ∎

The graphs G for which $\mathrm{CUT}(G) = \mathrm{MET}(G)$ have been characterized by Seymour [Sey81a], who calls this property the "sums of circuits" property (in the more general framework of binary matroids; see Section 2.2). Hence, Theorem 1.5.10 follows from Seymour's result using switching, as mentioned by Barahona and Mahjoub [BM86]. The proof we have given here for Theorem 1.5.10 is based on the work of Barahona [Bar83b].

As an application, the max-cut problem can be solved in polynomial time for the class of graphs with no K_5-minor.

Indeed, Barahona and Mahjoub [BM86] have shown that the separation problem for the system of inequalities (1.5.1) and (1.5.2) can be solved in polynomial time; the algorithm was described in Section **??**. Therefore, one can solve any linear programming problem over the polytope $\mathrm{MET}^{\Box}(G)$ in polynomial time (using the ellipsoid method; see [GLS88]).

Another way to see it is as follows ([Bar93]). Recall from Theorem **??** that the polytope $\mathrm{MET}^{\Box}(G)$ is the projection on the edge set of G of the polytope $\mathrm{MET}^{\Box}_n$. Therefore, if G has no K_5-minor and $w \in \mathbb{R}^E$, then the max-cut problem for (G, w) can be formulated as the linear programming problem:

$$\begin{array}{lll} \max & w^T x & \\ \text{s.t.} & x_{ij} - x_{ik} - x_{jk} \leq 0 & \text{for } 1 \leq i, j, k \leq n, \\ & x_{ij} + x_{ik} + x_{jk} \leq 2 & \text{for } 1 \leq i, j, k \leq n, \end{array}$$

which has $\binom{n}{2}$ variables and $4\binom{n}{3}$ constraints (after setting $w_{ij} := 0$ if ij is not an edge of G).

Barahona [Bar83b] shows that the max-cut problem in a graph with no K_5-minor can be solved inductively, in $O(n^4)$ time, using the above mentioned decomposition result of Wagner.

When G has no K_5 minor, the linear program:

$$\max(w^T x \mid x \text{ satisfies } (1.5.1), (1.5.2))$$

has an integer optimum solution for all $w \in \mathbb{Z}^E_+$. Moreover, if $w(\delta_G(v)) \in 2\mathbb{Z}$ for all $v \in V$, then the dual linear program too has an integer optimum solution. This fact will follow from the more general result in Theorem 2.2.10.

1.6 Max-Cut and Maximum Bipartite Subgraphs

A problem, which is closely related to the max-cut problem, is the following:

The maximum weight bipartite subgraph problem. *Given a graph $G = (V, E)$ and $w \in \mathbb{Q}^E$, find a bipartite subgraph (V, F) of G whose weight $w(F)$ is maximum.*

Clearly, we may assume that the weight function is nonnegative, as an edge with a negative weight will never be chosen in an optimal solution. Observe also that the (inclusionwise) maximal bipartite subgraphs of G are precisely the (maximal) cuts of G. Therefore, for nonnegative weights, the max-cut problem can be alternatively formulated as a maximum weight bipartite subgraph problem.

1.6.1 The Bipartite Subgraph Polytope

Let $\mathrm{BIP}^{\square}(G)$ denote the **bipartite subgraph polytope** of G, which is defined as the convex hull of the incidence vectors of the bipartite subgraphs of G, i.e.,

$$\mathrm{BIP}^{\square}(G) := \mathrm{Conv}(\{\chi^F \mid F \subseteq E \text{ such that } (V,F) \text{ is bipartite}\}).$$

Hence, the max-cut problem for the pair (G,w) where $w \in \mathbb{Q}_+^E$ can be formulated as the linear program: $\max(w^T x \mid x \in \mathrm{BIP}^{\square}(G))$.

The full linear description of the polytope $\mathrm{BIP}^{\square}(G)$ is not known in general. The facial structure of $\mathrm{BIP}^{\square}(G)$ has been studied in [BGM85], where several classes of facets are described. For instance, if C is an odd cycle in G (i.e., a cycle of odd cardinality), then the **odd cycle inequality**:

$$\sum_{e \in C} x_e \leq |C| - 1 \tag{1.6.1}$$

is valid for $\mathrm{BIP}^{\square}(G)$ (as a bipartite graph contains no odd cycle). Barahona [Bar80] shows that every odd cycle inequality (1.6.1) defines a facet of $\mathrm{BIP}^{\square}(G)$; he shows also that every inequality $x_e \geq 0$ and $x_e \leq 1$ ($e \in E$) defines a facet of $\mathrm{BIP}^{\square}(G)$. Let $\mathrm{ODD}^{\square}(G)$ denote the polytope in $\mathbb{R}^E$, which is defined by the system:

$$\begin{cases} x(C) \leq |C| - 1 & \text{for } C \text{ odd cycle,} \\ 0 \leq x_e \leq 1 & \text{for } e \in E. \end{cases} \tag{1.6.2}$$

Then,

$$\mathrm{BIP}^{\square}(G) \subseteq \mathrm{ODD}^{\square}(G).$$

Grötschel and Pulleyblank [GP81] call a graph G **weakly bipartite** if

$$\mathrm{BIP}^{\square}(G) = \mathrm{ODD}^{\square}(G).$$

The interest in this class of graphs lies in the fact that the separation problem for the system (1.6.2) can be solved in polynomial time. Therefore, using the ellipsoid method (see [GLS88]), one can solve the max-cut problem for weakly bipartite graphs and nonnegative edge weights in polynomial time.

The separation problem for the system (1.6.2) reads:

Instance: A vector $x \in \mathbb{R}^E$.

Question: Does x satisfy all the inequalities of the system (1.6.2) ? If not, find an inequality from (1.6.2) which is violated by x.

The following algorithm for solving this problem has been described by Grötschel and Pulleyblank [GP81]. First check whether $0 \leq x_e \leq 1$ for all $e \in E$. If not, then we have found a violated inequality. Hence, we can now suppose that $0 \leq x_e \leq 1$ for all $e \in E$. Set $w_e := 1 - x_e$ for $e \in E$. Then, $w \geq 0$. Moreover, x satisfies all odd cycle inequalities if and only if $w(C) \geq 1$ for all odd cycles C. Suppose that we know an algorithm permitting to find a shortest (with respect to the weights w) even cardinality path between any two nodes of G. Then, for every edge $e := uv$ with $w_e < 1$, one computes a shortest even path P joining the endnodes u, v of e and one checks whether $w(P) + w_e \geq 1$. If not, then we have found a violated odd cycle inequality for x. Hence, the separation problem can be solved by applying at most $|E|$ times a shortest even path algorithm.

A shortest even path between two given nodes u, v of G can be found in the following way. Let G_u (resp. G_v) denote the graph obtained from G by deleting the node u (resp. v); we suppose that G_u and G_v have disjoint node sets. Let H denote the graph consisting of the disjoint union of G_u and G_v, where in addition every node of G_u (except v) is connected by an edge to its copy in G_v. The edges of H coming from G_u or G_v keep their original weight, while the new edges receive weight zero. Then, obviously, the even paths from u to v in G correspond to the perfect matchings (of the same weight) in H. Hence, a shortest even path from u to v in G can be found by solving a minimum weight perfect matching problem in H.

1.6.2 Weakly Bipartite Graphs

By the discussion above, one can solve the max-cut problem in polynomial time for the pairs (G, w), where G is weakly bipartite and w is nonnegative. Hence arises now the question of identifying which graphs are weakly bipartite. In fact, the complete characterization of the class of weakly bipartite graphs is a hard, yet unsolved, open problem. Seymour [Sey77] has formulated a conjecture concerning this class; we will describe it below. Several classes of graphs are known to be weakly bipartite, in particular,

(i) bipartite graphs (this is obvious !),

(ii) graphs having two nodes whose deletion produces a bipartite graph [Bar83a],

(iii) planar graphs [Bar80],

(iv) graphs with no K_5-minor [Sey81a, FMU92].

For the results corresponding to (ii),(iii),(iv), see Theorems 1.6.3, 1.6.4, 1.6.5, respectively. Further classes of weakly bipartite graphs are described in the next subsection. We refer to the work of Gerards [Ger93] for a detailed survey on what is known about weakly bipartite graphs.

THEOREM 1.6.3. *Let G be a graph such that there exist two nodes v_1, v_2 whose deletion produces a bipartite graph. Then, G is weakly bipartite.*

PROOF. Let $G' = (V', E')$ denote the graph obtained from G by deleting nodes v_1, v_2. So, G' is bipartite with, say, bipartition (U, W). We construct a new graph $\tilde{G}$ in the following way. The node set of $\tilde{G}$ is $V' \cup \{s_1, s_2, t_1, t_2\}$ where s_1, s_2, t_1, t_2 are four additional distinct nodes. The edges of $\tilde{G}$ are the edges of E' together with
- the pairs (s_i, u) $(u \in U)$ for which $(v_i, u) \in E$, $i = 1, 2$,
- the pairs (t_i, w) $(w \in W)$ for which $(v_i, w) \in E$, $i = 1, 2$, and
- the pair (s_1, t_2) if $(v_1, v_2) \in E$.

Let $\mathcal{P}$ denote the collection of all (s_i, t_i)-paths in $\tilde{G}$, for $i \in \{1, 2\}$. By the two-commodity max-flow min-cut theorem of Hu [Hu63], the program: $\max(w^T x \mid x(P) \leq |P| - 1$ for $P \in \mathcal{P}, 0 \leq x \leq 1)$ has an integer optimum solution for every objective function w. It can be easily checked that the paths in $\mathcal{P}$ correspond exactly to the odd cycles in G. This shows that G is weakly bipartite. ∎

THEOREM 1.6.4. *Every planar graph is weakly bipartite.*

PROOF. Let G be a planar graph and G^* its plane dual. Then, the odd cycles in G correspond to the odd cuts in G^*, i.e., to the T_0-cuts in G^* where T_0 denotes the set of nodes of G^* having an odd degree. Hence, the result follows from Theorem 1.4.1. ∎

THEOREM 1.6.5. *Every graph with no K_5-minor is weakly bipartite.*

The proof we present below for Theorem 1.6.5 is due to Poljak [Pol92]. It is based on Theorem 1.6.6 below, which shows that the two polytopes $\mathrm{MET}^{\square}(G)$ and $\mathrm{ODD}^{\square}(G)$ yield the same result when optimizing a nonnegative linear objective function over them. This fact is obviously true when applied to the integral polytopes $\mathrm{CUT}^{\square}(G)$ and $\mathrm{BIP}^{\square}(G)$; hence, it extends to their respective linear relaxations.

THEOREM 1.6.6. [BCH92, Pol92] *Let $G = (V, E)$ be a graph and let $w \in \mathbb{Q}_+^E$. Then,*

$$\max_{x \in \mathrm{MET}^{\square}(G)} w^T x = \max_{x \in \mathrm{ODD}^{\square}(G)} w^T x.$$

We need the following lemma for the proof of Theorem 1.6.6.

LEMMA 1.6.7. [Pol92] *Let $G = (V, E)$ be a graph and let $y = (y_{C,F})$ be a nonnegative integer valued vector, indexed by the pairs (C, F), where C is a cycle of G, $F \subseteq C$ with $|F|$ odd. If y satisfies:*

$$\sum_{(C,F)|e\in F} y_{C,F} - \sum_{(C,F)|e\in C\setminus F} y_{C,F} \geq 0 \quad \textit{for all } e \in E, \tag{1.6.8}$$

then there exist $\tilde{z} = (\tilde{z}_e)_{e\in E}$ and $\tilde{y} = (\tilde{y}_C)_{C \text{ odd cycle}}$, both nonnegative integer valued and satisfying:

$$\sum_{(C,F)|e\in F} y_{C,F} - \sum_{(C,F)|e\in C\setminus F} y_{C,F} = \tilde{z}_e + \sum_{C \text{ odd cycle}|e\in C} \tilde{y}_C \quad \textit{for all } e \in E, \tag{1.6.9}$$

$$\sum_{e\in E} \tilde{z}_e + \sum_{C \text{ odd cycle}} \tilde{y}_C \leq \sum_{(C,F)} (|F| - 1) y_{C,F}. \tag{1.6.10}$$

PROOF. Suppose that the lemma does not hold and choose such y for which $k := \sum_{(C,F)} y_{C,F}$ is minimum. Then,

(a) y cannot be decomposed as the sum $y' + y''$ of two nonnegative integer vectors y', y'', each satisfying (1.6.8) (by the minimality assumption on y).

For each pair (C, F) where C is a cycle, $F \subseteq C$ with $|F|$ odd, consider the vector $a^{C,F} \in \mathbb{Z}^E$ defined by

$$a_e^{C,F} = \begin{cases} 1 & \text{if } e \in F, \\ -1 & \text{if } e \in C \setminus F, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{A}$ denote the list of these vectors $a^{C,F}$, where $a^{C,F}$ is repeated $y_{C,F}$ times. Then, $|\mathcal{A}| = k$. Set $w := \sum_{a\in\mathcal{A}} a$; then, $w \geq 0$ by (1.6.8). We order the vectors in $\mathcal{A}$ as a sequence $a_1, \ldots, a_k$ in the following way. The first vector a_1 is chosen arbitrarily. Suppose $a_1, \ldots, a_i$ have been chosen; set $s_i := a_1 + \ldots + a_i$. Then, $0 \leq s_i \leq w$ cannot hold. (Otherwise, let $y'_{C,F}$ denote the number of times the vector $a^{C,F}$ occurs in the list $(a_1, \ldots, a_i)$ and $y''_{C,F}$ the number of times $a^{C,F}$ occurs in $\mathcal{A}\setminus\{a_1, \ldots, a_i\}$. Then, y', y'' satisfy (1.6.8) and $y = y' + y''$, which contradicts (a) above.) Hence, there exists an edge e_i such that, either $s_i(e_i) < 0$, or $s_i(e_i) > w(e_i)$. We choose $a_{i+1} \in \mathcal{A}\setminus\{a_1, \ldots, a_i\}$ such that $a_{i+1}(e_i) = 1$ in the former case, and $a_{i+1}(e_i) = -1$ in the latter case (such a_{i+1} exists as $0 \leq \sum_{a\in\mathcal{A}} a = w$). We claim that

$$\sum_{e\in E} w_e \leq 1 + \sum_{1\leq j\leq k} (|F_j| - 1) \tag{1.6.11}$$

(where (C_j, F_j) is the pair indexing a_j). Induction on $i = 1, \ldots, k$ shows easily that

$$\sum_{e|s_i(e)>0} s_i(e) \leq 1 + \sum_{1\leq j\leq i} (|F_j| - 1).$$

This relation for $i = k$ yields (1.6.11), which can be reformulated as

$$\sum_{e\in E} w_e \leq 1 + \sum_{(C,F)} y_{C,F}(|F| - 1).$$

Suppose first that $\sum_{e\in E} w_e \leq \sum_{(C,F)} y_{C,F}(|F| - 1)$. Set $\tilde{y}_C := 0$ for each odd cycle C and $\tilde{z} := w$. Then, $\tilde{y}, \tilde{z}$ obviously satisfy (1.6.9) and (1.6.10).
Suppose now that $\sum_{e\in E} w_e = 1 + \sum_{(C,F)} y_{C,F}(|F| - 1)$. This implies, in particular, that $w(E)$ is odd (as each $|F|$ is even). Note that $w(\delta_G(S)) \in 2\mathbb{Z}$ for every $S \subseteq V$. Therefore, w can be decomposed as $w = 2u + \sum_{1\leq j\leq p} \chi^{C_j}$ where $u \in \mathbb{Z}_+^E$ and the C_j's are circuits of G. Moreover, at least one of the C_j's is odd as $w(E)$ is odd. This shows the existence of an odd circuit C_0 such that $w - \chi^{C_0} \geq 0$. Set $\tilde{y}_{C_0} := 1$, $\tilde{y}_C := 0$ for every odd cycle $C \neq C_0$, and $\tilde{z} := w - \chi^{C_0}$. Then, $\tilde{y}, \tilde{z}$ satisfy (1.6.9) and (1.6.10). ∎

Proof of Theorem 1.6.6. The inequality: $\max_{x\in \mathrm{MET}^\square(G)} w^T x \leq \max_{x\in \mathrm{ODD}^\square(G)} w^T x$ holds, as $\mathrm{MET}^\square(G) \subseteq \mathrm{ODD}^\square(G)$. We show the reverse inequality. It turns out to be convenient to work with the dual programs. Let (D_{MET}) denote the dual program of the linear program: $\max(w^T x \mid x \in \mathrm{MET}^\square(G))$ and let D^*_{MET} denote its optimum value; the linear program (D_{MET}) has already been displayed in relation (**??**). Let (D_{ODD}) denote the dual of the linear program: $\max(w^T x \mid x \in \mathrm{ODD}^\square(G))$; it reads:

$$\begin{array}{lll}
\min & \sum_{C \text{ odd cycle}}(|C| - 1)y_C + \sum_{e\in E} z_e & \\
\text{s.t.} & \sum_{C|e\in C} y_C + z_e \geq w_e & (e \in E), \\
 & y_C \geq 0 & (C \text{ odd cycle}), \\
 & z_e \geq 0 & (e \in E),
\end{array}$$

and let D^*_{ODD} denote its optimum value. We show that $D^*_{MET} \geq D^*_{ODD}$. We can suppose without loss of generality that w is integer valued. Let y, z be an optimal basic solution to the program (D_{MET}); then, y, z are integer valued. We choose such a solution y, z for which $\sum_{(C,F)} y_{C,F}$ is minimum. We claim that

$$z \leq w.$$

Indeed, suppose that $z_e > w_e$ for some edge $e \in E$. Then there exists a pair $(\tilde{C}, \tilde{F})$ such that $y_{\tilde{C},\tilde{F}} > 0$ (else, $z_e \longrightarrow z_e - 1$ would yield a solution to (D_{MET}) of smaller cost). We can redefine a solution y', z' to (D_{MET}) by setting $y'_{\tilde{C},\tilde{F}} := y_{\tilde{C},\tilde{F}} - 1$, $y'_{C,F} := y_{C,F}$ for all other pairs (C, F), $z'_e := z_e - 1$, $z'_f := z_f + 1$ for $f \in \tilde{F}$, $z'_g := z_g$ for all other edges g. Then, y', z' has the same cost as y, z, but $\sum_{C,F} y'_{C,F} < \sum_{C,F} y_{C,F}$, which contradicts our minimality assumption.

As $z \leq w$, we deduce that y satisfies (1.6.8). Hence, we can apply Lemma 1.6.7. Therefore, there exist $\tilde{z}, \tilde{y}$ satisfying (1.6.9) and (1.6.10). Then, $z + \tilde{z}$ and $\tilde{y}$ provide a feasible solution to the program (D_{ODD}), showing that $D^*_{MET} \geq D^*_{ODD}$. ∎

Proof of Theorem 1.6.5. Suppose G is a graph with no K_5-minor. We show that $\mathrm{ODD}^\square(G) = \mathrm{BIP}^\square(G)$, i.e., that the program $\max_{x \in \mathrm{ODD}^\square(G)} w^T x$ has an integer optimum solution for every $w \in \mathbb{Q}^E$. Let $w \in \mathbb{Q}^E$, set $F := \{e \in E \mid w_e < 0\}$, and define $w' \in \mathbb{Q}^E_+$ by setting $w'_e := w_e$ if $w_e \geq 0$ and $w'_e := 0$ otherwise. Then, $\max_{x \in \mathrm{ODD}^\square(G)} w^T x = \max_{x \in \mathrm{ODD}^\square(G)} (w')^T x$. By Theorem 1.6.6, $\max_{x \in \mathrm{ODD}^\square(G)} (w')^T x = \max_{x \in \mathrm{MET}^\square(G)} (w')^T x$ and, by Theorem 1.5.10, $\max_{x \in \mathrm{MET}^\square(G)} (w')^T x$ is reached at a cut $\delta_G(S)$ of G. Then, $B := \delta_G(S) \setminus F$ is a bipartite subgraph of G with $w(B) = w'(\delta_G(S))$. Therefore, B is an optimum solution for $\max_{x \in \mathrm{ODD}^\square(G)} w^T x$, which shows the result. ∎

1.6.3 Seymour's Conjecture

In order to formulate Seymour's conjecture about weakly bipartite graphs, we need the notion of signed graphs. A **signed graph** is a pair (G, Σ) consisting of a graph G and a subset $\Sigma \subseteq E$ of edges; edges in Σ are called **odd** and edges in $E \setminus \Sigma$ **even**. A subset $F \subseteq E$ is said to be **Σ-odd** (or simply odd) if $|F \cap \Sigma|$ is odd and **Σ-even** otherwise. We let $\mathcal{C}(G, \Sigma)$ denote the collection of Σ-odd cycles in G. Then, the signed graph (G, Σ) is said to be **weakly bipartite** if the polytope

$$\mathrm{ODD}^\square(G, \Sigma) := \{x \in \mathbb{R}^E \mid \begin{array}{ll} 0 \leq x_e \leq 1 & \text{for } e \in E, \\ x(C) \leq |C| - 1 & \text{for } C \in \mathcal{C}(G, \Sigma)\} \end{array}$$

is integral, i.e., if all its vertices are integral valued. In the case $\Sigma = E$, $\mathrm{ODD}^\square(G, \Sigma) = \mathrm{ODD}^\square(G)$ and we find the previously defined notion of weakly bipartite graph.

Hence arises the question of characterizing the signed graphs that are weakly bipartite. Note that the signed graph $(K_5, E(K_5))$ is not weakly bipartite, as the vector $(\frac{2}{3}, \ldots, \frac{2}{3})$ is a fractional vertex of $\mathrm{ODD}^\square(K_5, E(K_5))$. Seymour's conjecture states that $(K_5, E(K_5))$ is essentially the unique obstruction to weak bipartiteness. In order to formulate it in a precise way, we need to define minors of signed graphs.

Note first that, if $\delta_G(S)$ is a cut in G, then the signed graphs (G, Σ) and $(G, \Sigma \triangle \delta_G(S))$ have the same collection of odd cycles. Hence, weak bipartiteness is invariant under **re-signing**, which means replacing Σ by the symmetric difference $\Sigma \triangle \delta_G(S)$ with a cut. Let (G, Σ) be a signed graph and let $e \in E$ be an edge of G. **Deleting** the edge e in (G, Σ) produces the signed graph $(G, \Sigma) \backslash e := (G \backslash e, \Sigma \setminus \{e\})$. One may contract only even edges. Hence, if e is odd, i.e., if $e \in \Sigma$, first re-sign (G, Σ) in order to make e even. Then,

contracting the edge $e \in E \setminus \Sigma$ in (G, Σ) produces the signed graph $(G, \Sigma)/e := (G/e, \Sigma)$. A **signed minor** of (G, Σ) is the result of a series of re-signings, deletions and contractions. It is not difficult to see that the class of weakly bipartite signed graphs is closed under taking signed minors. Hence, if (G, Σ) is weakly bipartite, it does not admit $(K_5, E(K_5))$ as a signed minor. Seymour [Sey77, Sey81a] conjectures that this condition characterizes weakly bipartite graphs. See Section 2.3 for a more general formulation of this conjecture in the context of binary matroids.

CONJECTURE 1.6.12. *A signed graph (G, Σ) is weakly bipartite if and only if it does not admit $(K_5, E(K_5))$ as a signed minor.* ∎

The following results related to this conjecture were given by Gerards [Ger93]:

(i) If G has no K_5-minor, then (G, Σ) is weakly bipartite for every signing Σ (see Corollary 1.6.15).

(ii) If G has a node u such that the signed graph obtained by deleting u from (G, Σ) has no $(K_4, E(K_4))$ minor, then (G, Σ) is weakly bipartite ([Ger92a]).

(iii) Conjecture 1.6.12 holds for any graph (G, Σ) having the property that any two Σ-odd circuits meet.

1.6.4 The Semimetric Polytope and Weakly Bipartite Graphs

We have seen in Theorem 1.6.6 that we obtain the same results if we optimize a nonnegative linear objective function over the semimetric polytope $\mathrm{MET}^{\square}(G)$ or over the odd cycle constrained polytope $\mathrm{ODD}^{\square}(G)$. In fact, this property extends to arbitrary objective functions, using the polytopes $\mathrm{ODD}^{\square}(G, \Sigma)$ where Σ encodes the positions of the negative edge weights.

THEOREM 1.6.13. *Let $G = (V, E)$ be a graph and let $w \in \mathbb{Q}^E$. Set $\Sigma := \{e \in E \mid w_e \geq 0\}$ and $c_e := |w_e|$ for all $e \in E$. Then,*

$$\max_{x \in \mathrm{MET}^{\square}(G)} w^T x = \max_{x \in \mathrm{ODD}^{\square}(G, \Sigma)} c^T x + w(E \setminus \Sigma).$$

PROOF. Set $\bar{\Sigma} := E \setminus \Sigma$. Let $r_{\bar{\Sigma}} : \mathbb{R}^E \longrightarrow \mathbb{R}^E$ denote the mapping defined by $y = r_{\bar{\Sigma}}(x)$ where

$$y_e = \begin{cases} x_e & \text{if } e \in \Sigma, \\ 1 - x_e & \text{if } e \in E \setminus \Sigma. \end{cases}$$

(This is a switching mapping, as defined in Section **??**.) Then,

$$\max_{x \in \mathrm{MET}^{\square}(G)} w^T x = \max_{x \in r_{\bar{\Sigma}}(\mathrm{MET}^{\square}(G))} c^T x + w(E \setminus \Sigma).$$

Note that the polytope $r_{\bar{\Sigma}}(\mathrm{MET}^{\square}(G))$ is defined by the inequalities: $0 \leq x_e \leq 1$ for $e \in E$, together with the inequalities: $x(F) - x(C \setminus F) \leq |F| - 1$ for the pairs (C, F) where C is a cycle in G, $F \subseteq C$ with $|F| + |C \cap \bar{\Sigma}|$ odd. One can show that

$$\max_{x \in r_{\bar{\Sigma}}(\mathrm{MET}^{\square}(G))} c^T x = \max_{x \in \mathrm{ODD}^{\square}(G,\Sigma)} c^T x.$$

The proof of this result is identical to that of Theorem 1.6.6 (i.e., for the case $\Sigma = E$). This concludes the proof of Theorem 1.6.13. ∎

As an application, we obtain the following results:

THEOREM 1.6.14. *Let $G = (V, E)$ be a graph. The following assertions are equivalent.*

(i) *The polytope* $\mathrm{MET}^{\square}(G)$ *is integral* [3].

(ii) *For every signing $\Sigma \subseteq E$, the polytope* $\mathrm{ODD}^{\square}(G,\Sigma)$ *is integral.* ∎

COROLLARY 1.6.15. *Let $G = (V, E)$ be a graph. Then, (G, Σ) is weakly bipartite for every signing $\Sigma \subseteq E$ if and only if G has no K_5-minor.* ∎

This result will be extended in the more general setting of binary matroids; see Corollary 2.3.2. We conclude with a remark on balancing signed graphs.

REMARK 1.6.16. Let (G, Σ) be a signed graph. Then, (G, Σ) is said to be **balanced** if every cycle of G contains an even number of edges from Σ ("odd" edges); a subset $F \subseteq E$ is called a **balancing set** for (G, Σ) if changing the signs of the edges in F makes the graph balanced, i.e, if the signed graph $(G, F \triangle \Sigma)$ is balanced. Then, the **balancing set polytope** $\mathrm{BAL}^{\square}(G, \Sigma)$ is defined as the convex hull of the balancing sets in (G, Σ). As can be easily checked, (G, Σ) is balanced if and only if Σ is a cut in G. Therefore, the balancing sets for (G, Σ) are the sets of the form: $\Sigma \triangle \delta_G(S)$ for $S \subseteq V$ and the balancing set polytope $\mathrm{BAL}^{\square}(G, \Sigma)$ is the image of the cut polytope $\mathrm{CUT}^{\square}(G)$ under the switching mapping r_{Σ}. In particular, finding a balancing set of minimum cardinality is an NP-hard problem; it coincides with the maximum cardinality cut problem when $\Sigma = E$. (See, e.g., [AACE81] for bounds related to the minimum cardinality balancing set.) ∎

1.7 Other Polynomial Max-Cut Instances

[3] We recall that a polyhedron P is said to be *integral* if every nonempty face of P contains an integral vector. In particular, when P has vertices, then P is integral if all its vertices are integer valued.

Chapter 2

Cycle Polytopes of Binary Matroids

The family of cuts of a graph G has the property that it is closed under taking the symmetric difference. The same property holds for the family of Eulerian subgraphs of G. Binary matroids are structures that are precisely defined by this property of closeness under the symmetric difference. The main definitions are recalled in Section 2.1.

In Section 2.2, we consider the cycle polytopes of arbitrary binary matroids. Many of the features that we have seen for the cut or Eulerian subgraph polyhedra extend for arbitrary cycle polyhedra. Moreover, we will see that binary matroids provide a unifying framework for several of the results presented earlier. For instance, we shall see in Remark 2.3.3 how the linear description of the T-join polyhedra $\mathrm{JOIN}^{\geq}(G,T)$ (for various T) can, in fact, be derived from the linear description of the Eulerian subgraph polytope $\mathrm{EUL}^{\square}(G)$. We present a characterization of the binary matroids with the sums of circuits property, i.e., of those binary matroids for which the cycle polyhedra are fully described by the "trivial necessary conditions".

In Section 2.3 we make the link with some integrality properties such as the $\mathbb{Q}_+$- and $\mathbb{Z}_+$-max-flow min-cut properties.

We group in Sections 2.4 and 2.5 results related to the lattice and the integer cone generated by the cycle space of a binary matroid.

2.1 Preliminaries on Binary Matroids

We recall here the definitions about binary matroids[1] that we need.

2.1.1 Cycles, Dual, Representation Matrix

A **binary matroid** $\mathcal{M}$ consists of a pair $(E,\mathcal{C})$, where $E =: E(\mathcal{M})$ is a finite set, the **groundset** of $\mathcal{M}$, and $\mathcal{C} =: \mathcal{C}(\mathcal{M})$ is a collection of subsets of E that is closed under taking the symmetric difference, i.e., such that

$$C \triangle C' \in \mathcal{C} \quad \text{for all } C, C' \in \mathcal{C}.$$

[1] A good reference on matroids is the book by Welsh [Wel76]. We consider here only *binary* matroids, so we will restrict our definitions to this special class of matroids.

The members of $\mathcal{C}$ are called the **cycles** of $\mathcal{M}$ and $\mathcal{C}$ is the **cycle space** of $\mathcal{M}$. Note that $\emptyset$ is always a cycle.

Two typical examples of binary matroids can be constructed from graphs. Let $G = (V, E)$ be a graph. The binary matroid on E whose cycle space is the set of Eulerian subgraphs of G is called the **graphic matroid** of G and is denoted as $\mathcal{M}(G)$. The binary matroid on E whose cycle space is the set of cuts of G is called the **cographic matroid** of G and is denoted as $\mathcal{M}^*(G)$.

Let $\mathcal{M} = (E, \mathcal{C})$ be a binary matroid. Set

$$\mathcal{C}^* := \{D \subseteq E \mid |C \cap D| \text{ is even for all } C \in \mathcal{C}\}.$$

Then, $\mathcal{C}^*$ is obviously closed under the symmetric difference. Hence, $\mathcal{M}^* := (E, \mathcal{C}^*)$ is a binary matroid, called the **dual matroid** of $\mathcal{M}$. The members of $\mathcal{C}^*$ (the cycles of $\mathcal{M}^*$) are also called the **cocycles** of $\mathcal{M}$. One can check that the dual of $\mathcal{M}^*$ coincides with $\mathcal{M}$, i.e., $(\mathcal{M}^*)^* = \mathcal{M}$.

As an example, the dual of the graphic matroid $\mathcal{M}(G)$ of a graph G is its cographic matroid $\mathcal{M}^*(G)$.

The minimal nonempty cycles of $\mathcal{M}$ are called the **circuits** of $\mathcal{M}$; similarly, the minimal nonempty cocycles are called its **cocircuits**. Every nonempty cycle can be decomposed as a disjoint union of circuits. An element $e \in E$ is called a **loop** (resp. a **coloop**) of $\mathcal{M}$ if the set $\{e\}$ is a circuit (resp. a cocircuit) of $\mathcal{M}$. Two elements $e \neq f \in E$ are said to be **parallel** (resp. **coparallel**) if the set $\{e, f\}$ is a circuit (resp. a cocircuit) of $\mathcal{M}$. Hence, in the graphic matroid $\mathcal{M}(G)$ of a graph G, a loop is a loop in G, a coloop is a cut edge (isthmus, or bridge) of G, two edges are parallel in $\mathcal{M}(G)$ if they are parallel in G and coparallel if they form a cut of size 2.

Binary matroids can be alternatively viewed as linear subspaces of the binary space $GF(2)$ [2]. Indeed, for any subsets $C, C' \subseteq E$, $\chi^{C \triangle C'} = \chi^C + \chi^{C'}$ (modulo 2). Hence, the set family $\mathcal{C}$ is closed under the symmetric difference if and only if the set $\{\chi^C \mid C \in \mathcal{C}\}$ is a linear subspace of the binary space $GF(2)^E$. Hence, identifying a set and its incidence vector, the cycle spaces of binary matroids are nothing but the linear subspaces over $GF(2)$. In this terminology, the cocycle space $\mathcal{C}^*$ of $\mathcal{M}$ is the orthogonal complement of the cycle space $\mathcal{C}$ in $GF(2)^E$.

There is yet a third way of viewing binary matroids. If M is a zero-one matrix whose columns are indexed by a set E, then the set

$$\{x \in \{0, 1\}^E \mid Mx = 0 \text{ (modulo 2)}\}$$

is a linear subspace of $GF(2)^E$. Conversely, every linear subspace of $GF(2)$ arises as the set of solutions over $GF(2)$ of an equation: $Mx = 0$ (over $GF(2)$).

Any zero-one matrix M for which the cycle space of $\mathcal{M}$ is given as the solution set of the equation $Mx = 0$ (over $GF(2)$) is called a **representation matrix** of $\mathcal{M}$. Then, a set $C \subseteq E$ is a cycle of $\mathcal{M}$ if and only if the corresponding set of columns of M is linearly dependent over $GF(2)$. A set $I \subseteq E$ is said to be **independent** in $\mathcal{M}$ if the corresponding set of columns of M is linearly independent over $GF(2)$. The maximal independent sets in $\mathcal{M}$ are called the **bases** of $\mathcal{M}$. All bases have the same cardinality, equal to the **rank** of $\mathcal{M}$.

[2] $GF(2) := \{0, 1\}$ denotes the field with two elements.

2.1.2 Minors

Let $\mathcal{M} = (E, \mathcal{C})$ be a binary matroid and let $e \in E$. Set

$$\mathcal{C}\backslash e := \{C \subseteq E \setminus \{e\} \mid C \in \mathcal{C}\}, \ \ \mathcal{C}/e := \{C \setminus \{e\} \mid C \in \mathcal{C}\}.$$

In the language of binary spaces, $\mathcal{C}\backslash e$ arises from $\mathcal{C}$ by taking its intersection with the hyperplane $x_e = 0$, while $\mathcal{C}/e$ arises from $\mathcal{C}$ by taking its projection on $\mathbb{R}^{E\backslash\{e\}}$. Both $\mathcal{C}\backslash e$ and $\mathcal{C}/e$ are again binary spaces. Hence, $\mathcal{M}\backslash e := (E \setminus \{e\}, \mathcal{C}\backslash e)$ and $\mathcal{M}/e := (E \setminus \{e\}, \mathcal{C}/e)$ are both binary matroids. One says that $\mathcal{M}\backslash e$ is obtained from $\mathcal{M}$ by **deleting** the element e and that $\mathcal{M}/e$ is obtained from $\mathcal{M}$ by **contracting** the element e. The deletion and contraction operations commute with respect to taking duals, namely,

$$(\mathcal{M}\backslash e)^* = \mathcal{M}^*/e, \ \ (\mathcal{M}/e)^* = \mathcal{M}^*\backslash e.$$

If $\mathcal{N}$ is a binary matroid that can be obtained from $\mathcal{M}$ by a series of deletions and contractions, one says that $\mathcal{N}$ is a **minor** of $\mathcal{M}$. Every minor of $\mathcal{M}$ is of the form: $\mathcal{M}\backslash X/Y$, where X, Y are two disjoint subsets of E (as the deletion and contraction operations commute, i.e., $\mathcal{M}\backslash e/f = \mathcal{M}/f\backslash e$). Observe that the deletion and contraction operations, when applied to the graphic matroid $\mathcal{M}(G)$ of a graph G, correspond to the usual operations of deleting and contracting an edge in G, i.e,

$$(\mathcal{M}(G))\backslash e = \mathcal{M}(G\backslash e), \ (\mathcal{M}(G))/e = \mathcal{M}(G/e).$$

Hence, the class of graphic matroids is closed under taking minors; the same holds for the class of cographic matroids.

We now present several concrete examples of binary matroids, as we will need them later.

The Fano Matroid. The **Fano matroid** F_7 is the binary matroid on seven elements which is represented by the following matrix:

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

(the columns are indexed by the elements of $E = \{1, \ldots, 7\}$ as indicated). The circuits of F_7 are the seven sets:

$$124, 135, 167, 236, 257, 347, 456$$

together with their complements. (Here we denote a set $\{1, 2, 4\}$ by the string 124.) The cycles of F_7 are the sets: $\emptyset$, E, C and $E \setminus C$, for C circuit of F_7. Observe that the circuits of size 3 of F_7 can also be viewed as the lines of the Fano plane, which is shown in Figure 2.1.1.

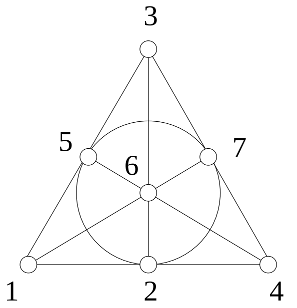


Figure 2.1.1: The Fano plane

The **dual Fano matroid** F_7^* is the dual of the Fano matroid F_7. It is represented, e.g., by the following matrix:

$$\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

The circuits of F_7^* are the sets:

$$3567, 2467, 2345, 1457, 1346, 1256, 1237$$

(i.e., the complements of the lines in the Fano plane). The only additional cycle is $\emptyset$.

The Matroid R_{10}. Let R_{10} denote the binary matroid on ten elements which is represented by the 5×10 matrix whose columns are all the binary vectors with three ones and two zeros, i.e, the matrix:

$$\begin{array}{c} \begin{array}{cccccccccc} e_{34} & e_{35} & e_{45} & e_{23} & e_{24} & e_{25} & e_{13} & e_{14} & e_{15} & e_{12} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

It is convenient to suppose that the columns of the above matrix are indexed by the set E consisting of the 10 edges of the complete graph K_5 on $\{1,2,3,4,5\}$ (here, e_{ij} denotes the edge between nodes i, j in K_5). Then, one sees easily that the cycles of R_{10} correspond to the cycles in K_5 having an even length, and the cocycles of R_{10} correspond to the sets: $\emptyset$, E, $\delta_{K_5}(S)$, and $E \setminus \delta_{K_5}(S)$ for $S \subseteq \{1,2,3,4,5\}$. Observe that the matroid R_{10} is self dual, i.e., that R_{10}^* is isomorphic to R_{10}.

The Binary Projective Space $\mathcal{P}_r$. Let $\mathcal{P}_r$ denote the binary matroid which is represented by the matrix W_r, which is the $r \times (2^r - 1)$ matrix whose columns are all possible nonzero binary vectors of length r. For instance, $\mathcal{P}_1$ is the matroid consisting just of one coloop, and $\mathcal{P}_3$ coincides with the Fano matroid F_7. $\mathcal{P}_r$ is the binary projective space of dimension $r-1$. Its dual matroid $\mathcal{P}_r^*$ is represented by the matrix $[I_{2^r-r-1}|A_r]$, where the rows of A_r are all possible binary vectors of length r except the zero vector and the unit vectors. The following can be checked about the cycles of $\mathcal{P}_r^*$: $\mathcal{P}_r^*$ has 2^r cycles; each nonempty cycle of $\mathcal{P}_r^*$ has cardinality 2^{r-1} (and, hence, is a circuit). More information about $\mathcal{P}_r$ will be given in Section 2.4.

2.2 The Cycle Polytope and Cone

let $\mathcal{M} = (E, \mathcal{C})$ be a binary matroid. The **cycle polytope** of $\mathcal{M}$ is the polytope $\mathrm{CYC}^\square(\mathcal{M})$, which is defined as the convex hull of the incidence vectors of the cycles of $\mathcal{M}$, i.e.,

$$\mathrm{CYC}^\square(\mathcal{M}) := \mathrm{Conv}(\{\chi^C \mid C \in \mathcal{C}\}).$$

The **cycle cone** of $\mathcal{M}$ is the cone $\mathrm{CYC}(\mathcal{M})$, which is defined as the conic hull of the incidence vectors of the cycles of $\mathcal{M}$, i.e.,

$$\mathrm{CYC}(\mathcal{M}) := \{\sum_{C \in \mathcal{C}} \lambda_C \chi^C \mid \lambda_C \geq 0 \text{ for all } C \in \mathcal{C}\}.$$

Note that, for the graphic and cographic matroids, the cycle polyhedra coincide, respectively, with the Eulerian subgraph and the cut polyhedra; namely,

$$\mathrm{CYC}^\square(\mathcal{M}(G)) = \mathrm{EUL}^\square(G),\ \mathrm{CYC}(\mathcal{M}(G)) = \mathrm{EUL}(G), \text{ and}$$

$$\mathrm{CYC}^\square(\mathcal{M}^*(G)) = \mathrm{CUT}^\square(G),\ \mathrm{CYC}(\mathcal{M}^*(G)) = \mathrm{CUT}(G).$$

We present in this section several results related to the cycle polyhedra. We follow essentially the work of Barahona and Grötschel [BG86], and Grötschel and Truemper [GT89b, GT89a].

2.2.1 Switching Symmetries

A first basic property of the cycle polyhedra is that they enjoy a lot of symmetries, namely, the switching symmetries ([BG86]). We remind a few facts, already presented in Section **??**. Given a subset $C \subseteq E$, the switching mapping $r_C : \mathbb{R}^E \longrightarrow \mathbb{R}^E$ is defined by $y = r_C(x)$ where

$$y_e = \begin{cases} 1 - x_e & \text{if } e \in C, \\ x_e & \text{if } e \in E \setminus C. \end{cases}$$

As the cycle space $\mathcal{C}$ is closed under the symmetric difference, the switching mapping r_C preserves the cycle polytope $\mathrm{CYC}^\square(\mathcal{M})$ for every $C \in \mathcal{C}$, i.e.,

$$r_C(\mathrm{CYC}^\square(\mathcal{M})) = \mathrm{CYC}^\square(\mathcal{M}).$$

The switching mapping acts on inequalities in the following way: If x satisfies the inequality: $v^T x \leq v_0$, then its image $y = r_C(x)$ satisfies the inequality: $(v^C)^T y \leq v_0 - v(C)$, where $v_e^C = -v_e$ if $e \in C$ and $v_e^C = v_e$ if $e \in E \setminus C$.

As an application, all the facets of the polytope $\mathrm{CYC}^\square(\mathcal{M})$ can be derived from the list of facets of $\mathrm{CYC}^\square(\mathcal{M})$ containing any given vertex and, in particular, from the list of facets of $\mathrm{CYC}^\square(\mathcal{M})$ containing the origin, that is, from the list of facets of the cycle cone $\mathrm{CYC}(\mathcal{M})$. (Recall Proposition **??**.)

Let $\mathcal{M} = (E, \mathcal{C})$ be a binary matroid with representation matrix M. Then, the polytope

$$\mathrm{CYC}^{\square}(\mathcal{M}, b) := \mathrm{Conv}(\{x \in \{0,1\}^E \mid Mx = b \text{ (modulo 2)}\})$$

can be easily deduced from the cycle polytope $\mathrm{CYC}^{\square}(\mathcal{M})$ of $\mathcal{M}$ (here, b is an arbitrary 0-1 vector). Indeed, let $C_0 \subseteq E$ such that χ^{C_0} is a solution of the equation $Mx = b$ (modulo 2). Then, $\mathrm{CYC}^{\square}(\mathcal{M}, b)$ is the image of the cycle polytope $\mathrm{CYC}^{\square}(\mathcal{M})$ under the switching mapping r_{C_0}. In particular, the facial structure of $\mathrm{CYC}^{\square}(\mathcal{M}, b)$ can be easily deduced from that of $\mathrm{CYC}^{\square}(\mathcal{M})$. As an example, the switchings of the Eulerian subgraph polytope of a graph G are the T-join polytopes of G. As another example, the switchings of the cut polytope of G are its balancing set polytopes; recall Remark 1.6.16.

2.2.2 The Sums of Circuits Property

Some valid inequalities for the cycle polytope $\mathrm{CYC}^{\square}(\mathcal{M})$ can be easily defined as follows. First, the inequalities: $0 \leq x_e \leq 1$ (for $e \in E$) are trivially valid for $\mathrm{CYC}^{\square}(\mathcal{M})$. Let D be a cocycle of $\mathcal{M}$ and let $F \subseteq D$ with $|F|$ odd; then, the **cocycle inequality**:

$$x(F) - x(D \setminus F) \leq |F| - 1 \tag{2.2.1}$$

is valid for $\mathrm{CYC}^{\square}(\mathcal{M})$ (this follows from the fact that every cycle has an even intersection with the cocycle D). Note that the cocycle inequalities (2.2.1) for a given cocycle D and various F are equivalent, under switching. Let $\mathrm{MET}^{\square}(\mathcal{M})$ denote the polytope in $\mathbb{R}^E$, which is defined by the inequalities: $0 \leq x_e \leq 1$ ($e \in E$) together with the cocycle inequalities (2.2.1) for $D \in \mathcal{C}^*$, $F \subseteq D$ with $|F|$ odd. Similarly, define the cone:

$$\mathrm{MET}(\mathcal{M}) := \{x \in \mathbb{R}^E_+ \mid x_e - x(D \setminus \{e\}) \leq 0 \text{ for all } D \in \mathcal{C}^*, e \in D\}.$$

In the case when $\mathcal{M}$ is the cographic matroid $\mathcal{M}^*(G)$ of a graph G, then $\mathrm{MET}^{\square}(\mathcal{M}^*(G)) = \mathrm{MET}^{\square}(G)$ and $\mathrm{MET}(\mathcal{M}^*(G)) = \mathrm{MET}(G)$ are the usual semimetric polyhedra of the graph G; this explains our notation. Observe that, in the description of the polyhedra $\mathrm{MET}^{\square}(\mathcal{M})$ and $\mathrm{MET}(\mathcal{M})$, we can restrict ourselves to the cocycle inequalities (2.2.1) where D is a cocircuit (else, D is a disjoint union of cocircuits and the corresponding cocycle inequality follows obviously from other cocircuit inequalities).

LEMMA 2.2.2. *Let $X \subseteq E$. Then, χ^X lies on an extreme ray of the cone* $\mathrm{MET}(\mathcal{M})$ *if and only if X is a circuit in $\mathcal{M}$.*

PROOF. Let C be a circuit in $\mathcal{M}$. Suppose that $\chi^C = x + y$ where $x, y \in \mathrm{MET}(\mathcal{M})$. Then, $x_e = 0$ for $e \notin C$ and $x_e = x_f$ for all $e \neq f \in C$ (to see it, use the fact that there exists a cocircuit D such that $C \cap D = \{e, f\}$). This shows that χ^C lies on an extreme ray of $\mathrm{MET}(\mathcal{M})$. Conversely, let $X \subseteq E$ such that χ^X lies on an extreme ray of $\mathrm{MET}(\mathcal{M})$. Consider the matroid $\mathcal{M} \backslash (E \setminus X)$; it has no coloops. Applying Lemma 2.2.12, there exists an integer $r \geq 1$ and a list of circuits $C_1, \ldots, C_k$ in $\mathcal{M} \backslash (E \setminus X)$ covering every element of X exactly r times. So, $r\chi^X = \sum_{i=1,\ldots,k} \chi^{C_i}$. This implies that X is a circuit. ∎

We have the inclusions:

$$\mathrm{CYC}^{\square}(\mathcal{M}) \subseteq \mathrm{MET}^{\square}(\mathcal{M}), \ \mathrm{CYC}(\mathcal{M}) \subseteq \mathrm{MET}(\mathcal{M}).$$

Hence arises the question of characterizing the binary matroids for which equality holds. Due to switching,

$$\mathrm{CYC}^{\square}(\mathcal{M}) = \mathrm{MET}^{\square}(\mathcal{M}) \Longleftrightarrow \mathrm{CYC}(\mathcal{M}) = \mathrm{MET}(\mathcal{M}).$$

The binary matroids $\mathcal{M}$ for which equality: $\mathrm{CYC}(\mathcal{M}) = \mathrm{MET}(\mathcal{M})$ holds are said to have the **"sums of circuits property"** by Seymour [Sey81a].

Observe that none of the binary matroids F_7^*, R_{10}, or $\mathcal{M}^*(K_5)$ has the sums of circuits property. Indeed, the vector $(\frac{2}{3},\ldots,\frac{2}{3})$ is a fractional vertex of the polytope $\mathrm{MET}^{\square}(\mathcal{M})$ for $\mathcal{M} = \mathcal{M}^*(K_5)$ or F_7^*; and the vector $(\frac{3}{4},\ldots,\frac{3}{4})$ is a fractional vertex of the polytope $\mathrm{MET}^{\square}(R_{10})$. The following result shows that the exclusion of these three minors F_7^*, R_{10}, and $\mathcal{M}^*(K_5)$ characterizes the sums of circuits property.

THEOREM 2.2.3. [Sey81a] *Let $\mathcal{M}$ be a binary matroid. Then, $\mathcal{M}$ has the sums of circuits, i.e.,* $\mathrm{CYC}(\mathcal{M}) = \mathrm{MET}(\mathcal{M})$ *or, equivalently,* $\mathrm{CYC}^{\square}(\mathcal{M}) = \mathrm{MET}^{\square}(\mathcal{M})$, *if and only if $\mathcal{M}$ does not have F_7^*, $\mathcal{M}^*(K_5)$, or R_{10} as a minor.*

This result was proved by Seymour in [Sey81a], where he considers the sums of circuits property along with several other properties related to multicommodity flows; Theorem 2.2.3 follows there from more general considerations. Among the essential tools used in [Sey81a] are some decomposition results for binary matroids with some prescribed excluded minors. We present below a direct proof of Theorem 2.2.3 - also based on these decomposition results. A proof along the same lines was given by Grötschel and Truemper [GT89a].

Let us first introduce the notion of k-sums for binary matroids. Let $\mathcal{M}_i = (E_i, \mathcal{C}_i)$ be a binary matroid, for $i = 1, 2$. We consider the binary matroid $\mathcal{M}_1 \diamond \mathcal{M}_2 := (E_1 \cup E_2, \mathcal{C}_1 \triangle \mathcal{C}_2)$, where

$$\mathcal{C}_1 \triangle \mathcal{C}_2 := \{C_1 \triangle C_2 \mid C_1 \in \mathcal{C}_1,\ C_2 \in \mathcal{C}_2\},$$

and the binary matroid

$$\mathcal{M}_1 \triangle \mathcal{M}_2 := (\mathcal{M}_1 \diamond \mathcal{M}_2) \backslash (E_1 \cap E_2),$$

i.e., $\mathcal{M}_1 \triangle \mathcal{M}_2$ is the binary matroid on the set $E_1 \triangle E_2$ whose cycles are the subsets of $E_1 \triangle E_2$ of the form $C_1 \triangle C_2$ for $C_i \in \mathcal{C}_i$, $i = 1, 2$. We consider the cases:

(i) $E_1 \cap E_2 = \emptyset$; then, $\mathcal{M}_1 \diamond \mathcal{M}_2 = \mathcal{M}_1 \triangle \mathcal{M}_2$;

(ii) $E_1 \cap E_2 = \{e_0\}$ and e_0 is not a loop in $\mathcal{M}_1$ nor in $\mathcal{M}_2$;

(iii) $|E_1 \cap E_2| = 3$ and $E_1 \cap E_2$ is a circuit in both $\mathcal{M}_1$ and $\mathcal{M}_2$.

Note that, when applied to graphic matroids, the construction $\mathcal{M}_1 \diamond \mathcal{M}_2$ corresponds in the cases (i),(ii),(iii) to the clique k-sum operation for graphs. By analogy, let us call $\mathcal{M}_1 \diamond \mathcal{M}_2$ the **clique k-sum** of $\mathcal{M}_1$ and $\mathcal{M}_2$ where $k = 1, 2, 3$ in the cases (i),(ii),(iii), respectively. The binary matroid $\mathcal{M}_1 \triangle \mathcal{M}_2$ is called the **k-sum** of $\mathcal{M}_1$ and $\mathcal{M}_2$, where $k = 1, 2, 3$ in the cases (i),(ii),(iii), respectively. If $\mathcal{M}$ is the k-sum of $\mathcal{M}_1$ and $\mathcal{M}_2$, it is said to be a **proper k-sum** if $\mathcal{M}_1$ and $\mathcal{M}_2$ are both proper minors of $\mathcal{M}$ (i.e., minors distinct from $\mathcal{M}$). We will use the following observation: The cocycles of the clique k-sum $\mathcal{M}_1 \diamond \mathcal{M}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ are the sets of the form $D_1 \cup D_2$, where D_i is a cocycle of $\mathcal{M}_i$ for

$i = 1, 2$ and $D_1 \cap E_2 = D_2 \cap E_1$. Seymour [Sey80] (see also [Sey81a]) proved the following decomposition result.

THEOREM 2.2.4. *Every binary matroid with no F_7, $\mathcal{M}(K_5)$, or R_{10} minor may be obtained by means of proper k-sums ($k = 1, 2, 3$) from graphic matroids with no $\mathcal{M}(K_5)$ minor, cographic matroids, and copies of F_7^*.* ∎

The program for the proof of Theorem 2.2.3 is now quite clear (and analogous to that of Theorem 1.5.10): In view of Theorem 2.2.4, one should see how the sums of circuits property behaves under the k-sum operation. Note, however, that the 3-sum operation does *not* preserve the sums of circuits property. The following counterexample is given in [GT89a]: Consider the graphs G_1, G_2 from Figure 2.2.5. Then, $\mathcal{M}_i := \mathcal{M}^*(G_i)$ has the sums of circuits property for $i = 1, 2$, but their 3-sum does not have it (as G_1, G_2 do not have K_5 as a minor, but the graph G - obtained by identifying the cut formed by the three edges in bold in G_1 and G_2 and then removing it - has a K_5 minor).

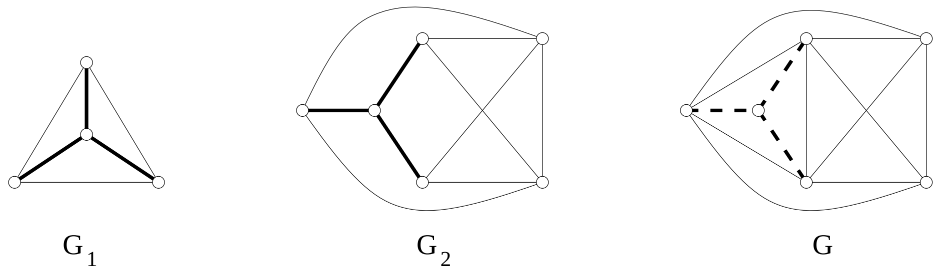


Figure 2.2.5

But the 3-sum can yet be applied in the following way: Let $\mathcal{M}$ be the 3-sum of $\mathcal{M}_1$ and $\mathcal{M}_2$; if both $\mathcal{M}_1^*$ and $\mathcal{M}_2^*$ have the sums of circuits property then so does $\mathcal{M}^*$ (see Proposition 2.2.7).

We now give several results: Proposition 2.2.6 (the sums of circuits property is closed under taking minors), Proposition 2.2.7 (behaviour under the k-sum operation), Proposition 2.2.8 (F_7 has the sums of circuits property), which together with Theorem 1.4.5 (graphic matroids have the some of circuits property) and Theorem 1.5.10 (cographic matroids with no $\mathcal{M}^*(K_5)$ minor have the sums of circuits property), will imply Theorem 2.2.3.

PROPOSITION 2.2.6. *Let $\mathcal{M}$ be a binary matroid on E and let $e \in E$. If $\mathcal{M}$ has the sums of circuits property then so do both $\mathcal{M} \backslash e$ and $\mathcal{M}/e$.*

PROOF. Set $\mathcal{N} := \mathcal{M} \backslash e$ or $\mathcal{M}/e$. Given $x \in \mathrm{MET}(\mathcal{N})$, we have to show that $x \in \mathrm{CYC}(\mathcal{N})$. For this, we extend x to a vector $y \in \mathbb{R}^E$ in such a way that $y \in \mathrm{MET}(\mathcal{M})$. In the case of the deletion minor, set $y_e := 0$. Then, $y \in \mathrm{MET}(\mathcal{M}) = \mathrm{CYC}(\mathcal{M})$. Hence, $y = \sum_C \alpha_C \chi^C$ where $\alpha_C \geq 0$, C cycle of $\mathcal{M}$ not containing e. Hence, $x = \sum_C \alpha_C \chi^C \in \mathrm{CYC}(\mathcal{M} \backslash e)$. In the case of the contraction minor, a suitable value of y_e can be easily found by using the assumption that $x \in \mathrm{MET}(\mathcal{M}/e)$. Hence, $y = \sum_C \alpha_C \chi^C$ where $\alpha_C \geq 0$, C cycle of $\mathcal{M}$.

Therefore, $x = \sum_C \alpha_C \chi^{C \setminus \{e\}} \in \mathrm{CYC}(\mathcal{M}/e)$. ∎

PROPOSITION 2.2.7. *Let $\mathcal{M}$ be the k-sum of the binary matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, where $k = 1, 2, 3$. If $\mathcal{M}_1^*$ and $\mathcal{M}_2^*$ have the sum of circuits property, then $\mathcal{M}^*$ too has the sums of circuits property.*

PROOF. [3] Let $\mathcal{N} := \mathcal{M}_1 \diamond \mathcal{M}_2$ denote the clique k-sum of $\mathcal{M}_1$ and $\mathcal{M}_2$. As $\mathcal{M}$ is a minor of $\mathcal{N}$, in view of Proposition 2.2.6, it suffices to show that $\mathcal{N}^*$ has the sums of circuits property. For this, let $x \in \mathrm{MET}(\mathcal{N}^*)$; we show that $x \in \mathrm{CYC}(\mathcal{N}^*)$. Let x_i denote the projection of x on $\mathbb{R}^{E_i}$ for $i = 1, 2$. Obviously, $x_i \in \mathrm{MET}(\mathcal{M}_i^*)$ (as the cycle space of $\mathcal{M}_i$ is contained in the cycle space of $\mathcal{N}$). Hence, $x_i \in \mathrm{CYC}(\mathcal{M}_i^*)$, i.e., x_i can be decomposed as

$$x_i = \sum_{j \in J^{(i)}} \alpha_j^{(i)} \chi^{D_j^{(i)}}$$

where $\alpha_j^{(i)} \geq 0$, $D_j^{(i)}$ cocycle of $\mathcal{M}_i$, for all j, and for $i = 1, 2$. In the case $k = 1$, we deduce that

$$x = \sum_{i=1,2} \sum_{j \in J^{(i)}} \alpha_j^{(i)} \chi^{D_j^{(i)}},$$

which shows that $x \in \mathrm{CYC}(\mathcal{N}^*)$. Consider now the case $k = 2$, i.e., $E_1 \cap E_2 = \{e_1\}$. Let us partition the index set $J^{(i)}$ as $J_0^{(i)} \cup J_1^{(i)}$, where $J_1^{(i)}$ consists of the indices $j \in J^{(i)}$ for which the cocycle $D_j^{(i)}$ contains e_1. Note that $\sum_{j \in J_1^{(1)}} \alpha_j^{(1)} = x_1(e_1) = x(e_1) = x_2(e_1) = \sum_{j \in J_1^{(2)}} \alpha_j^{(2)}$. Then,

$$x = \frac{1}{x(e_1)} \sum_{j_1 \in J_1^{(1)},\ j_2 \in J_1^{(2)}} \alpha_{j_1}^{(1)} \alpha_{j_2}^{(2)} \chi^{D_{j_1}^{(1)} \cup D_{j_2}^{(2)}} + \sum_{i=1,2} \sum_{j \in J_0^{(i)}} \alpha_j^{(i)} \chi^{D_j^{(i)}},$$

which shows that $x \in \mathrm{CYC}(\mathcal{N}^*)$. Consider finally the case $k = 3$, i.e., $E_1 \cap E_2 := \{e_1, e_2, e_3\}$ is a circuit in both $\mathcal{M}_1$ and $\mathcal{M}_2$. Then, a cocycle of $\mathcal{M}_i$ meets $E_1 \cap E_2$ in zero or two elements. This permits us to partition the index set $J^{(i)}$ as $J_0^{(i)} \cup J_{12}^{(i)} \cup J_{13}^{(i)} \cup J_{23}^{(i)}$, where $J_0^{(i)}$ consists of the indices j for which $D_j^{(i)} \cap \{e_1, e_2, e_3\} = \emptyset$, and $J_{ab}^{(i)}$ consists of the indices j for which $D_j^{(i)} \cap \{e_1, e_2, e_3\} = \{e_a, e_b\}$ for $ab = 12, 13, 23$. As $x_1(e) = x(e) = x_2(e)$ for $e \in E_1 \cap E_2$, we obtain that, for $i = 1, 2$,

$$\sum_{j \in J_{12}^{(i)}} \alpha_j^{(i)} = \frac{1}{2}(x(e_1) + x(e_2) - x(e_3)) =: m_{12}, \quad \sum_{j \in J_{13}^{(i)}} \alpha_j^{(i)} = \frac{1}{2}(x(e_1) + x(e_3) - x(e_2)) =: m_{13},$$

[3] The proof we give here is analogous to that given earlier for Proposition 1.5.7. We show, in fact, that $\mathcal{N}^*$ has the sums of circuits property, where $\mathcal{N}$ is the clique k-sum of $\mathcal{M}_1$ and $\mathcal{M}_2$; using Proposition 2.2.6, this implies that $\mathcal{M}^*$ has the sums of circuits property as $\mathcal{M}$ is a minor of $\mathcal{N}$. Proving directly that $\mathcal{M}^*$ has the sums of circuits property - as is done in [GT89a] - is a more difficult task as one faces the following problem: Given $x \in \mathrm{MET}(\mathcal{M}^*)$, extend x on $E_1 \cap E_2$ so as to obtain, by projection, a vector of $\mathrm{MET}(\mathcal{M}_i^*)$ for $i = 1, 2$. Working with the clique k-sum permits to avoid this difficulty. One more difference between the proof here and the proof from [GT89a] is that we work with the semimetric and cycle cones rather than with polytopes as they are easier to handle.

$$\sum_{j \in J_{23}^{(i)}} \alpha_j^{(i)} = \frac{1}{2}(x(e_2) + x(e_3) - x(e_1)) =: m_{23}.$$

We can also suppose that $\sum_{j \in J_0^{(i)}} \alpha_j^{(i)}$ is a constant m_0, for $i = 1, 2$ (adding if necessary some multiple of $\chi^{\emptyset}$ to the decomposition of x_1 or x_2). Then,

$$x = \frac{1}{m_0} \sum_{j_1 \in J_0^{(1)},\ j_2 \in J_0^{(2)}} \alpha_{j_1}^{(1)} \alpha_{j_2}^{(2)} \chi^{D_{j_1}^{(1)} \cup D_{j_2}^{(2)}} + \sum_{1 \le a < b \le 3} \frac{1}{m_{ab}} \sum_{j_1 \in J_{ab}^{(1)},\ j_2 \in J_{ab}^{(2)}} \chi^{D_{j_1}^{(1)} \cup D_{j_2}^{(2)}}.$$

This shows that $x \in \mathrm{CYC}(\mathcal{N}^*)$. ∎

Proposition 2.2.8. *The Fano matroid F_7 has the sums of circuits property.*

Proof. This can be checked, e.g., using computer. ∎

When $\mathcal{M}$ has the sums of circuits property, the linear program:

$$\begin{array}{lll} \max & w^T x & \\ \text{s.t.} & x(F) - x(D \setminus F) \le |F| - 1 & (F \subseteq D,\ D \text{ cocycle},\ |F| \text{ odd}) \\ & 0 \le x_e \le 1 & (e \in E) \end{array} \tag{2.2.9}$$

has an integer optimum solution for every $w \in \mathbb{Z}^E$. In fact, under some parity condition, the dual linear program too has an integer optimum solution. This fact will follow from results on multicommodity flows; see Proposition 3.5.12 in Section 3.2.

Theorem 2.2.10. *Let $\mathcal{M}$ be a binary matroid with the sums of circuits property, i.e., with no F_7^*, $\mathcal{M}^*(K_5)$, or R_{10} minor. Let $w \in \mathbb{Z}^E$ such that $w(C) \in 2\mathbb{Z}$ for all cycles C in $\mathcal{M}$. Then, the dual linear program of* (2.2.9) *has an integer optimum solution.* ∎

2.2.3 Cocycle Facets

We now turn to the characterization of the inequalities $0 \le x \le 1$ and of the cocycle inequalities (2.2.1) that define facets of $\mathrm{CYC}^{\square}(\mathcal{M})$. Let us start with an observation about the dimension of the cycle polytope.

If $e \in E$ is a coloop of $\mathcal{M}$ then the cycle space is contained in the hyperplane defined by: $x_e = 0$; if $e \neq f \in E$ are coparallel in $\mathcal{M}$ then the cycle space is contained in the hyperplane defined by: $x_e = x_f$. Define a **coparallel class** of $\mathcal{M}$ as a maximal subset $F \subseteq E$ such that every two elements of F are coparallel. In view of these observations, we can restrict our attention to studying the cycle space of the binary matroid $\bar{\mathcal{M}}$, where $\bar{\mathcal{M}}$ is obtained from $\mathcal{M}$ by
- deleting the coloops,
- and contracting $\{e_1, \ldots, e_p\}$ for each coparallel class $\{e_0, e_1, \ldots, e_p\}$.

Then, $\bar{\mathcal{M}}$ has no coloops and no coparallel elements.

LEMMA 2.2.11. [BG86] *Let $\bar{\mathcal{M}}$ be a binary matroid with no coloops and no coparallel elements. Then, its cycle space is full dimensional.*

PROOF. By Lemma 2.2.12, there exists a list $\{C_1, \ldots, C_p\}$ of circuits in $\bar{\mathcal{M}}$ and an integer $r \geq 1$ such that $\sum_{1 \leq i \leq p} \chi^{C_i} = r\chi^E$. For $e \in E$, as $\bar{\mathcal{M}} \backslash e$ has no coloops, we can find a list $\{C'_1, \ldots, C'_q\}$ of circuits of $\bar{\mathcal{M}}$ and $s \geq 1$ such that $\sum_{1 \leq j \leq q} \chi^{C'_j} = s\chi^{E \backslash \{e\}}$. This shows that $\chi^{\{e\}}$ belongs to the linear span of the cycle space for every $e \in E$. Hence, the cycle space is full dimensional. ∎

LEMMA 2.2.12. [Sey81a] *Let $\mathcal{M}$ be a binary matroid on E without coloops. There exist an integer $r \geq 1$ and a list $\mathcal{L}$ of (possibly repeated) circuits in $\mathcal{M}$ that cover every element of E exactly r times.*

PROOF. Call a list as in Lemma 2.2.12 an r-cover. We show that an r-cover exists by induction on $|E|$. Suppose that $|E| \geq 2$. If there exist two coparallel elements e and f, the result follows by the induction assumption applied to $\mathcal{M}/f$. Hence, we can suppose that $\mathcal{M} \backslash e$ has no coloops for all $e \in E$. By the induction assumption, $\mathcal{M} \backslash e$ has an r_e-cover. for some $r_e \geq 1$. Set $r := \prod_{e \in E} r_e$ and, for $e \in E$, let $\mathcal{L}_e$ be an r-cover for $\mathcal{M} \backslash e$. Then, $\cup_{e \in E} \mathcal{L}_e$ is an R-cover for $\mathcal{M}$, for $R = r(|E| - 1)$. ∎

Clearly, if an element $e \in E$ belongs to a cocycle of size 3, say $\{e, f, g\}$, then the inequality $x_e \geq 0$ does not define a facet of $\mathrm{CYC}^{\square}(\mathcal{M})$, as it follows from the two cocycle inequalities:

$$-x_e - x_f + x_g \leq 0, \; -x_e + x_f - x_g \leq 0.$$

This is, in fact, the only case when $x_e \geq 0$ is not facet defining.

THEOREM 2.2.13. [BG86] *Let $\mathcal{M}$ be a binary matroid without coloops and without coparallel elements and let $e \in E$. The inequality $x_e \geq 0$ (or $x_e \leq 1$) defines a facet of $\mathrm{CYC}^{\square}(\mathcal{M})$ if and only if e does not belong to a cocycle of size 3.* ∎

EXAMPLE 2.2.14. [GT89b] **The cycle polytope of the binary matroid $\mathcal{P}_r^*$.** The binary matroid $\mathcal{P}_r^*$ has been defined earlier; its groundset has $2^r - 1$ elements, $\mathcal{P}_r^*$ has $2^r - 1$ nonempty cycles, each of size 2^{r-1}. Moreover, the cycle polytope of $\mathcal{P}_r^*$ is full dimensional (as $\mathcal{P}_r^*$ has no coloops and no coparallel elements). Therefore, $\mathrm{CYC}^{\square}(\mathcal{P}_r^*)$ is a simplex, whose facets are defined by the inequality:

$$\sum_{e \in E} x_e \leq 2^{r-1}$$

together with its switchings by the $2^r - 1$ nonempty cycles. We deduce, in particular, the linear description of the cycle polytope of the dual Fano matroid $F_7^* = \mathcal{P}_3^*$; $\mathrm{CYC}^{\square}(F_7^*)$ is defined by the inequality:

$$\sum_{1\leq i\leq 7} x_i \leq 4$$

together with its switchings by the seven circuits of F_7^*. Hence, the cocycle inequalities do not define facets of the cycle polytope of F_7^*, or $\mathcal{P}_r^*$. ∎

Given a cocircuit D, an element $e \in E \setminus D$ is called a **chord** of D if there exist two cocircuits D_1 and D_2 such that $D_1 \cap D_2 = \{e\}$ and $D_1 \triangle D_2 = D$.

THEOREM 2.2.15. [BG86] *Let $\mathcal{M}$ be a binary matroid without coloops and without coparallel elements and suppose that $\mathcal{M}$ has no F_7^* minor. Let D be a cocycle of $\mathcal{M}$. Then, the cocycle inequality* (2.2.1) *defines a facet of* $\mathrm{CYC}^{\square}(\mathcal{M})$ *if and only if D is a cocircuit without a chord.* ∎

Grötschel and Truemper [GT89b] describe some other procedures for constructing facets of the cycle polytope $\mathrm{CYC}^{\square}(\mathcal{M})$: Suppose that $\mathcal{M}$ is a binary matroid that contains a contraction minor $\mathcal{N}$ of the form $\mathcal{P}_k^*$ (for some $k \geq 1$) and suppose that $\mathcal{N}$ is a maximal such contraction minor of $\mathcal{M}$. Then, every facet of $\mathrm{CYC}^{\square}(\mathcal{N})$ yields - by zero-lifting - a facet of $\mathrm{CYC}^{\square}(\mathcal{M})$. It is also shown in [GT89b] how to find, for every element $e \in E$, a maximal contraction minor of $\mathcal{M}$ containing e and of the form $\mathcal{P}_k^*$. (Note that $\mathcal{M}/(E\setminus\{e\})$ is $\mathcal{P}_1^*$.)

We conclude with a remark showing that every cycle polytope can be obtained as the projection of a simplex.

REMARK 2.2.16. We have seen in Example 2.2.14 above that the cycle polytope of the binary matroid $\mathcal{P}_k^*$ has a very simple linear description as it is a simplex. In fact, as observed in [GT89b], the cycle polytope of every binary matroid $\mathcal{M}$ can be obtained as the projection of some simplex $\mathrm{CYC}^{\square}(\mathcal{P}_k^*)$. Therefore, the linear description of $\mathrm{CYC}^{\square}(\mathcal{M})$ can be deduced - at least theoretically - from that of $\mathrm{CYC}^{\square}(\mathcal{P}_k^*)$ by applying the Fourier-Motzkin elimination method (see, e.g., [Sch86, Zie95]). This method is, however, not practical if many variables have to be eliminated.

Let us now see how $\mathrm{CYC}^{\square}(\mathcal{M})$ arises as a projection of some $\mathrm{CYC}^{\square}(\mathcal{P}_k^*)$. Suppose $\mathcal{M}$ is a binary matroid on E. If every element of E is a loop then every subset of E is a cycle and, thus, $\mathrm{CYC}^{\square}(\mathcal{M}) = [0,1]^E$. Otherwise, we can suppose that $\mathcal{M}$ has no coloops, no coparallel elements and that it has rank $r \geq 1$. Then, $\mathcal{M}$ has a representation matrix of the form $[I_r|A]$, where A is a binary $r \times k$ matrix with $k := |E| - r \geq 2$. Hence, A is a row submatrix of the matrix A_k, where $[I_{2^k-k-1}|A_k]$ represents the matroid $\mathcal{P}_k^*$. If Y denotes the subset corresponding to the rows of A_k that are not present in A then $\mathcal{M}$ arises as the contraction minor: $\mathcal{M} = \mathcal{P}_k^*/Y$ of $\mathcal{P}_k^*$. Therefore, the polytope $\mathrm{CYC}^{\square}(\mathcal{M})$ can be obtained from the polytope $\mathrm{CYC}^{\square}(\mathcal{P}_k^*)$ by projecting out the variables x_e for $e \in Y$. ∎

2.2.4 The Maximum Weight Cycle Problem

Given a binary matroid $\mathcal{M} = (E, \mathcal{C})$, the **maximum weight cycle problem** consists of finding a cycle $C \in \mathcal{C}$ whose weight $w(C) = \sum_{e \in C} w_e$ is maximum. It can be formulated as a linear programming problem over the polytope $\mathrm{CYC}^{\square}(\mathcal{M})$. This problem is NP-hard as it contains the max-cut problem as a special instance.

Grötschel and Truemper [GT89a] have shown that the maximum weight cycle problem can be solved in polynomial time for several classes of binary matroids (one supposes that $\mathcal{M}$ is given by a representation matrix). This is the case, in particular, for the following classes of binary matroids:

(i) $\mathcal{M}$ does not have F_7 nor $\mathcal{M}^*(K_5)$ as a minor,

(ii) $\mathcal{M}$ does not have F_7^* nor $\mathcal{M}^*(K_5)$ as a minor,

(iii) $\mathcal{M}$ has the sums of circuits property,

(iv) $\mathcal{M}$ is graphic, or cographic with no $\mathcal{M}^*(K_5)$ minor.

Note that we know already the result for the class (iv) by the preceding sections. The main tools for proving the above result are some decomposition results for the matroids in question, and the following results concerning the separation problem for the polytope $\mathrm{MET}^{\square}(\mathcal{M})$. Namely, Grötschel and Truemper show that the separation problem for the system consisting of the inequalities: $0 \leq x_e \leq 1$ ($e \in E$) and the inequalities (2.2.1) for D cocircuit of $\mathcal{M}$, can be solved in polynomial time for the following classes of matroids:

(i) $\mathcal{M}$ does not have F_7 as a minor,

(ii) $\mathcal{M}$ does not have F_7^* as a minor,

(iii) $\mathcal{M}$ has the sums of circuits property,

(iv) $\mathcal{M}$ is graphic or cographic.

In fact, they show that, using decomposition techniques, the separation problem for the above classes of matroids can be solved efficiently if one knows a polynomial separation algorithms in the graphic and cographic cases. In the latter two cases, such algorithm were given, respectively, by Padberg and Rao [PR82] and by Barahona and Mahjoub [BM86] (the latter algorithm is presented in Section **??**).

2.3 Some Related Polyhedra

2.3.1 Odd Cycle Constrained Polyhedra

Let $\mathcal{M}$ be a binary matroid on E and let Σ be a subset of E. Consider the polytope:

$$\begin{array}{rll} \mathrm{ODD}^{\square}(\mathcal{M}, \Sigma) := \{x \in \mathbb{R}^E \mid & 0 \leq x_e \leq 1 & \forall e \in E, \\ & x(D) \leq |D| - 1 & \forall D \ \text{ cocircuit of } \mathcal{M} \text{ with } |D \cap \Sigma| \text{ odd}\}. \end{array}$$

This extends the definition given in Section 1.6 in the context of signed graphs, namely, $\mathrm{ODD}^{\square}(\mathcal{M}^*(G), \Sigma) = \mathrm{ODD}^{\square}(G, \Sigma)$. We have the following chain of inclusions:

$$\mathrm{CYC}^{\square}(\mathcal{M}) \subseteq \mathrm{MET}^{\square}(\mathcal{M}) \subseteq \mathrm{ODD}^{\square}(\mathcal{M}, E).$$

We have seen some results in Section 1.6 which show how the polyhedra $\mathrm{MET}^{\square}(G)$ and $\mathrm{ODD}^{\square}(G,\Sigma)$ relate; see Theorems 1.6.6, 1.6.13 and 1.6.14. These results extend in an obvious way to the case of arbitrary binary matroids; the proofs are completely identical.

THEOREM 2.3.1. *Let $\mathcal{M}$ be a binary matroid on E and let $w \in \mathbb{Q}^E$. Set $\Sigma := \{e \in E \mid w_e \geq 0\}$ and $c_e := |w_e|$ for $e \in E$. Then,*

$$\max_{x \in \mathrm{MET}^{\square}(\mathcal{M})} w^T x = \max_{x \in \mathrm{ODD}^{\square}(\mathcal{M},\Sigma)} c^T x + w(E \setminus \Sigma).$$

∎

COROLLARY 2.3.2. *Let $\mathcal{M}$ be a binary matroid on E. The following assertions are equivalent.*

(i) *$\mathcal{M}$ has the sums of circuits property.*

(ii) *$\mathcal{M}$ does not have F_7^*, $\mathcal{M}^*(K_5)$, or R_{10} as a minor.*

(iii) *For every signing $\Sigma \subseteq E$, the polytope $\mathrm{ODD}^{\square}(\mathcal{M},\Sigma)$ is integral.* ∎

REMARK 2.3.3. (i) Let us apply Corollary 2.3.2 to graphic matroids. Let $G = (V,E)$ be a graph. As the graphic matroid of G has the sums of circuits property, we deduce that the polytope $\mathrm{ODD}^{\square}(\mathcal{M}(G),\Sigma)$ is integral for every signing Σ. Equivalently, the polyhedron:

$$\{x \in \mathbb{R}_+^E \mid x(\delta_G(S)) \geq 1 \text{ for all } \Sigma\text{-odd cuts } \delta_G(S)\}$$

is integral for every signing Σ. Given $\Sigma \subseteq E$, let T denote the set of nodes having an odd degree in (V,Σ); then, Σ is a T-join. Observe that a cut $\delta_G(S)$ is Σ-odd if and only if it is a T-cut, i.e., if $|S \cap T|$ is odd. Therefore, by Lemma 1.3.1, the above polyhedron coincides with the T-join polyhedron $\mathrm{JOIN}^{\square}(G,T)$. Hence, we obtain here the linear description of the T-join polyhedron (presented in Theorem 1.4.1) as a simple application of the linear description of the Eulerian subgraph cone (presented in Theorem 1.4.5).
(ii) The equivalence (i) $\iff$ (iii) in Corollary 2.3.2 follows also easily from some results on multicommodity flows established in [Sey81a]. More precisely, it will follow from Lemma 3.5.10 and the equivalence (ii) $\iff$ (v) in Theorem 3.5.2 (which is an easy part in this theorem). ∎

An important open question is the characterization of the pairs $(\mathcal{M},\Sigma)$ for which the polytope $\mathrm{ODD}^{\square}(\mathcal{M},\Sigma)$ is integral. For the class of cographic matroids, this contains the question of characterizing the weakly bipartite graphs. We present three examples of pairs $(\mathcal{M},\Sigma)$ for which the polytope $\mathrm{ODD}^{\square}(\mathcal{M},\Sigma)$ is not integral.

(i) The polytope $\mathrm{ODD}^{\square}(F_7^*,E)$ (where E is the groundset of F_7^*) is not integral. Indeed, it is defined by the inequalities: $x(L) \leq 2$ where the sets L are the lines of the Fano plane. Hence, the vector $(\frac{2}{3},\ldots,\frac{2}{3})$ is a fractional vertex of $\mathrm{ODD}^{\square}(F_7^*,E)$.

(ii) The polytope $\mathrm{ODD}^{\square}(\mathcal{M}^*(K_5),E)$ (where E is the groundset of $\mathcal{M}^*(K_5)$, i.e., the edge set of K_5) is not integral. Indeed, it is defined by the inequalities: $x(C) \leq |C|-1$ where the sets C are the odd circuits of K_5; hence, the vector $(\frac{2}{3},\ldots,\frac{2}{3})$ is again a fractional vertex.

(iii) The third example is defined from the matroid R_{10}. Recall that we may suppose that the groundset of R_{10} is the edge set of K_5. Let Σ denote the edge set of a 5-circuit in K_5. Then, the polytope $\mathrm{ODD}^{\square}(R_{10}, \Sigma)$ is not integral. Indeed, the Σ-odd cocircuits in R_{10} correspond to the edge sets $E(K_5) \setminus \delta_{K_5}(S)$ for $S \subseteq \{1, \ldots, 5\}$ with $|S| = 1, 2$. Hence, the vector $(\frac{3}{4}, \ldots, \frac{3}{4})$ is a fractional vertex of $\mathrm{ODD}^{\square}(R_{10}, \Sigma)$.

A conjecture by Seymour states that these are the only minimal obstructions to the integrality property of the polytope $\mathrm{ODD}^{\square}(\mathcal{M}, \Sigma)$; we formulate it below in terms of the max-flow min-cut property.

2.3.2 The $\mathbb{Q}_+$-Max-Flow Min-Cut Property

For convenience, let us introduce some definitions. A **clutter** on a groundset E is a collection $\mathcal{O}$ of subsets of E such that $A \not\subseteq B$ for all $A, B \in \mathcal{O}$. The **blocker** of $\mathcal{O}$ is the clutter $b(\mathcal{O})$ consisting of the minimal subsets of E meeting every member of $\mathcal{O}$. Note that $b(b(\mathcal{O})) = \mathcal{O}$. Following Seymour [Sey77], one says that the clutter $\mathcal{O}$ has the **$\mathbb{Q}_+$-max-flow min-cut property** [4] (the **$\mathbb{Q}_+$-MFMC property** for short, or the **weak MFMC property**) if the polyhedron:

$$\{x \in \mathbb{R}_+^E \mid x(C) \geq 1 \text{ for all } C \in \mathcal{O}\}$$

is integral. The $\mathbb{Q}_+$-MFMC property is preserved under taking blockers, i.e.,

$$\text{(2.3.4)} \qquad \mathcal{O} \text{ has the } \mathbb{Q}_+\text{-MFMC property} \implies b(\mathcal{O}) \text{ has the } \mathbb{Q}_+\text{-MFMC property.}$$

Given a binary matroid $\mathcal{M}$ on E and $\Sigma \subseteq E$, let $\mathcal{O}(\mathcal{M}, \Sigma)$ denote the collection of Σ-odd circuits of $\mathcal{M}$, i.e., of the circuits C of $\mathcal{M}$ such that $|C \cap \Sigma|$ is odd. Every clutter of the form $\mathcal{O}(\mathcal{M}, \Sigma)$ is called a **binary clutter**.

Binary clutters can be alternatively defined as the ports of binary matroids. Given a matroid $\mathcal{N}$ on a set $E \cup \{\ell\}$, its **ℓ-port** is defined as the clutter $\{C \subseteq E \mid C \cup \{\ell\} \text{ is a circuit of } \mathcal{M}\}$. Observe that the ℓ-port of $\mathcal{N}$ coincides with the binary clutter $\mathcal{O}(\mathcal{N}/\ell, \Sigma)$, where Σ is any subset of E such that $\Sigma \cup \{\ell\}$ is a cocycle of $\mathcal{N}$. Conversely, every binary clutter $\mathcal{O}(\mathcal{M}, \Sigma)$ can be realized as the ℓ-port of the binary matroid $\mathcal{S}_\ell(\mathcal{M}, \Sigma)$, which is defined as the binary matroid on $E \cup \{\ell\}$ represented by the matrix

$$\begin{array}{c} \begin{array}{cc} \ell & E \end{array} \\ \begin{pmatrix} 1 & \chi^{\Sigma} \\ 0 & M \end{pmatrix} \end{array},$$

where M is a matrix representing $\mathcal{M}$. Note that taking the blocker corresponds to duality, i.e., if $\mathcal{O}$ is the ℓ-port of $\mathcal{N}$ then $b(\mathcal{O})$ is the ℓ-port of $\mathcal{N}^*$.

To make the link with the previous definitions, note that $\mathcal{O}(\mathcal{M}, \Sigma)$ has the $\mathbb{Q}_+$-MFMC property if and only if the polytope $\mathrm{ODD}^{\square}(\mathcal{M}^*, \Sigma)$ is integral. The binary clutters corresponding to the three examples (i),(ii),(iii) above are, respectively:

[4] Let A denote the matrix whose rows are the incidence vectors of $\mathcal{O}$. When $\mathcal{O}$ has the $\mathbb{Q}_+$-MFMC property, i.e., when the polyhedron $\{x \geq 0 \mid Ax \geq 1\}$ is integral, one also says that the matrix A is **ideal**. This terminology reflects the analogy with *perfect* matrices, which correspond to the case when the polytope $\{x \geq 0 \mid Ax \leq 1\}$ is integral.

(i) the clutter $\mathcal{L}_7 := \mathcal{O}(F_7, E(F_7))$ (consisting of the lines of the Fano plane),

(ii) the clutter $\mathcal{O}_5 := \mathcal{O}(\mathcal{M}(K_5), E(K_5))$ (consisting of the odd circuits of K_5),

(iii) and its blocker $b(\mathcal{O}_5)$ (consisting of the complements of the cuts in K_5).

Each of the clutters $\mathcal{L}_7$, $\mathcal{O}_5$, $b(\mathcal{O}_5)$ does not have the $\mathbb{Q}_+$-MFMC property. The following conjecture is due to Seymour [Sey77, Sey81a]; it contains Conjecture 1.6.12.

CONJECTURE 2.3.5. *A binary clutter has the $\mathbb{Q}_+$-MFMC property if and only if it does not have $\mathcal{L}_7$, $\mathcal{O}_5$, or $b(\mathcal{O}_5)$ as a minor* [5]*. Equivalently, given a binary matroid $\mathcal{N}$ on $E \cup \{\ell\}$, its ℓ-port has the $\mathbb{Q}_+$-MFMC property if and only if $\mathcal{N}$ does not have $\mathcal{S}_\ell(F_7, E(F_7))$, $\mathcal{S}_\ell(\mathcal{M}(K_5), E(K_5))$, or its dual, as a minor* [6] *using* [7] *ℓ.* ∎

For general (nonbinary) clutters the problem of characterizing the $\mathbb{Q}_+$-MFMC property seems even more difficult. No conjecture - at least in terms of a list of minimal forbidden minors - is known. In fact, Cornuéjols and Novick [CN94] have described a list of several thousands clutters on ten elements that do not have the $\mathbb{Q}_+$-MFMC property (and are minimal with respect to this fact). On the positive side, structural results for the minimum counterexamples for the $\mathbb{Q}_+$-MFMC property have been given by Lehman [Leh90]; see also Padberg [Pad90] and Seymour [Sey90].

The following fractional version of the $\mathbb{Q}_+$-MFMC property is studied in [GL95]. Given an integer $d \geq 1$, a clutter $\mathcal{O}$ is said to be **box $\frac{1}{d}$-integral** if every vertex of the polytope

$$\{x \in \mathbb{R}_+^E \mid x(C) \geq 1 (\text{ for } C \in \mathcal{O}),\ x_e = a_e (\text{ for } e \in I)\}$$

is $\frac{1}{d}$-integral, for all $I \subseteq E$, and all $\frac{1}{d}$-integral $a_e \in [0,1]$ $(e \in I)$. (A vector is said to be $\frac{1}{d}$-integral if all its components belong to $\frac{1}{d}\mathbb{Z}$.) For $d = 1$, box 1-integrality coincides with the $\mathbb{Q}_+$-MFMC property. Gerards and Laurent [GL95] show that a binary clutter is box $\frac{1}{d}$-integral for some $d \geq 2$ (or for all $d \geq 2$) if and only if it does not have any of the clutters Q_6 and Q_7 as a minor. (Q_6 is the port of F_7^* and Q_7 consists of the circuits of F_7 containing a given element.) An application to the semimetric polytope is given in [LP95].

2.3.3 The $\mathbb{Z}_+$-Max-Flow Min-Cut Property

Given a clutter $\mathcal{O}$ on E, one says that $\mathcal{O}$ has the **$\mathbb{Z}_+$-MFMC property** (or is **Mengerian**) if the linear program:

$$\begin{array}{ll} \min & w^T x \\ & x(C) \geq 1 \quad (C \in \mathcal{O}) \\ & x_e \geq 0 \qquad (e \in E) \end{array}$$

[5] Let $\mathcal{O}$ be a clutter on E. Deleting and contracting the element $e \in E$ in $\mathcal{O}$ produces, respectively, the clutters $\mathcal{O}\backslash e := \{C \subseteq E \setminus \{e\} \mid C \in \mathcal{O}\}$ and $\mathcal{O}/e$ consisting of the minimal members of $\{C \setminus \{e\} \mid C \in \mathcal{O}\}$. A minor of $\mathcal{O}$ is any clutter obtained from $\mathcal{O}$ by a series of deletions and/or contractions.

[6] The matroid $\mathcal{S}_\ell(F_7, E(F_7))$ is also known as $AG(2,3)$; it is self dual. Note the correspondance between the excluded minors for the sums of circuits property and the (conjectured) excluded minors for the $\mathbb{Q}_+$-MFMC property: if $\mathcal{N}$ is $\mathcal{S}_\ell(F_7, E(F_7))$, $\mathcal{S}_\ell(\mathcal{M}(K_5), E(K_5))$, or its dual, then $\mathcal{N}/\ell$ is F_7, $\mathcal{M}(K_5)$, or R_{10} ($=R_{10}^*$), respectively.

[7] One says that a minor of $\mathcal{N}$ **uses** ℓ if it is of the form $\mathcal{N}\backslash X/Y$ where $\ell \notin X \cup Y$.

and its linear programming dual both have an integer optimizing solution for every $w \in \mathbb{Z}_+^E$. Hence, the $\mathbb{Z}_+$-MFMC property implies the $\mathbb{Q}_+$-MFMC property. An equivalent definition is as follows.

A w-**packing** in $\mathcal{O}$ is a collection $\{C_1, \ldots, C_p\}$ of (possibly repeated) members of $\mathcal{O}$ such that every element $e \in E$ belongs to at most w_e of them. Then, the clutter $\mathcal{O}$ has the $\mathbb{Z}_+$-MFMC property if and only if the following combinatorial min-max relation holds: For every $w \in \mathbb{Z}_+^E$, $\min(w(D) \mid D \in b(\mathcal{O}))$ is equal to the maximum cardinality of a w-packing in $\mathcal{O}$.

Consider the clutter[8] Q_6 on six elements defined by:

$$Q_6 := \{356, 246, 145, 123\}.$$

Then, Q_6 has the $\mathbb{Q}_+$-MFMC property, but not the $\mathbb{Z}_+$-MFMC property (to see it, use the weight function $w \equiv 1$). A deep result of Seymour shows that the exclusion of Q_6 characterizes the $\mathbb{Z}_+$-MFMC property within the class of binary clutters.

THEOREM 2.3.6. [Sey77] *A binary clutter has the $\mathbb{Z}_+$-MFMC property* [9] *if and only if it has no Q_6-minor. Equivalently, given a binary matroid $\mathcal{N}$ on $E \cup \{\ell\}$, its ℓ-port has the $\mathbb{Z}_+$-MFMC property if and only if $\mathcal{N}$ does not have F_7^* as a minor using ℓ.* ∎

Some decomposition results for the class of matroids having no F_7^* minor using a given element have been given by Tseng and Truemper [TT86]; see the exposition by Truemper [Tru92]. One way of proving Theorem 2.3.6 is by using these decompositions techniques; see [Tru92].

In conclusion, let us see how the above results apply for some concrete examples of binary clutters.

st-Paths and st-Cuts. Let $G = (V, E)$ be a graph and $s \neq t \in V$. Let G' denote the graph obtained from G by adding one new edge between s and t, denoted as ℓ. Then, the ℓ-port of the graphic matroid $\mathcal{M}(G')$ is precisely the set of paths from s to t in G. The ℓ-port of the cographic matroid $\mathcal{M}^*(G')$ consists of the minimal st-cuts in G (the minimal cuts separating the nodes s and t).

Both clutters of st-paths and st-cuts are Mengerian. We know that the clutter of st-paths is Mengerian by Menger's theorem [10] (see Theorem 3.1.6). One can easily check that the clutter of minimal T-cuts is Mengerian. (Indeed, given $w \in \mathbb{Z}_+^E$, let $d(v)$ denote the shortest (with respect to w) length of a path from s to every node v. For every integer $i = 0, 1, \ldots, d(s) - 1$, set $S_i := \{v \in V \mid d(v) \leq i\}$. Then, the cuts $\delta_G(S_i)$ $(i \leq d(s) - 1)$ form a w-packing of cardinality $d(s)$, which is precisely the minimum weight of an st-path.)

T-Joins and T-Cuts. Let $G = (V, E)$ be a graph and $T \subseteq V$ of even cardinality. The pair $[G, T]$ is sometimes called a **graft** (cf. [Sey80]). Let A_G denote the node-edge incidence

[8] Q_6 is the port of F_7^*. Alternatively, Q_6 can be viewed as the collection of odd circuits of the graph K_4.

[9] Each of the clutters $\mathcal{L}_7$, $\mathcal{O}_5$ and $b(\mathcal{O}_5)$ contains Q_6 as a minor; hence, Conjecture 2.3.5 is consistent with Theorem 2.3.6.

[10] Hence the terminology "Mengerian".

matrix of G and let $\mathcal{G}(G,T)$ denote the binary matroid on $E \cup \{\ell\}$, which is represented by the matrix:

$$\begin{array}{cc} E & \ell \\ \end{array}$$
$$\begin{pmatrix} A_G & \chi^T \end{pmatrix};$$

$\mathcal{G}(G,T)$ is called a **graft matroid**. Then, the ℓ-port of $\mathcal{G}(G,T)$ consists of the minimal T-joins and the ℓ-port of its dual $(\mathcal{G}(G,T))^*$ consists of the minimal T-cuts.

The clutters of minimal T-joins and T-cuts have the $\mathbb{Q}_+$-MFMC property (recall Theorem 1.4.1), but they are not always Mengerian. Consider the grafts $\overline{K_4} := [K_4, V(K_4)]$, and $\overline{K_{3,2}} := [K_{3,2}, T]$ where T consists of all nodes of $K_{3,2}$ except one node of degree 3; see Figure 2.3.7.

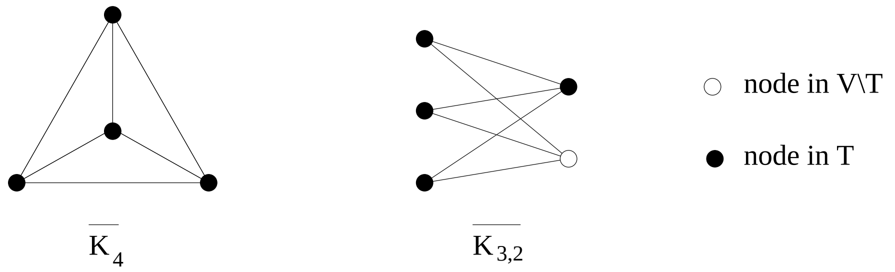


Figure 2.3.7

Note that the clutter of minimal T-cuts in $\overline{K_4}$ and the clutter of minimal T-joins in $\overline{K_{3,2}}$ both coincide with the clutter Q_6. Hence they are not Mengerian. Theorem 2.3.6 can be specialized as follows:

(i) The clutter of minimal T-cuts of the graph G is Mengerian if and only if the graft $[G,T]$ does not "reduce" to $\overline{K_4}$.

(ii) The clutter of minimal T-joins of G is Mengerian if and only if the graft $[G,T]$ does not "reduce" to $\overline{K_{3,2}}$.

Here, by reduction of a graft, we mean applying a sequence of deletions and contractions of edges, where
- deleting an edge e from $[G,T]$ produces the graft $[G\backslash e, T]$,
- contracting the edge $e := uv$ from $[G,T]$ produces the graft $[G/e, T']$, where $T' := T \setminus \{u,v\}$ if $|T \cap \{u,v\}|$ is even and $T' := T \setminus \{u,v\} \cup \{u^*\}$ if $|T \cap \{u,v\}|$ is odd and u^* is the node of G/e to which the edge e is contracted,

and possibly deleting isolated nodes of $V \backslash T$. (These operations correspond to deletions and contractions in the graft matroid.) We refer to Gerards [Ger89] for a detailed treatment.

When the clutter of minimal T-cuts is Mengerian then the minimum cardinality of a T-join is equal to the maximum number of pairwise disjoint T-cuts, i.e., with the terminology of Section 1.3, $\tau(G,T) = \nu(G,T)$. In particular, if G has no K_4-minor then, by (i) above, $\tau(G,T) = \nu(G,T)$ for all $T \subseteq V$ of even cardinality.

Σ-Odd Circuits and Σ-Coboundaries. Let (G,Σ) be a signed graph. We remind that $\mathcal{S}_\ell(\mathcal{M}(G),\Sigma)$ is the binary matroid on $E \cup \{\ell\}$, which is represented by the matrix:

$$\begin{array}{c} \begin{array}{cc} \ell & E \end{array} \\ \begin{pmatrix} 1 & \chi^\Sigma \\ 0 & A_G \end{pmatrix}. \end{array}$$

The ℓ-port of $\mathcal{S}_\ell(\mathcal{M}(G),\Sigma)$ is the clutter of the Σ-odd circuits of G, while the ℓ-port of its dual consists of all minimal sets of the form $\Sigma\triangle\delta_G(S)$ for $S\subseteq V$. A set $\Sigma\triangle\delta_G(S)$ is called a **Σ-coboundary** [11] of (G,Σ).

Consider the signed graphs $\widetilde{K_4} := (K_4, E(K_4))$, and $\widetilde{K_3^2} := (K_3^2,\Sigma)$ where K_3^2 is the graph on three nodes with two parallel edges between any two nodes and Σ is the edge set of a triangle; see Figure 2.3.8.

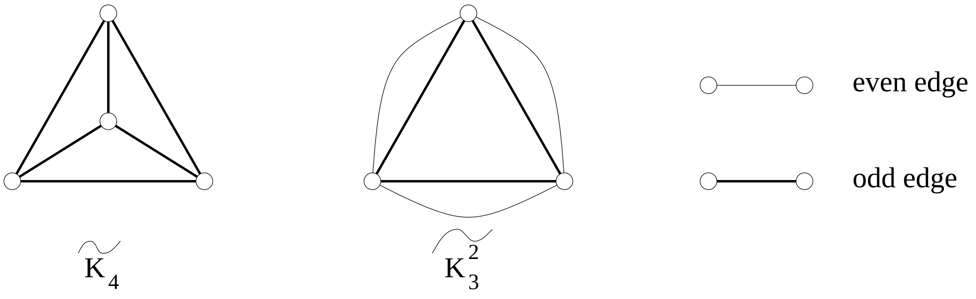


Figure 2.3.8

Note that the collection of odd circuits in $\widetilde{K_4}$ and the collection of Σ-coboundaries in $\widetilde{K_3^2}$ are both isomorphic to Q_6. Theorem 2.3.6 can be specialized as follows:

(i) The clutter of Σ-odd circuits of (G,Σ) is Mengerian if and only if (G,Σ) does not have $\widetilde{K_4}$ as a signed minor.

(ii) The clutter of minimal Σ-coboundaries in (G,Σ) is mengerian if and only if (G,Σ) does not have $\widetilde{K_3^2}$ as a signed minor.

We refer to Gerards [Ger89] for a detailed treatment.

2.4 Other Hulls of Cycles: The Cycle Lattice

We now mention several results related to other hulls of cycles. Given a binary matroid $\mathcal{M}=(E,\mathcal{C})$ and a set $\mathbb{K}\subseteq\mathbb{R}$, set

$$\mathbb{K}(\mathcal{M}) := \{\sum_{C\in\mathcal{C}} \lambda_C \chi^C \mid \lambda_C\in\mathbb{K} \text{ for all } C\in\mathcal{C}\}.$$

We consider the cases when $\mathbb{K}=\mathbb{Z},\mathbb{Z}_+,\mathbb{R}_+$; then, $\mathbb{Z}(\mathcal{M})$, $\mathbb{Z}_+(\mathcal{M})$, $\mathbb{R}_+(\mathcal{M}) = \mathrm{CYC}(\mathcal{M})$ denote, respectively, the **lattice**, the **integer cone**, and the cone generated by the cycles of $\mathcal{M}$.

The cycle cone has been considered in Section 2.2. The lattice and the integer cone of the cographic matroid of the complete graph K_n have been considered in detail in Chapter 4. ($\mathbb{Z}(\mathcal{M}^*(K_n))$ is described in Section **??**; the most part of Chapter 4 is devoted to the study of the hypercube embeddable semimetrics, that is, the elements of the integer cone $\mathbb{Z}_+(\mathcal{M}^*(K_n))$.) Some other concrete examples of cycle lattices have been described in Section **??**. Namely, given a graph $G=(V,E)$ and $T\subseteq V$ of even cardinality, let $\mathcal{M}$ denote the matroid on E whose cycle space is the set of even T-cuts, i.e., the cuts $\delta_G(S)$

[11] The Σ-coboundaries are the same objects as the balancing sets for (G,Σ), defined in Remark 1.6.16.

such that $|S \cap T|$ is even (or, equivalently, such that $|\delta_G(S) \cap H|$ is even, where H is a given T-join). The lattice $\mathbb{Z}(\mathcal{M})$ is described in Propositions **??** and **??**; its description uses equations modulo the integers 2,4,8.

We consider now the lattice $\mathbb{Z}(\mathcal{M})$ and the integer cone $\mathbb{Z}_+(\mathcal{M})$ of an arbitrary binary matroid $\mathcal{M}$. Consider the following conditions:

$$x_e = 0 \;\; \forall e \text{ coloop in } \mathcal{M}, \;\; x_e = x_f \;\; \forall e, f \text{ coparallel in } \mathcal{M}; \tag{2.4.1}$$

$$x(D) \in 2\mathbb{Z} \;\; \forall D \text{ cocycle of } \mathcal{M}; \tag{2.4.2}$$

$$x_e - x(D \setminus e) \leq 0 \;\; \forall D \text{ cocycle of } \mathcal{M}, e \in D. \tag{2.4.3}$$

As was seen in Lemma 2.2.11, the linear space $\mathbb{R}(\mathcal{M})$ spanned by the cycle space $\mathcal{C}$ is described by condition (2.4.1), i.e.,

$$\mathbb{R}(\mathcal{M}) = \{x \in \mathbb{R}^E \mid x \text{ satisfies } (2.4.1)\}.$$

The following inclusions hold obviously:

$$\mathbb{R}_+(\mathcal{M}) \subseteq \{x \in \mathbb{R}^E_+ \mid x \text{ satisfies } (2.4.3)\}; \tag{2.4.4}$$

$$\mathbb{Z}(\mathcal{M}) \subseteq \{x \in \mathbb{Z}^E \mid x \text{ satisfies } (2.4.1) \text{ and } (2.4.2)\}; \tag{2.4.5}$$

$$\mathbb{Z}_+(\mathcal{M}) \subseteq \mathbb{Z}(\mathcal{M}) \cap \mathbb{R}_+(\mathcal{M}) \subseteq \{x \in \mathbb{Z}^E_+ \mid x \text{ satisfies } (2.4.2) \text{ and } (2.4.3)\}. \tag{2.4.6}$$

The matroids for which equality holds in (2.4.4), i.e., the matroids with the sums of circuits property, have beeen characterized in Theorem 2.2.3. By analogy, if equality holds in (2.4.5), one says that $\mathcal{M}$ has the **lattice of circuits property**. If $\mathbb{Z}_+(\mathcal{M}) = \{x \in \mathbb{Z}^E_+ \mid x \text{ satisfies } (2.4.2) \text{ and } (2.4.3)\}$ then $\mathcal{M}$ is said to have the **circuit cover property**. Finally, when $\mathbb{Z}_+(\mathcal{M}) = \mathbb{Z}(\mathcal{M}) \cap \mathbb{R}_+(\mathcal{M})$, i.e., when the cycle space $\mathcal{C}$ is a Hilbert basis, we say that $\mathcal{M}$ has the **Hilbert basis property**.

We group here results related to the cycle lattice $\mathbb{Z}(\mathcal{M})$ and the lattice of circuits property. The integer cone $\mathbb{Z}_+(\mathcal{M})$ and the circuit cover property will be considered in the next section.

2.4.1 Binary Matroids Whose Minors All Have the Lattice Property

Let $\mathcal{M} = (E, \mathcal{C})$ be a binary matroid. We start with an easy but helpful observation for studying the lattice of circuits property.

Lemma 2.4.7. *$\mathcal{M}$ has the lattice of circuits property if and only if*

$$2\chi^F \in \mathbb{Z}(\mathcal{M}) \;\; \textit{for every coparallel class } F \textit{ of } \mathcal{M}. \tag{2.4.8}$$

Proof. Given $x \in \mathbb{Z}^E$, write x as $x = y + 2z$, where $y \in \{0,1\}^E$ and $z \in \mathbb{Z}^E$. Then, x satisfies (2.4.1) and (2.4.2) if and only if y belongs to the cycle space of $\mathcal{M}$ and z is an integer combination of (incidence vectors of) coparallel classes. ∎

This permits, for instance, to check that graphic and cographic matroids have the lattice of circuits property. Indeed, relation (2.4.8) holds in both cases. This is obvious in the cographic case: If $F := \{e_1, \ldots, e_p\}$ is the set of parallel edges joining two nodes u and v in a graph G, then $2\chi^F = \chi^{\delta_G(u)} + \chi^{\delta_G(v)} - \chi^{\delta_G(\{u,v\})}$. In the graphic case, this follows by applying Menger's theorem. More generally, if $\mathcal{M}$ is a binary matroid with no F_7^* minor then (2.4.8) holds, as we can then apply the analogue of Menger's theorem, provided by Theorem 2.3.6.

THEOREM 2.4.9. *Every binary matroid $\mathcal{M}$ with no F_7^* minor has the lattice of circuits property.*

PROOF. Let $F := \{e, e_1, \ldots, e_p\}$ $(p \geq 0)$ be a coparallel class in $\mathcal{M}$ and consider the matroid $\mathcal{M}' := \mathcal{M}/\{e_1, \ldots, e_p\}$. As $\mathcal{M}'$ has no F_7^* minor (and, thus, no F_7^* minor using the element e), we deduce from Theorem 2.3.6 that $\min(|D| \mid D \cup \{e\}$ is a cocircuit of $\mathcal{M}')$ is equal to the maximum number, say k, of circuits $C_1', \ldots, C_k'$ of $\mathcal{M}'$ such that $C_i' \cap C_j' = \{e\}$. Note that $|D| \geq 2$ if $D \cup \{e\}$ is a cocircuit of $\mathcal{M}'$. Therefore, there exist two circuits C_1', C_2' in $\mathcal{M}'$ such that $C_1' \cap C_2' = \{e\}$. Then, $C_i := C_i' \cup \{e_1, \ldots, e_p\}$ is a circuit of $\mathcal{M}$ (for $i = 1, 2$) and $C_1 \cap C_2 = F$. This shows that $2\chi^F = \chi^{C_1} + \chi^{C_2} - \chi^{C_1 \triangle C_2} \in \mathbb{Z}(\mathcal{M})$. Hence, (2.4.8) holds. ∎

An elementary proof of Theorem 2.4.9, not using Theorem 2.3.6, was given by Cunningham [Cun77]; we present it below (see Theorem 2.4.10).

Observe that the dual Fano matroid F_7^* does not have the lattice of circuits property. (Indeed, the vector $(2, \ldots, 2)$ does not belong to $\mathbb{Z}(F_7^*)$ while it obviously satisfy (2.4.1) and (2.4.2).) However, there exist binary matroids having F_7^* as a minor and yet having the lattice of circuits property. For instance, one can check that the binary matroid S_8, which is represented by the matrix:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

has the lattice of circuits property, and that $S_8 \backslash 1 = F_7^*$. In other words, the lattice of circuits property is *not* preserved under the deletion operation. One can check that it is preserved under the contraction operation. Hence, the exclusion of F_7^* as a minor characterizes the class of binary matroids which have the lattice of circuits property, as well as all their minors.

THEOREM 2.4.10. *Let $\mathcal{M}$ be a binary matroid. The following assertions are equivalent.*

(i) *$\mathcal{M}$ and all its minors have the lattice of circuits property.*

(ii) *$\mathcal{M}$ does not have F_7^* as a minor.*

Proof. (ii) $\Longrightarrow$ (i). Suppose $\mathcal{M}=(E,\mathcal{C})$ has no F_7^* minor and does not have the lattice of circuits property. Suppose that $\mathcal{M}$ is minor minimal with respect to these properties, i.e., that every proper minor of $\mathcal{M}$ has the lattice of circuits property. Let $x \in \mathbb{Z}^E$ satisfying (2.4.1) and (2.4.2) and such that $x \notin \mathbb{Z}(\mathcal{M})$.

Claim 2.4.11. *$2\chi^{\{e\}} \notin \mathbb{Z}(\mathcal{M})$ for all $e \in E$, and $|D| \geq 3$ for every nonempty cocycle D of $\mathcal{M}$.*

Proof. Given $e \in E$, consider the matroid $\mathcal{M}/e$. Let x' denote the projection of x on $\mathbb{R}^{E\setminus\{e\}}$. Then, $x' \in \mathbb{Z}(\mathcal{M}/e)$ as x' satisfies the conditions (2.4.1) and (2.4.2) relative to $\mathcal{M}/e$. Hence, $x' = \sum_{C\in\mathcal{C}, e\notin C} \lambda_C \chi^C + \sum_{C\in\mathcal{C}, e\in C} \mu_C \chi^{C\setminus\{e\}}$, where $\lambda_C, \mu_C \in \mathbb{Z}$. Set $y := \sum_C \lambda_C \chi^C + \sum_C \mu_C \chi^C$. Then, $y \in \mathbb{Z}(\mathcal{M})$ and $x - y = a\chi^{\{e\}}$ (for some $a \in \mathbb{Z}$) satisfies (2.4.1) and (2.4.2). The element e is not a loop of $\mathcal{M}$ (else, $x \in \mathbb{Z}(\mathcal{M})$). Hence, there exists a cocycle D of $\mathcal{M}$ containing e; using (2.4.2) with D, we deduce that a is an even integer. Therefore, $2\chi^{\{e\}} \notin \mathbb{Z}(\mathcal{M})$ (else, $x \in \mathbb{Z}(\mathcal{M})$). The above also shows that no cocircuit of $\mathcal{M}$ has cardinality ≤ 2. ∎

Claim 2.4.12. *Any two elements of E belong to a common cocycle of size* 3.

Proof. Let $e \neq f \in E$ and suppose that every cocycle of $\mathcal{M}$ containing $\{e,f\}$ has size ≥ 4. Then, the matroid $\mathcal{M}\backslash e$ has no coloops. If g, g' are two coparallel elements in $\mathcal{M}\backslash e$, then $\{e,g,g'\}$ is a cocycle in $\mathcal{M}$ which does not contain f. Hence, the vector $2\chi^{\{f\}}$ satisfies the conditions (2.4.1) and (2.4.2) relative to $\mathcal{M}\backslash e$. Therefore, $2\chi^{\{f\}}$ belongs to $\mathbb{Z}(\mathcal{M}\backslash e)$ and, thus, to $\mathbb{Z}(\mathcal{M})$, which contradicts Claim 2.4.11. ∎

We can now conclude the proof. Fix $e_0 \in E$. By Claims 2.4.11 and 2.4.12, there is a one-to-one mapping $\varphi_{e_0} : E \setminus \{e_0\} \longrightarrow E \setminus \{e_0\}$ such that the set $\{e_0, e, \varphi_{e_0}(e)\}$ is a cocycle for every e. Moreover, φ_{e_0} is an involution. This permits to order the elements of E as $\{e_0, e_1, f_1, e_2, f_2, \ldots, e_k, f_k\}$ in such a way that $\{e_0, e_i, f_i\}$ is a cocycle for every $i = 1, \ldots, k$. If $k = 1$, then $E = \{e_0, e_1, f_1\}$ is the unique nonempty cocycle of $\mathcal{M}$; hence, the cycles are $\{e_0, e_1\}$, $\{e_0, f_1\}$, and $\{e_1, f_1\}$, from which one deduces that $\mathcal{M}$ has the lattice of circuits property. If $k = 2$, i.e., $|E| = 5$, then we obtain a contradiction as there is no cocycle of size 3 containing e_1 and e_2. Hence, we can suppose that $k \geq 3$, i.e., that $|E| \geq 7$. We show that $\mathcal{M}$ contains F_7^* as a minor.

Indeed, we can suppose without loss of generality that the cocycle of size 3 containing e_1 and e_2 is the set $\{e_1, e_2, e_3\}$. Set $E_0 := \{e_0, e_1, e_2, e_3, f_1, f_2, f_3\}$. Then, the following sets are all cocycles of $\mathcal{M}$: $C := \{e_0, e_1, f_1\}$, $\{e_0, e_2, f_2\}$, $\{e_0, e_3, f_3\}$, $\{e_1, e_2, e_3\}$, $\{e_1, f_2, f_3\}$, $\{e_2, f_1, f_3\}$, $\{e_3, f_1, f_2\}$, as well as their complements $E_0 \setminus C$ with respect to E_0. One can check easily that $\mathcal{M}/(E \setminus E_0)$ is the dual Fano matroid F_7^*. ∎

2.4.2 The Cocycle Lattice of the Binary Projective Space $\mathcal{P}_r$

Before proceeding to the study of the cycle lattice of an arbitrary matroid we consider in detail the case of the projective space $\mathcal{P}_r$. We remind that $\mathcal{P}_r$ is represented by the matrix W_r, which is the $r \times (2^r - 1)$ matrix whose columns are all nonzero binary vectors of $GF(2)^r$; its groundset is the set $P_r := GF(2)^r \setminus \{0\}$ of columns of W_r. We let $L(\mathcal{P}_r) := \mathbb{Z}(\mathcal{P}_r^*)$ denote the cocycle lattice of $\mathcal{P}_r$, i.e., the lattice generated by the cocycles of $\mathcal{P}_r$, and $L^*(\mathcal{P}_r) := (L(\mathcal{P}_r))^*$ denote its dual lattice. [12] The lattice $L(\mathcal{P}_r)$ can be easily described.

PROPOSITION 2.4.13. *The lattice $L(\mathcal{P}_r) = \mathbb{Z}(\mathcal{P}_r^*)$ consists of the vectors $x \in \mathbb{Z}^{P_r}$ satisfying the conditions:*

(i) $x(P_r) = 0$ (*modulo* 2^{r-1}),

(ii) $x(D) = 0$ (*modulo* 2^{r-2}) *for every cocircuit D of $\mathcal{P}_r$.*

PROOF. Necessity of the conditions (i)-(ii) follows from the fact that $|D| = 2^{r-1}$, $|D \cap D'| = 2^{r-2}$ for all distinct cocircuits D, D' of $\mathcal{P}_r$. Conversely, let $x \in \mathbb{Z}^{P_r}$ satisfying (i),(ii). Then,

$$x = \sum_{D \text{ cocircuit of } \mathcal{P}_r} \left(\frac{x(D)}{2^{r-2}} - \frac{x(P_r)}{2^{r-1}} \right) \chi^D$$

which shows that $x \in L(\mathcal{P}_r)$. ∎

In particular, the lattice $\mathbb{Z}(F_7^*)$ (the cocycle lattice of F_7) is characterized by the conditions: $\sum_{1 \le i \le 7} x_i = 0$ (modulo 4), and $x(L) = 0$ (modulo 2) for every line L of the Fano plane.

By Proposition 2.4.13, $2^{r-1}\mathbb{Z}^{P_r} \subseteq L(\mathcal{P}_r)$, i.e., $L^*(\mathcal{P}_r) \subseteq \frac{1}{2^{r-1}}\mathbb{Z}^{P_r}$. We describe below a set of generators for $L^*(\mathcal{P}_r)$. For this, we need some definitions.

A subset $F \subseteq P_r$ is called a **flat** of $\mathcal{P}_r$ if the addition of any element to F increases its rank. In other words, a flat of rank k is of the form: $\text{lin}_2(a_1, \ldots, a_k) \setminus \{0\}$, where $a_1, \ldots, a_k$ are linearly independent (over $GF(2)$) elements of P_r. Hence, every flat of rank k is isomorphic to $\mathcal{P}_k$. The only flat of rank r is the set P_r. The flats of rank $r-1$ in $\mathcal{P}_r$ are called its **hyperplanes**; they are precisely the sets $P_r \setminus C$ where C is a cocircuit of $\mathcal{P}_r$. The flats of rank 3 are isomorphic to $\mathcal{P}_3 = F_7$; we refer to them as to the Fano planes of $\mathcal{P}_r$. The flats of rank 2 are the **lines** of $\mathcal{P}_r$.

If F is a flat of rank k in $\mathcal{P}_r$, then the vector $\frac{1}{2^{k-1}}\chi^F$ belongs to $L^*(\mathcal{P}_r)$ (as $F \cap H$ is a flat of rank $k-1$ for every hyperplane H). In fact, it follows from Theorem 2.4.15 below that these vectors generate the whole lattice $L^*(\mathcal{P}_r)$, i.e., that

$$L^*(\mathcal{P}_r) = \mathbb{Z}\left(2^{1-r(F)}\chi^F \mid F \text{ is a flat of rank } r(F) \text{ in } \mathcal{P}_r\right).$$

LEMMA 2.4.14. *Let H be a hyperplane in $\mathcal{P}_r$. If $x \in L^*(\mathcal{P}_r)$ then $2x_{|H} \in L^*(H)$.*

[12] We remind that for a lattice L in $\mathbb{R}^k$ its **dual lattice** is $L^* := \{x \in \mathbb{R}^k \mid x^T y \in \mathbb{Z} \text{ for all } x \in L\}$. Then, $(L^*)^* = L$. We shall often use the following observation: Given an integer N, $N\mathbb{Z}^k \subseteq L$ if and only if $L^* \subseteq \frac{1}{N}\mathbb{Z}^k$.

PROOF. We check that $2x(H\setminus H_0)\in\mathbb{Z}$ for every flat H_0 of rank $r-2$ contained in H. Let H_1 and H_2 be two hyperplanes in $\mathcal{P}_r$ such that $H_i=H_0\cup A_i$, where A_1,A_2 partition the set $P_r\setminus H$. Then, $x(P_r\setminus H_1)=x(H\setminus H_0)+x(A_2)\in\mathbb{Z}$, $x(P_r\setminus H_2)=x(H\setminus H_0)+x(A_1)\in\mathbb{Z}$, and $x(P_r\setminus H)=x(A)+x(B)\in\mathbb{Z}$, which implies that $2x(H\setminus H_0)\in\mathbb{Z}$. ∎

THEOREM 2.4.15. [LS93] *Suppose that $x\in L^*(\mathcal{P}_r)$ and that 2^kx is integer valued. Then,*

$$x\in\mathbb{Z}\left(\frac{1}{2^{r(F)-1}}\chi^F \mid F \textit{ is a flat of rank } r(F)\le k+1\right).$$

PROOF. The proof is by induction on r. The case when $r\le 2$ is obvious. Let x satisfying the assumptions of Theorem 2.4.15. Let H_1 be a hyperplane in $\mathcal{P}_r$. Then,

(a) We can suppose that $x_{|H_1}=0$.

Indeed, by Lemma 2.4.14, $2x_{|H_1}\in L^*(H_1)$. By the induction assumption, we deduce that

$$2x_{|H_1}=\sum_i\lambda_i 2^{1-r(F_i)}\chi^{F_i}$$

where the λ_i's are integers and the F_i's are flats in H_1 of rank $r(F_i)\le k$. Let $e\notin H_1$ and let G_i denote the flat spanned by $F_i\cup\{e\}$. Set

$$y:=x-\sum_i\lambda_i 2^{1-r(G_i)}\chi^{G_i}.$$

As 2^ky is integer and $y\in L^*(\mathcal{P}_r)$, it suffices to show that Theorem 2.4.15 holds for y. This shows the assertion (a).

Let H_2 be another hyperplane in $\mathcal{P}_r$. Then,

(b) We can suppose that $x_{|H_1}=0$ and $x_{|H_2}=0$.

Indeed, let x satisfying the assumptions of Theorem 2.4.15 and $x_{|H_1}=0$. Repeating the above argument with H_2 and an element $e_2\in H_1\setminus H_2$, we find a vector:

$$z:=x-\sum_i\lambda_i\frac{1}{2^{r(G_i)-1}}\chi^{G_i},$$

where the λ_i's are integers, each flat G_i is a flat of rank $r(G_i)\le k+1$ such that $e_2\in G_i$ and $G_i\cap H_2$ has rank $r(G_i)-1$. Moreover, $z_{|H_2}=0$. We show that we can also suppose that $z_{|H_1}=0$. First, $z_f=0$ for all $f\in H_1\setminus H_2$, $f\neq e_2$; indeed, such f belongs to none of the sets G_i. Let H_3 be a hyperplane in $\mathcal{P}_r$ containing $P_r\setminus(H_1\cup H_2)$, namely, $H_3=P_r\setminus(H_1\triangle H_2)$. Since $z\in L^*(\mathcal{P}_r)$, $z_{e_2}=z(P_r\setminus H_3)\in\mathbb{Z}$. Now, the vector $z':=z-z_{e_2}\chi^{\{e_2\}}$ satisfies the assumptions of Theorem 2.4.15 and its restriction to H_1 and H_2 is zero. This shows the assertion (b).

We conclude the proof by showing that $x_{|H_3}\in L^*(H_3)$, where x is as in (b) and H_3 as defined above. Then, the result will follow by the induction assumption. Let H be a hyperplane in H_3 and let H' be the hyperplane in $\mathcal{P}_r$ spanned by H and the element e_2. Then, $x(H_3\setminus H)=x(\mathcal{P}_r\setminus H')\in\mathbb{Z}$. ∎

2.4.3 The Cocycle Lattice $L(\mathcal{M})$

We now study the cycle lattice of an arbitrary binary matroid $\mathcal{M}$. For the results presented below, we follow the work of Lovász and Seress [LS93, LS95].

It turns out to be more convenient to work with the cocycle space $\mathbb{Z}(\mathcal{M}^*)$ of $\mathcal{M}$, rather than its cycle space. To simplify notation we let, following [LS93], $L(\mathcal{M}) := \mathbb{Z}(\mathcal{M}^*)$ denote the cocycle space of $\mathcal{M}$. Then, its dual lattice $L^*(\mathcal{M}) := (L(\mathcal{M}))^*$ consists of the vectors x such that $x(D) \in \mathbb{Z}$ for all cocycles D of $\mathcal{M}$.

In what follows we always suppose that $\mathcal{M}$ is a binary matroid of rank r with no loops and no parallel elements. (This is no loss of generality, as the lattice $L(\mathcal{M})$ can be easily deduced from the lattice $L(\bar{\mathcal{M}})$, where $\bar{\mathcal{M}}$ is obtained from $\mathcal{M}$ by contracting loops and deleting $\{e_1, \ldots, e_p\}$ for each parallel class $\{e_0, e_1, \ldots, e_p\}$.) Then, $\mathcal{M}$ has a representation matrix B which is a column submatrix of the matrix W_r (representing $\mathcal{P}_r$). In other words, $\mathcal{M}$ can be described simply by specifying its groundset as a subset of the set P_r of columns of W_r. For this reason we will now use the same letter M for denoting the groundset of the matroid $\mathcal{M}$. In particular, $P_r = GF(2)^r \setminus \{0\}$ denotes the groundset of $\mathcal{P}_r$. Moreover, given a subset $M \subseteq P_r$ we may speak of the matroid M which means, of course, the matroid represented by the column submatrix of W_r indexed by M.

In particular, $\mathcal{M}$ is a deletion minor of $\mathcal{P}_r$, namely, $\mathcal{M} = \mathcal{P}_r \backslash (P_r \setminus M)$. Hence, the cocycle space of $\mathcal{M}$ is the projection on $\mathbb{R}^M$ of the cocycle space of $\mathcal{P}_r$. Therefore,

PROPOSITION 2.4.16. *Let $\mathcal{M}$ be a binary matroid of rank r on $M \subseteq P_r$. The cocycle lattice $L(\mathcal{M})$ of $\mathcal{M}$ is the projection of the cocycle lattice $L(\mathcal{P}_r)$ of $\mathcal{P}_r$ on $\mathbb{R}^M$. Given $x \in \mathbb{R}^M$ and $x' := (x, 0, \ldots, 0) \in \mathbb{R}^{P_r}$, then $x \in L^*(\mathcal{M})$ if and only if $x' \in L^*(\mathcal{P}_r)$.* ∎

Let $k(\mathcal{M})$ denote the smallest integer k such that $2^k \mathbb{Z}^M \subseteq L(\mathcal{M})$ or, equivalently, $L^*(\mathcal{M}) \subseteq \frac{1}{2^k}\mathbb{Z}^M$. Then, $k(\mathcal{P}_r) = r - 1$ by the discussion above, which implies that

$$k(\mathcal{M}) \leq r - 1. \tag{2.4.17}$$

Note that $k(\mathcal{M}) = 0$ if and only if $\mathcal{M}$ is a free matroid (i.e., $\emptyset$ is the only cycle). Moreover, the case $k(\mathcal{M}) = 1$ corresponds precisely to the lattice of circuits property. Namely,

LEMMA 2.4.18. *The following assertions are equivalent.*

(i) *$\mathcal{M}^*$ has the lattice of circuits property.*

(ii) *$2\mathbb{Z}^M \subseteq L(\mathcal{M})$.*

(iii) *$L^*(\mathcal{M}) \subseteq \frac{1}{2}\mathbb{Z}^M$.* ∎

We present below several results leading, in particular, to a characterization of the binary matroids $\mathcal{M}$ for which $k(\mathcal{M}) = 1$ (see Theorem 2.4.27) and $k(\mathcal{M}) = 2$ (see Theorem 2.4.28).

We need some more definitions. Given two vectors $a, b \in GF(2)^n$, $a \oplus b$ denotes their sum modulo 2 and $a \circ b$ denotes the vector $(a_1 b_1, \ldots, a_n b_n)$. Given two sets of vectors A and B, $A \circ B := \{a \circ b \mid a \in A, b \in B\}$. We set $A^{\circ k} := A \circ \ldots \circ A$ (k times). Moreover, we let $\mathrm{lin}_2(A) := GF(2)(A)$ denote the linear subspace over $GF(2)$ spanned by A; hence, $\mathrm{lin}_2(A)$ consists of all possible sums of members of A. If U is a matrix with set of rows

A, then its **k-extension matrix** is the matrix $U^{(k)}$ whose set of rows is $A^{\circ k}$. Note that $A^{\circ 1} = A$ and $U^{(1)} = U$. The following lemmas from [LS93] indicate some useful properties of the operations over the integers and the operations modulo 2. Lemma 2.4.19 can be checked by direct inspection.

LEMMA 2.4.19. *Given a set $A = \{a_1, \ldots, a_r\}$ of binary vectors, we have:*
(i) $a_1 \oplus \ldots \oplus a_r = \sum_{1\le t\le r}(-2)^{t-1}\sum_{1\le i_1<\ldots<i_t\le r}(a_{i_1}\circ\ldots\circ a_{i_t})$.
(ii) $2^{r-1}a_1\circ\ldots\circ a_r = \sum_{1\le t\le r}(-1)^{t-1}\sum_{1\le i_1<\ldots<i_t\le r}(a_{i_1}\oplus\ldots\oplus a_{i_t})$.
(iii) $\mathbb{Z}(\mathrm{lin}_2(A)) = \mathbb{Z}(A \cup 2A^{\circ 2} \cup 4A^{\circ 4} \cup \ldots \cup 2^{r-1}A^{\circ r})$. ∎

LEMMA 2.4.20. *Let $b_1, \ldots, b_r \in \mathbb{Z}^n$ and $\widetilde{b_i} = b_i$ (modulo 2), $\widetilde{b_i} \in GF(2)^n$, for $i = 1, \ldots, r$. If $\mathrm{lin}_2(\widetilde{b}_1, \ldots, \widetilde{b}_r) = GF(2)^n$ and $2^N\mathbb{Z}^n \subseteq \mathbb{Z}(b_1, \ldots, b_r)$ for some $N \ge 0$, then $\mathbb{Z}(b_1, \ldots, b_r) = \mathbb{Z}^n$.*

PROOF. Clearly, $r \ge n$. We can suppose that $\widetilde{b}_1, \ldots, \widetilde{b}_n$ are linearly independent over $GF(2)$. Hence, $\delta := \det(b_1, \ldots, b_n)$ is an odd integer. Let $x \in \mathbb{Z}^n$. By Cramer's rule, we deduce that $\delta x \in \mathbb{Z}(b_1, \ldots, b_n) \subseteq \mathbb{Z}(b_1, \ldots, b_r)$. On the other hand, $2^N x \in \mathbb{Z}(b_1, \ldots, b_r)$. As δ and 2^N are relatively prime this implies that $x \in \mathbb{Z}(b_1, \ldots, b_r)$. ∎

LEMMA 2.4.21. *Let A be a set of linearly independent (over $GF(2)$) vectors in $GF(2)^n$. If $2^s\mathbb{Z}^n \subseteq \mathbb{Z}(\mathrm{lin}_2(A))$, then $\mathrm{lin}_2(A^{\circ 2^s}) = GF(2)^n$.*

PROOF. The proof is by induction on $s \ge 0$. The result holds obviously for $s = 0$. Suppose that $s \ge 1$ and that $2^s\mathbb{Z}^n \subseteq \mathbb{Z}(\mathrm{lin}_2(A))$. Applying Lemma 2.4.19 (iii), we have

$$\begin{aligned}\mathbb{Z}(\mathrm{lin}_2(A^{\circ 2})) &= \mathbb{Z}(A^{\circ 2} \cup 2A^{\circ 4} \cup \ldots \cup 2^{j-1}A^{\circ(2j)} \cup \ldots)\\ &\supseteq \mathbb{Z}(A^{\circ 2} \cup 2A^{\circ 3} \cup \ldots \cup 2^{j-1}A^{\circ(j+1)} \cup \ldots).\end{aligned}$$

We show that

$$\mathbb{Z}(A^{\circ 2} \cup 2A^{\circ 3} \cup \ldots \cup 2^{j-1}A^{\circ(j+1)} \cup \ldots) \supseteq 2^{s-1}\mathbb{Z}^n;$$

then, by the induction assumption, $\mathrm{lin}_2(A^{\circ 2^s}) = \mathrm{lin}_2((A^{\circ 2})^{\circ 2^{s-1}}) = GF(2)^n$.

Let $x \in \mathbb{Z}^n$. Then, $2^s x \in \mathbb{Z}(\mathrm{lin}_2(A))$; hence, letting $a_1, a_2, \ldots$ denoting the elements of A,

$$2^s x = \sum_i \lambda_i a_i + 2\sum_{ij}\lambda_{ij}a_i\circ a_j + 4\sum_{ijk}\lambda_{ijk}a_i\circ a_j\circ a_k + \ldots,$$

where all λ's are integer. This implies that the λ_i's are even integers (as the vectors in A are linearly independent). By dividing by 2 the above relation, we obtain that $x \in \mathbb{Z}(A^{\circ 2} \cup 2A^{\circ 3} \cup \ldots \cup 2^{j-1}A^{\circ(j+1)} \cup \ldots)$. ∎

Let $\mathcal{F}_{r,k}$ [13] denote the binary space which is spanned by the flats of rank k in $\mathcal{P}_r$, i.e., $\mathcal{F}_{r,k}$ consists of all possible symmetric differences $F_1 \triangle \ldots \triangle F_p$ where the F_i's are flats of

[13] The binary spaces $\mathcal{F}_{r,k}$ are known in coding theory as the punctured Reed-Muller codes; see Chapter 13 in [MS77].

rank k in $\mathcal{P}_r$. Hence, $\mathcal{F}_{r,k}$ is the cycle space of a binary matroid on P_r; we denote this matroid as $\mathcal{P}_r^k$. For example, $\mathcal{P}_r^1$ has rank 0 (as every element is a flat of rank 1, implying that every subset is a cycle of $\mathcal{P}_r^1$); $\mathcal{P}_r^2 = \mathcal{P}_r$ (as the flats of rank 2 are the sets $\{a, b, a \oplus b\}$ for $a \neq b \in P_r$ and their symmetric differences are all cycles of $\mathcal{P}_r$); $\mathcal{P}_r^r$ has P_r as unique circuit (as P_r is the unique flat of rank r).

A basis $\mathcal{L}_k$ (over $GF(2)$) for the binary space $\mathcal{F}_{r,k}$ can be constructed in the following way (see [LS93]). Let $\{e_1, \ldots, e_r\}$ be a set of r linearly independent (over $GF(2)$) vectors in P_r. Then, $\mathcal{L}_k$ consists of the following flats F of rank k: F is spanned by the vectors $e_{i_1}, \ldots, e_{i_{k-1}}$ and f_F, where $1 \leq i_1 < i_2 < \ldots < i_{k-1} \leq r-1$ and f_F is a nonzero element of $\text{lin}_2(e_{i_{k-1}+1}, \ldots, e_r)$. It is shown in [LS93] that every set $F \in \mathcal{F}_{r,k}$, $|F| = n$, can be expressed as a symmetric difference of at most $n\binom{r-1}{k-1}$ members of $\mathcal{L}_k$ and that such a decomposition can be found in time $O(n\binom{r+k-2}{k})$.

Let $\bar{\mathcal{F}}_{r,k}$ denote the binary space spanned by the sets $P_r \setminus F$, where F is a flat of rank k in $\mathcal{P}_r$. Equivalently, $\bar{\mathcal{F}}_{r,k}$ consists of all sets of the form: $F_1 \triangle \ldots \triangle F_p$, where the F_i's are flats of rank k in $\mathcal{P}_r$ and p is an *even* integer. One can check that the binary space $\bar{\mathcal{F}}_{r,r-k+1}$ coincides with $\mathcal{F}_{r,k}^{\perp}$, the orthogonal complement of $\mathcal{F}_{r,k}$ in $GF(2)^{P_r}$. In other words, the cocycle space of the binary matroid $\mathcal{P}_r^k$ is precisely $\bar{\mathcal{F}}_{r,r-k+1}$. Note that $\mathcal{F}_{r,k} \supseteq \mathcal{F}_{r,k+1}$ and $\bar{\mathcal{F}}_{r,k} \supseteq \bar{\mathcal{F}}_{r,k+1}$. The following result provides a representation matrix for the matroid $\mathcal{P}_r^k$.

PROPOSITION 2.4.22. [LS93] *The k-extension $W_r^{(k)}$ of the matrix W_r is a representation matrix of the matroid $\mathcal{P}_r^{k+1}$.* ∎

COROLLARY 2.4.23. *Given a subset $M \subseteq P_r$ and let B denote the column submatrix of the matrix W_r indexed by M. The following assertions are equivalent.*

(i) *M is independent in the matroid $\mathcal{P}_r^3$.*

(ii) *M does not contain a binary sum of Fano planes in $\mathcal{P}_r$, i.e., $F \not\supseteq F_1 \triangle \ldots \triangle F_p$ where each F_i is a flat of rank 3 in $\mathcal{P}_r$.*

(iii) *The matrix $B^{(2)}$ (the 2-extension matrix of B) has full column rank $|M|$.* ∎

As above, given $M \subseteq P_r$, let B denote the column submatrix of W_r indexed by M, let $\mathcal{M}$ denote the binary matroid represented by B, and let $A := \{a_1, \ldots, a_r\}$ denote the set of rows of B. Applying Lemma 2.4.19 (iii), we have

$$L(\mathcal{M}) = \mathbb{Z}(\text{lin}_2(A)) = \mathbb{Z}(A \cup 2A^{\circ 2} \cup \ldots \cup 2^{r-1}A^{\circ r}). \tag{2.4.24}$$

Moreover, one can check that $\mathbb{Z}(A^{\circ r}) = \mathbb{Z}^M$, which yields again relation (2.4.17). In fact, $k(\mathcal{M}) \leq s-1$, if any $s+1$ vectors in A have a zero coordinate-wise product. (Indeed, this implies that $\mathbb{Z}(A^{\circ s}) = \mathbb{Z}^M$. To see it, let $S_1, \ldots, S_r$ denote the sets whose incidence vectors are $a_1, \ldots, a_r$. Then, each cell in the Venn diagram of $S_1, \ldots, S_r$ has at most one element (else, B would have two identical columns). Using the inclusion-exclusion formula, one can express each vector $\chi^{\{e\}}$ ($e \in M$) as an integer combination of members of $A^{\circ s}$.) As an extension, we have the following upper bound.

THEOREM 2.4.25. [LS93] *If $\text{lin}_2(A^{\circ(s+1)}) = GF(2)^M$, then $2^s\mathbb{Z}^M \subseteq L(\mathcal{M})$.*

PROOF. Using (2.4.24) and (2.4.17), we have

$$2^{r-1}\mathbb{Z}^M \subseteq L(\mathcal{M}) \subseteq \mathbb{Z}(A^{\circ(s+1)} \cup 2A^{\circ(s+2)} \cup \ldots \cup 2^{r-s-1}A^{\circ r}).$$

Using the assumption and Lemma 2.4.20, we deduce that

$$\mathbb{Z}(A^{\circ(s+1)} \cup 2A^{\circ(s+2)} \cup \ldots \cup 2^{r-s-1}A^{\circ r}) = \mathbb{Z}^M.$$

Therefore,

$$L(\mathcal{M}) \supseteq \mathbb{Z}(2^s A^{\circ(s+1)} \cup \ldots \cup 2^{r-1}A^{\circ r}) = 2^s\mathbb{Z}^M.$$

∎

Lemma 2.4.21 can be seen as a (weak) converse to Theorem 2.4.25. However, in the cases $s = 0, 1$, Lemma 2.4.21 shows that equivalence holds in Theorem 2.4.25.

COROLLARY 2.4.26. *If $M \subseteq P_r$ and $k(\mathcal{M}) \geq k$, then M is dependent in the matroid $\mathcal{P}_r^{k+1}$.*

PROOF. Suppose that M is independent in $\mathcal{P}_r^{k+1}$. Then, by Proposition 2.4.22, the matrix $B^{(k)}$ has full column rank $|M|$. Hence, $GF(2)^M = \text{lin}_2(A^{\circ k})$ (where A denotes the set of rows of B, and B is the submatrix of W_r indexed by M). Therefore, by Theorem 2.4.25, $2^{k-1}\mathbb{Z}^M \subseteq L(\mathcal{M})$, implying that $k(\mathcal{M}) \leq k - 1$. ∎

2.4.4 Binary Matroids with $k(\mathcal{M}) \leq 2$

We can now summarize the results that have been obtained for the binary matroids with $k(\mathcal{M}) = 1$.

THEOREM 2.4.27. [LS93] *Let $\mathcal{M}$ be a binary matroid of rank r on $M \subseteq P_r$ and let B be the column submatrix of W_r indexed by M. We remind that $L(\mathcal{M})$ is the lattice $\mathbb{Z}(\mathcal{M}^*)$ spanned by the cocycles of $\mathcal{M}$. The following assertions are equivalent.*

(i) *$\mathcal{M}^*$ has the lattice of circuits property.*

(ii) $2\mathbb{Z}^M \subseteq L(\mathcal{M})$.

(iii) $L^*(\mathcal{M}) \subseteq \frac{1}{2}\mathbb{Z}^M$.

(iv) $L^*(\mathcal{M}) = \{x \in \mathbb{R}^M \mid x = y + \frac{1}{2}\chi^C$ *where* $y \in \mathbb{Z}^M$ *and* C *is a cycle of* $\mathcal{M}\}$.

(v) *The 2-extension matrix $B^{(2)}$ of B has full column rank $|M|$.*

(vi) *M does not contain the binary sum of a set of Fano planes (flats of rank 3 in $\mathcal{P}_r$).*

PROOF. (i) $\iff$ (ii) $\iff$ (iii) follows from Lemma 2.4.18; (iii) $\iff$ (iv) is obvious; (v) $\iff$ (vi) follows from Corollary 2.4.23; (v) $\implies$ (ii) follows from Theorem 2.4.25, and (ii) $\implies$ (v) from Lemma 2.4.21. ∎

In particular, given a binary matroid $\mathcal{M}$ represented by a matrix P, one can check in time polynomial in the size of P whether $\mathcal{M}$ has the lattice of circuits property.

By Corollary 2.4.26 we know that $k(\mathcal{M}) \leq k$ if M is independent in the binary matroid $\mathcal{P}_r^{k+2}$. This implication is an equivalence in the case $k = 1$ by Theorem 2.4.27 above. However, equivalence does not hold if $k \geq 2$. The following counterexample is given in [LS95]: Let F_1, F_2, F_3 be three flats of rank 4 in $\mathcal{P}_r$ ($r \geq 6$) such that $F_1 \cap F_2 \cap F_3 = \emptyset$ and $F_i \cap F_j$ has rank 2 for $1 \leq i < j \leq 3$. Set $M := F_1 \triangle F_2 \triangle F_3$. Then, the set M is dependent in $\mathcal{P}_r^4$; however, one can check that $k(\mathcal{M}) = 2$.

The binary matroids with $k(\mathcal{M}) = 2$ are characterized in Theorem 2.4.28 below. We need some notation. Let $\mathcal{M}$ be a binary matroid of rank r on the set $M \subseteq P_r$. Let B denote the column submatrix of W_r indexed by M and let $B^{(2)}$ be its 2-extension. Let V_M denote the subspace of $GF(2)^{r+\binom{r}{2}}$, which is spanned by the columns of $B^{(2)}$.

Let $\{e_1, \ldots, e_r\} \subseteq M$ be a subset of r linearly independent (over $GF(2)$) vectors in M and let $\mathcal{L}_k$ denote the corresponding basis of the space $\mathcal{F}_{r,k}$.

Given a subset $F \subseteq M$ such that $F \in \mathcal{F}_{r,4}$ with decomposition: $F := \triangle_{J \in \mathcal{J}} J$ where $\mathcal{J} \subseteq \mathcal{L}_4$, in the basis $\mathcal{L}_4$, set

$$Y_F := \{e \in P_r \setminus M : |\{J \in \mathcal{J} \mid e \in J\}| = 2(\text{ modulo } 4)\}$$

and let y_F denote the 2-extension of the vector $\oplus_{y \in Y_F} y$.

THEOREM 2.4.28. [LS95] *Suppose that $k(\mathcal{M}) \geq 2$. The following assertions are equivalent.*

(i) $k(\mathcal{M}) = 2$.

(ii) *For all $F \in \mathcal{F}_{r,4}$ such that $\emptyset \neq F \subseteq M$, $y_F \notin V_M$.*

(iii) *For all $F \in \mathcal{F}_{r,4}$ such that $\emptyset \neq F \subseteq M$ and all $X \subseteq M$, $X \cup Y_F \notin \mathcal{F}_{r,3}$.*

PROOF. (ii) $\Longleftrightarrow$ (iii) is clear.
(i) $\Longrightarrow$ (iii). Let $F \in \mathcal{F}_{r,4}$, $\emptyset \neq F \subseteq M$, and suppose that there exists $X \subseteq M$ such that $X \cup Y_F \in \mathcal{F}_{r,3}$. We construct a vector $x \in L^*(\mathcal{M})$ such that $4x \notin \mathbb{Z}^M$, which contradicts the assumption that $k(\mathcal{M}) = 2$. Let $F := \triangle_{J \in \mathcal{J}} J$ where $\mathcal{J} \subseteq \mathcal{L}_4$, be the decomposition of F in the basis $\mathcal{L}_4$, and let $X \cup Y_F := \triangle_{K \in \mathcal{K}} K$ where $\mathcal{K} \subseteq \mathcal{L}_3$, be the decomposition of $X \cup Y_F$ in the basis $\mathcal{L}_3$. Set

$$z_1 := \frac{1}{8} \sum_{J \in \mathcal{J}} \chi^J, \; z_2 := z_1 + \frac{1}{4} \sum_{K \in \mathcal{K}} \chi^K.$$

Then, for $e \in P_r \setminus M$, $z_1(e)$ has denominator 4 if $e \in Y_F$ and denominator ≤ 2 otherwise. Hence, $z_2(e)$ has denominator ≤ 2 in every element $e \in P_r \setminus M$. By adding to z_2 the vectors $\frac{1}{2}\chi^L$ for some suitably chosen lines L, we obtain a vector $x \in L^*(\mathcal{P}_r)$ such that $x_e \in \mathbb{Z}$ for all $e \in P_r \setminus M$ while x_e has denominator 8 for some $e \in M$. Then, the projection $x_{|M}$ of such x on $\mathbb{R}^M$ belongs to $L^*(\mathcal{M})$ and is such that $4x_{|M} \notin \mathbb{Z}^M$. We add vectors $\frac{1}{2}\chi^L$ to z_2 in the following manner. Recall that $e_1, \ldots, e_r$ is a basis of $GF(2)^{P_r}$ which is contained in M. For all $e \in P_r \setminus M$ for which $z_2(e)$ has denominator 2 and such that $e \notin \mathrm{lin}_2(e_1, \ldots, e_{r-1})$, consider the line $L := \{e, e_r, e \oplus e_r\}$ and add $\frac{1}{2}\chi^L$ to z_2. Call z_3 the resulting vector. Now, $z_3(e)$ is integral for every $e \in P_r \setminus (M \cup \mathrm{lin}_2(e_1, \ldots, e_{r-1}))$. Next, for all $e \in P_r \setminus M$ for which $z_3(e)$ has denominator 2 and such that $e \in \mathrm{lin}_2(e_1, \ldots, e_{r-1}) \setminus \mathrm{lin}_2(e_1, \ldots, e_{r-2})$, consider the line $L := \{e, e_{r-1}, e \oplus e_{r-1}\}$ and add $\frac{1}{2}\chi^L$ to z_3. (Note that this does not

affect the previously defined integral values of z_3, as $e \oplus e_{r-1} \in \mathrm{lin}_2(e_1, \ldots, e_{r-2})$ if $e \in \mathrm{lin}_2(e_1, \ldots, e_{r-1}) \setminus \mathrm{lin}_2(e_1, \ldots, e_{r-2})$.) Reiterating the process, we obtain a vector $x \in L^*(\mathcal{P}_r)$ which is integral outside of M.
(iii) $\Longrightarrow$ (i). Suppose that $k(\mathcal{M}) > 2$. Then, there exists $x \in L^*(\mathcal{M})$ such that $8x \in \mathbb{Z}^M$ and $4x \notin \mathbb{Z}^M$. Hence, the set $F := \{e \in M \mid x_e$ has denominator $8\}$ belongs to $\mathcal{F}_{r,4}$, by Theorem 2.4.15 and Proposition 2.4.16. Let $F := \triangle_{J \in \mathcal{J}} J$ where $\mathcal{J} \subseteq \mathcal{L}_4$, be its decomposition in the basis $\mathcal{L}_4$ of $\mathcal{F}_{r,4}$. Set

$$x' := x - \frac{1}{8} \sum_{J \in \mathcal{J}} \chi^J,$$

where we denote again by x the vector $(x, 0, \ldots, 0) \in \mathbb{R}^{P_r}$. By Theorem 2.4.15, x can be expressed as an integer combination of vectors $2^{1-k}\chi^G$ where G is a flat of rank $k \leq 3$. In particular, the set consisting of the positions $e \in P_r$ for which $x'(e)$ has denominator 4 belongs to $\mathcal{F}_{r,3}$. On the other hand, for $e \in P_r \setminus M$, $x'(e)$ has denominator 4 precisely if the number of J's containing e is 2 modulo 4, i.e., if $e \in Y_F$. Therefore, $Y_F \cup X \in \mathcal{F}_{r,3}$ for some subset $X \subseteq M$. ∎

As an application, one can check whether $k(\mathcal{M}) = 2$ in time polynomial in $n = |M|$ (see [LS95]). This can be done as follows: Let $F_1, \ldots, F_s$ be a basis for the space $\{F \subseteq M \mid F \in \mathcal{F}_{r,4}\}$. Compute the vectors y_{F_i} for $i = 1, \ldots, s$ (this can be done as one can compute in polynomial time the decomposition of F_i in the basis $\mathcal{L}_4$). Finally, observe that if $F = \triangle_{i \in I} F_i$ for $I \subseteq \{1, \ldots, s\}$ then $y_F \in V_M$ if and only if $\oplus_{i \in I} y_{F_i} \in V_M$, which can be checked in polynomial time.

There is no known similar characterization for the class of binary matroids with $k(\mathcal{M}) = k$ when $k \geq 3$. However, using an extension of the above arguments, Lovász and Seress [LS95] show that, given an integer k and a matroid $\mathcal{M}$ on $n = |M|$ elements, one can decide whether $k(\mathcal{M}) \geq k$ in time polynomial in k, n.

We conclude with a generalization of Theorem 2.4.10 to the binary matroids with $k(\mathcal{M}) \geq 1$. We first state a preliminary result.

Lemma 2.4.29. *Let M be a nonempty subset of P_r, let B denote the column submatrix of W_r indexed by M, and let $\mathcal{M}$ be the binary matroid on M represented by B. If $M \in \mathcal{F}_{r,k}$, then $\mathcal{M}$ contains $\mathcal{P}_k$ as a minor.*

Proof. The proof is by induction on $r \geq k$. If $r = k$ then $M = P_r$, i.e., $\mathcal{M} = \mathcal{P}_r$ and the result holds trivially. Suppose $r > k$ and let $e \in M$. We can partition M as $M = \{e\} \cup X \cup (X \oplus e) \cup Y$, where X, Y are disjoint, $X \oplus e := \{x \oplus e \mid x \in X\}$, and $y \oplus e \notin M$ for all $y \in Y$. Consider the binary matroid $\mathcal{N} := \mathcal{M}/e \backslash (X \oplus e)$ on the set $N := X \cup Y$. Hence, $\mathcal{N}$ has no loops and no parallel elements. Let H be a hyperplane in $\mathcal{P}_r$ that contains the set N. We show that $N \in \mathcal{F}_{r-1,k}$, i.e., that N is a cocycle of the binary matroid $\mathcal{P}^k_{r-1}$ (with H in the role of $\mathcal{P}_{r-1}$). Then, by the induction assumption, $\mathcal{N}$ contains a $\mathcal{P}_k$ minor and, thus, $\mathcal{M}$ too has a $\mathcal{P}_k$ minor. As the cocycle space of $\mathcal{P}^k_{r-1}$ is spanned by the sets $H \setminus F$, where F is a flat of rank $r - k$ contained in H, it suffices to check that $|N \cap (H \setminus F)|$ is even for every such F. Let G denote the flat in $\mathcal{P}_r$ spanned by $F \cup \{e\}$, with rank $r - k + 1$. Then, $|M \cap (P_r \setminus G)|$ is even, as $M \in \mathcal{F}_{r,k}$. The result now follows after observing that $|N \cap (H \setminus F)|$ and $|M \cap (P_r \setminus G)|$ have the same parity. ∎

THEOREM 2.4.30. [LS93] *Let $\mathcal{M}$ be a binary matroid. The following assertions are equivalent.*

(i) *$\mathcal{M}$ does not have $\mathcal{P}_{k+1}$ as a minor.*

(ii) *$k(\mathcal{N}) \leq k-1$ for every minor $\mathcal{N}$ of $\mathcal{M}$.*

PROOF. (ii) $\Longrightarrow$ (i) follows from the fact that $k(\mathcal{P}_{k+1}) = k$. Conversely, suppose that $k(\mathcal{M}) \geq k$. We suppose that $\mathcal{M}$ is a binary matroid of rank r on $M \subseteq P_r$. Then, by Corollary 2.4.26, M is dependent in the matroid $\mathcal{P}_r^{k+1}$. Therefore, $M \supseteq F_1 \triangle \ldots \triangle F_p$, where the F_i's are flats of rank $k+1$ in $\mathcal{P}_r$. By Lemma 2.4.29, every such set $F_1 \triangle \ldots \triangle F_p$ contains $\mathcal{P}_{k+1}$ as a minor. Hence, $\mathcal{M}$ too contains $\mathcal{P}_{k+1}$ as a minor. ∎

2.5 Other Hulls of Cycles: The Circuit Cover Property

In this section we present results related to the integer cone $\mathbb{Z}_+(\mathcal{M})$ of a binary matroid $\mathcal{M}$. We remind from relation (2.4.6) that

$$\mathbb{Z}_+(\mathcal{M}) \subseteq \mathbb{Z}(\mathcal{M}) \cap \mathbb{R}_+(\mathcal{M}) \subseteq Q(\mathcal{M}),$$

where we set

$$\begin{aligned} Q(\mathcal{M}) := \{x \in \mathbb{Z}_+^E \mid\ & x(D) = 0 \text{ (modulo 2) for } D \text{ cocycle of } \mathcal{M}, \\ & x_e - x(D \setminus \{e\}) \leq 0 \text{ for } D \text{ cocycle of } \mathcal{M}, e \in D\}. \end{aligned}$$

We recall that $\mathcal{M}$ is said to have the circuit cover property if $\mathbb{Z}_+(\mathcal{M}) = Q(\mathcal{M})$, and the Hilbert basis property if $\mathbb{Z}_+(\mathcal{M}) = \mathbb{Z}(\mathcal{M}) \cap \mathbb{R}_+(\mathcal{M})$. The latter terminology is justified by the fact that, when equality $\mathbb{Z}_+(\mathcal{M}) = \mathbb{Z}(\mathcal{M}) \cap \mathbb{R}_+(\mathcal{M})$ holds, then the cycle space of $\mathcal{M}$ forms a Hilbert basis (see, e.g., [Sch86] for details on Hilbert bases). The next lemma indicates how the two properties relate.

LEMMA 2.5.1. *Let $\mathcal{M}$ be a binary matroid. the following assertions are equivalent.*

(i) *$\mathcal{M}$ has the circuit cover property.*

(ii) *$\mathcal{M}$ has the sums of circuits property and the Hilbert basis property.*

PROOF. (i) $\Longrightarrow$ (ii). We check that $\mathcal{M}$ has the sums of circuits property. For this let $x \in \mathrm{MET}(\mathcal{M})$; we can suppose without loss of generality that x is rational valued (as the result for arbitrary x will then follow by continuity). Let k be an integer such that $kx \in 2\mathbb{Z}^E$. Then, $kx \in Q(\mathcal{M})$. Therefore, $kx \in \mathbb{Z}_+(\mathcal{M})$ which implies that $x \in \mathbb{R}_+(\mathcal{M})$.
(ii) $\Longrightarrow$ (i) follows from the fact that the lattice $\mathbb{Z}(\mathcal{M})$ is characterized by the parity condition: $x(D)$ even for all cocycles D of $\mathcal{M}$, as $\mathcal{M}$ has no F_7^* minor (because $\mathcal{M}$ has the sums of circuits property; recall Theorems 2.2.3 and 2.4.10). ∎

There is a vast literature on questions related to the integer cone $\mathbb{Z}_+(\mathcal{M})$ of a binary matroid $\mathcal{M}$ and, in particular, for graphic matroids. A vector x belongs to $\mathbb{Z}_+(\mathcal{M})$ if there is a list of (possibly repeated) circuits (or cycles) of $\mathcal{M}$ covering every element e exactly x_e times; such a list is called a **circuit cover** of x.

As we will see below, the binary matroids with the circuit cover property are fully characterized. Yet, if $\mathcal{M}$ does not have the circuit cover property, it is of interest to identify some classes of vectors x for which the condition $x \in \mathbb{Z}(\mathcal{M}) \cap \mathbb{R}_+(\mathcal{M})$ suffices to ensure that x has a circuit cover. For instance, the following is known (see Proposition 4.2 in [God93]): Suppose $\mathcal{M}$ has no F_7^* minor and $x = r\chi^E$ for some integer $r \neq 2$. Then, x has a circuit cover if and only if $x \in \mathbb{Z}(\mathcal{M}) \cap \mathbb{R}_+(\mathcal{M})$. Interestingly, the case $r = 2$ remains open. For graphic matroids, the notorious *circuit double cover conjecture* asserts that in a bridgeless [14] graph $G = (V, E)$ the vector $2\chi^E$ has a circuit cover, i.e., there exists a list of circuits in G covering every edge exactly twice. See, e.g., the survey by Jaeger [Jae85] for more information on this conjecture.

We remind that the four-colour theorem is equivalent to the statement that every bridgeless graph can be decomposed as the union of two cycles. (It will be used in this form in the proof of Theorem 2.5.2.)

Similar questions have been studied for other combinatorial objects than cycles in matroids, in particular, for perfect matchings in graphs. For instance, the cone and the lattice generated by the perfect matchings of a graph have been characterized by Lovász [Lov87]. As in the case of cycle spaces, the integer cone of perfect matchings is not yet fully understood. There are several open conjectures. One of them, due to Lovász, asserts that if a graph does not contain the Petersen graph P_{10} as a minor then its perfect matchings form a Hilbert basis. (Note that this conjecture is not yet settled for planar graphs and that it would imply the four-colour theorem.) We refer, e.g., to the survey by Goddyn [God93] for more details on questions, results and references around the topic of circuit covers.

We now present a characterization of the binary matroids having the circuit cover property. Note that, for graphic matroids, the circuit cover property and the Hilbert basis property coincide, as all graphic matroids have the sums of circuits property. Seymour [Sey79] proved that this property holds for the graphic matroids of planar graphs.

THEOREM 2.5.2. [Sey79] *The graphic matroid of a planar graph has the circuit cover property.*

This result was extended by Alspach, Goddyn and Zhang [AGZ94] who completely characterized the graphic matroids with the circuit cover property. Let P_{10} denote the **Petersen graph**; see Figure 2.5.3.

[14] A **bridge** in a graph is a cut of size 1.

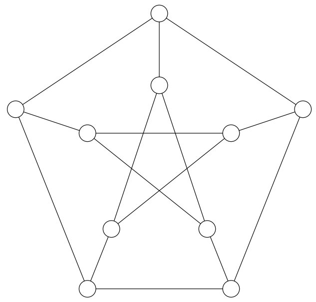


Figure 2.5.3: The Petersen graph P_{10}

The graphic matroid $\mathcal{M}(P_{10})$ does not have the circuit cover property. (Indeed, consider the vector x taking value 2 on a perfect matching of P_{10} and value 1 on the remaining edges. To see that x has no circuit cover one can check, for instance, that $x - \chi^C \notin \mathbb{R}_+(\mathcal{M}(P_{10}))$ for every circuit C of P_{10}.) The exclusion of P_{10} as a minor characterizes the graphs with the circuit cover property.

THEOREM 2.5.4. [AGZ94] *The graphic matroid $\mathcal{M}(G)$ of a graph G has the circuit cover property if and only if G does not have the Petersen graph P_{10} as a minor.* ∎

We present below the proof of Seymour [Sey79] for Theorem 2.5.2. We do not give the proof of Alspach, Goddyn and Zhang for Theorem 2.5.4 as it is quite technically involved. It is in some sense an extension of Seymour's proof, but several steps are much more complicated. (For instance, at some point in the proof of Theorem 2.5.2, we use the "splitting off" operation, which preserves planarity of graphs, but does not preserve the class of graphs with no Petersen minor.)

PROOF OF THEOREM 2.5.2. We use the following notation. Given a graph $G = (V, E)$ and a vector $p \in \mathbb{R}^E$, we say that the pair (G, p) is *admissible* if $p \in Q(\mathcal{M}(G))$, i.e., if p satisfies the obvious necessary conditions for membership in $\mathbb{Z}_+(\mathcal{M}(G))$.

Suppose that Theorem 2.5.2 does not hold. Let (G, p) be a counterexample, i.e., (G, p) is admissible, G is planar, and $p \notin \mathbb{Z}_+(\mathcal{M}(G))$. We choose such a counterexample for which the quantity $n(G, p) := \sum_{e \in E}(1 + p(e))^2$ is minimum. Then, G is connected.

The main part of the proof consists in proving that $p \in \{1, 2\}^E$. Suppose that this is the case. Set $F := \{e \in E \mid p(e) = 1\}$. Then, F is a cycle in G (as $p(B) = |F \cap B| + 2|(E \setminus F) \cap B|$ is even for every cut B). Note that G has no cut edge (as $p(e) = 0$ for every cut edge). By applying the four-colour theorem we obtain that the edge set of G can be covered by two cycles C_1 and C_2. Hence, the three cycles C_1, C_2 and $C_3 := C_1 \triangle C_2$ cover every edge of G exactly twice. This implies that $p = \sum_{i=1,2,3} \chi^{F \triangle C_i}$, contradicting our assumption.

We now proceed to show that $p(e) = 1, 2$ for all edges $e \in E$. We start with some observations. A cut B in G is said to be *tight* if $p(e) = p(B \setminus \{e\})$ for some edge $e \in B$. A *trivial* cut is a cut of the form $\delta_G(S)$ where $|S| = 1, |V| - 1$.

(a) $p(e) \neq 0$ *for all* $e \in E$. (Indeed, if $p(e) = 0$ then $(G \backslash e, p_{|E \setminus \{e\}})$ would be a smaller counterexample.)

(b) *Every tight cut is trivial.*
For suppose that $\delta_G(S)$ is a nontrivial tight cut. As $p > 0$ this cut is a minimal cut, i.e., the subgraphs of G induced by S and $V \setminus S$ are both connected. Let G_1 denote the graph obtained by contracting all nodes of $V \setminus S$ to a single node; similarly, G_2 is obtained by contracting S to a single node. Let p_i denote the projection of p on the edge set of G_i. Then, $x_i \in Q(\mathcal{M}(G_i))$ for $i = 1, 2$. By minimality of $n(G,p)$ we deduce that p_i admits a circuit cover in G_i. It is then easy to construct a circuit cover for p. (See the proof of Theorem 1.4.5 where a similar argument has already been used.)

(c) *There is no vertex $v \in V$ for which $p(e) = 1$ for all edges $e \in \delta_G(v)$.*
For suppose v is such a vertex. Let $e_1 := vw_1$ and $e_2 := vw_2$ be two edges adjacent to v that occur consecutively when looking at the star $\delta_G(v)$ in a plane embedding of G. Let G' denote the graph obtained from G by removing the edges e_1 and e_2 and adding a new edge $e_0 := w_1w_2$. (This operation is known as the *splitting off* operation.) Then, G' is again planar. Define p' on the edge set of G' by setting $p'(e_0) := 1$ and $p'(e) := p(e)$ for all other edges. We claim that (G', p') is admissible. It is clear that $p'(B')$ is even for every cut B' of G'. Suppose that $p' \notin \mathbb{R}_+(\mathcal{M}(G'))$ and let B' be a cut of G', $e \in B'$ such that $2p'(e) > p'(B')$. Then, $2p'(e) \geq p'(B') + 2$ as $p'(B')$ is even. This implies that $p'(e) \geq 2$ and, thus, $e \neq e_0$. Say, $B' = \delta_{G'}(S)$ where S is a subset of V containing v. Let $B := \delta_G(S)$ be the corresponding cut in G. Note that

$$B = \begin{cases} B' \cup \{e_1, e_2\} & \text{if } w_1, w_2 \notin S, \\ B' & \text{if } w_1, w_2 \in S, \\ B' \setminus \{e_0\} \cup \{e_1\} & \text{if } w_1 \notin S, w_2 \in S, \\ B' \setminus \{e_0\} \cup \{e_2\} & \text{if } w_1 \in S, w_2 \notin S. \end{cases}$$

Hence, $p(B) = p'(B')$ except $p(B) = p'(B') + 2$ in the case when $w_1, w_2 \notin S$. We have the inequalities:

$$p(B) \leq p'(B') + 2,\ 2p(e) = 2p'(e) \leq p(B),\ p'(B') + 2 \leq 2p'(e).$$

Hence equality holds throughout. In particular, B is a tight cut. Hence, by (b), B is trivial. This implies that $S = \{v\}$. Therefore, the edge e is adjacent to v, implying that $p(e) = 1$, in contradiction with the above observation that $p(e) \geq 2$.

(d) *Let $e_0 := xy$ be an edge with $p(e_0) \geq 2$ and such that $p(e_0)$ is minimum. Set $p_0 := p - 2\chi^{\{e_0\}}$. Then, p_0 has a circuit cover.*
To see it, add an edge e_1 in parallel with e_0 thus creating a new planar graph G'. Define p' on G' by setting $p'(e_0) = p(e_0) - 1$, $p'(e_1) = 1$ and $p'(e) = p(e)$ on all other edges. Then, (G', p') is admissible. As $n(G', p') < n(G, p)$, we deduce that p' has a circuit cover L'. If L' would not use the circuit $C_0 := \{e_0, e_1\}$ one would easily obtain from it a circuit cover for p. Hence, L' uses C_0. By replacing e_1 by e_0 in every circuit $C \neq C_0$ of L' we obtain a list of circuits of G covering e_0 $p(e_0) - 2$ times and every other edge e exactly $p(e)$ times. Therefore, p_0 has a circuit cover.

Let L be a circuit cover of p_0; write L as $L = L_1 \cup L_2$, where the circuits of L_1 do not contain the edge e_0 while those of E_2 do. We construct an auxiliary directed graph G_L in the following way. Its node set is the node set V of G. For distinct nodes $u, v \in V$, there is an arc $u \to v$ in G_L if, either u, v lie on a common circuit of L_1, or there is a circuit of L_2 which passes through u, y, x, v in that order (or the reverse).

(e) *There is a directed path from x to y in the digraph G_L.*
Indeed, let U denote the set of nodes $u \in V$ for which there exists a directed path from x to u in G_L. Hence, $x \in U$. Suppose that $y \notin U$. Then, every circuit of L_1 is fully contained in U or in $V \setminus U$. Moreover, a circuit of L_2 contains besides e_0 exactly one more edge which is cut by the partition $(U, V \setminus U)$. Therefore, every circuit of L satisfies the equality: $x(e_0) = x(\delta_G(U) \setminus \{e_0\})$. Hence, the vector p_0 too satisfies this equality, which implies that $p(e_0) - 2 = p(\delta_G(U) \setminus \{e_0\})$, contradicting the fact that $p \in \mathbb{R}_+(\mathcal{M}(G))$.

Let $x =: v_0 \to v_1 \to v_2 \to \dots v_{k-1} \to v_k := y$ be a directed path from x to y in G_L of shortest length k. Each arc $v_{i-1} \to v_i$ can be labeled by a circuit C_i of L. We obtain a list $L_{xy} := (C_1, \dots, C_k)$ of circuits which is contained in L. Let M consist of the distinct members from L_{xy} and set $q_M := \sum_{C \in M} \chi^C$ and $q := q_M + 2\chi^{\{e_0\}}$.

We claim that (G, q) is admissible. It is clear that $q \in \mathbb{Z}(\mathcal{M}(G))$. Suppose that $q \notin \mathbb{R}_+(\mathcal{M}(G))$ and let B be a cut in G, $e \in B$ such that $2q(e) \geq q(B) + 2$. As $q_M \in \mathbb{R}_+(\mathcal{M}(G))$, we deduce that $e = e_0$ and $q_M(e_0) = q_M(B \setminus \{e_0\})$. Say $B := \delta_G(S)$ where S is a subset of V with $x \in S$, $y \notin S$. Then, every circuit C of M either is disjoint from B, or meets B in two edges one of them being e_0. As $x \in S$, $y \notin S$, there is an index $i = 1, \dots, k-1$ such that $v_{i-1} \in S$, $v_i \notin S$. Hence, the corresponding circuit C_i meets $\delta_G(S)$, which implies that $C_i \in L_2$. As the nodes v_{i-1}, y, x, v_i occur in that order along C_i we deduce that C_i meets $\delta_G(S)$ in at least four edges, yielding a contradiction.

If $n(G, q) < n(G, p)$ then q has a circuit cover given, say, by a list N of circuits. Then, $N \cup (L \setminus M)$ is a list of circuits providing a circuit cover for p. Hence, $n(G, q) = n(G, p)$, which implies that $p = q$. Therefore, $M = L_{xy} = L$, i.e., the circuits $C_1, \dots, C_k$ are pairwise distinct and $L = \{C_1, \dots, C_k\}$.

Next, we observe that every node v_i, $i = 1, \dots, k-1$, belongs to at most two circuits of L_1, namely, C_i and C_{i+1}. (Indeed, if $C \in L_1$ contains v_i and v_j with $j > i$, then $i \to j$ is an arc in G_L. Hence, $j = i + 1$ by minimality of the length k of an xy-path in G_L.) Therefore, $\sum_{C \in L_1} |C \cap \delta_G(v_i)| \leq 4$ for $i = 1, \dots, k-1$. On the other hand, L_2 counts $p(e_0) - 2$ circuits, which yields: $\sum_{C \in L_2} |C \cap \delta_G(v_i)| \leq 2(p(e_0) - 2)$. Therefore, by summing the above two relations, we obtain that

$$p(\delta_G(v_i)) \leq 2(p(e_0) - 2) + 4 = 2p(e_0)$$

(as $e_0 \notin \delta_G(v_i)$ for $i = 1, \dots, k-1$). By (b), there exists an edge $e \in \delta(v_i)$ such that $p(e) > 1$. By the choice of e_0, $p(e) \geq p(e_0)$. As $p \in \mathbb{R}_+(\mathcal{M}(G))$, we have $2p(e) \leq p(\delta_G(v_i))$. Therefore, $p(\delta_G(v_i)) = 2p(e) = 2p(e_0)$ and equality holds in the inequalities above. In particular, every node v_i belongs to two circuits of L_1. This shows that all circuits $C_1, \dots, C_k$ belong to L_1 and, thus, $L_2 = \emptyset$. Hence, $p(e_0) = 2$.

Finally, note that every node $v \in V$ belongs to at most two circuits of $L = L_1$. (Indeed, if v belongs to C_i and C_j with $i < j$, then G_L contains the arcs $v_{i-1} \to v$ and $v \to v_j$, which implies that $j = i + 1$ by minimality of the length k of an xy-path in G_L.) This shows that $p(e) \leq 2$ for all edges $e \in E$, thus concluding the proof. ∎

The following result gives a complete characterization of the binary matroids with the circuit cover property, thus settling a conjecture of Seymour [Sey81a]; it was shown by Fu and Goddyn [FG95].

THEOREM 2.5.5. [FG95] *A binary matroid $\mathcal{M}$ has the circuit cover property if and only if $\mathcal{M}$ does not have F_7^*, $\mathcal{M}^*(K_5)$, R_{10}, or $\mathcal{M}(P_{10})$ as a minor.*

The proof we give below is very similar to the proof given earlier for the characterization of the sums of circuits property. In a first step, we give some results showing how the Hilbert basis property behaves under taking minors; Proposition 2.5.6 is easy and Proposition 2.5.7 is a direct extension of a result from [Lau95] given in the context of cographic matroids. The behaviour of the circuit cover property under taking minors follows as an application.

PROPOSITION 2.5.6. *Let $\mathcal{M}$ be a binary matroid on E and let $e \in E$. If $\mathcal{M}$ has the Hilbert basis property then so does $\mathcal{M}\backslash e$.* ∎

PROPOSITION 2.5.7. *Let $\mathcal{M}$ be a binary matroid on E and let $e \in E$. Suppose that $\mathcal{M}$ does not have F_7^* as a minor and that $\mathcal{M}$ satisfies the following two conditions:*

(i) *$v(C) \in 2\mathbb{Z}$ for every cycle C of $\mathcal{M}$,*

(ii) *$v_e \in \{0, \pm 1\}$*

for every $v \in \mathbb{R}^E$ for which the inequality $v^T x \leq 0$ defines a facet of the cone $\mathbb{R}_+(\mathcal{M})$. If $\mathcal{M}$ has the Hilbert basis property then so does $\mathcal{M}/e$.

PROOF. Let $x \in \mathbb{Z}(\mathcal{M}/e) \cap \mathbb{R}_+(\mathcal{M}/e)$; we show that $x \in \mathbb{Z}_+(\mathcal{M}/e)$. For this we extend x to a vector $y \in \mathbb{R}^E$ by setting $y_e := \alpha$, where α is determined below. Clearly, $y \in \mathbb{R}_+(\mathcal{M})$ if and only if

$$\alpha_1 := \max(\sum_{f \in E, f \neq e} v_f x_f \mid v_e = -1) \leq \alpha \leq \alpha_2 := \min(-\sum_{f \in E, f \neq e} v_f x_f \mid v_e = 1)$$

where, in the maximum and in the minimum, v runs over the vectors for which the inequality $v^T z \leq 0$ defines a facet of $\mathbb{R}_+(\mathcal{M})$. By the assumption that $\mathcal{M}$ has no F_7^* and from Theorem 2.4.10, we know that membership in the lattice $\mathbb{Z}(\mathcal{M})$ is ensured by the parity condition (2.4.2) and the same holds for the minors of $\mathcal{M}$. Hence, $y \in \mathbb{Z}(\mathcal{M})$ if and only if α and $x(D_0 \setminus \{e\})$ have the same parity, where D_0 is any given cocycle of $\mathcal{M}$ containing e. Let us take $\alpha := \alpha_1$; we verify that it has the correct parity. Say, $\alpha = \alpha_1 = \sum_{f \in E, f \neq e} v_f x_f$. Consider the vector $z \in \mathbb{R}^E$ extending x by setting $z_e := 1$ if $x(D_0 \setminus \{e\})$ is odd and $z_e := 0$ otherwise. Then, one can easily check that $z \in \mathbb{Z}(\mathcal{M})$. By the assumption (i), we deduce that $v^T z \in 2\mathbb{Z}$. Therefore, $\alpha = \sum_{f \in E, f \neq e} v_f x_f \equiv z_e \equiv x(D_0 \setminus \{e\})$ (modulo 2). ∎

COROLLARY 2.5.8. *The circuit cover property is preserved under taking minors.*

PROOF. The result follows from Propositions 2.5.6 and 2.5.7, after observing that the conditions of Proposition 2.5.7 are fulfilled as $\mathcal{M}$ has the sums of circuits property. ∎

PROPOSITION 2.5.9. *Let $\mathcal{M}$ be the k-sum of the binary matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, for $k = 1, 2, 3$. If both $\mathcal{M}_1^*$ and $\mathcal{M}_2^*$ have the circuit cover property then so does $\mathcal{M}^*$.*

PROOF. The proof is analogous to that of Proposition 2.2.7 for the sum of circuits property. We keep the same notation and give only the main lines. Let $\mathcal{N} := \mathcal{M}_1 \diamond \mathcal{M}_2$ denote the clique k-sum of $\mathcal{M}_1$ and $\mathcal{M}_2$. In view of Corollary 2.5.8, it suffices to show that $\mathcal{N}^*$ has the circuit cover property. Let $x \in Q(\mathcal{N}^*)$; we show that $x \in \mathbb{Z}_+(\mathcal{N}^*)$. The projection x_i of x on $\mathbb{R}^{E_i}$ belongs to $Q(\mathcal{M}_i^*) = \mathbb{Z}_+(\mathcal{M}_i^*)$. Hence,

$$x_i = \sum_{j \in J^{(i)}} \chi^{D_j^{(i)}}$$

where the $D_j^{(i)}$ are (not necessarily distinct) cocycles of $\mathcal{M}_i$, for $i = 1, 2$. (Hence, we are in the same situation as in the proof of Proposition 2.2.7 if we let there all $\alpha_j^{(i)}$'s be equal to 1.) In the case $k = 1$, we consider the same partition of $J^{(i)}$ into $J_0^{(i)} \cup J_1^{(i)}$. As $|J_1^{(1)}| = |J_1^{(2)}| =: m$ we can write $J_1^{(i)} := \{j_h^{(i)} \mid h = 1, \ldots, m\}$ for $i = 1, 2$ and we can then pair the cocycles indexed by these two sets. This yields the following decomposition of x:

$$x = \sum_{1 \le h \le m} \chi^{D_h^{(1)} \cup D_h^{(2)}} + \sum_{i=1,2} \sum_{j \in J_0^{(i)}} \chi^{D_j^{(i)}}.$$

In the case $k = 3$, let the sets $J_0^{(i)}$ and $J_{ab}^{(i)}$ be defined in the same way as in the proof of Proposition 2.2.7. Then, $|J_0^{(i)}| =: m_0$, $|J_{ab}^{(i)}| =: m_{ab}$ for $ab = 12, 13, 23$, and $i = 1, 2$. Again one obtains a decomposition of x by suitably pairing the cocycles, namely,

$$x = \sum_{1 \le h \le m_0} \chi^{D_h^{(1)} \cup D_h^{(2)}} + \sum_{ab=12,13,23} \sum_{1 \le h \le m_{ab}} \chi^{D_h^{(1)} \cup D_h^{(2)}}.$$

In all cases we have shown that $x \in \mathbb{Z}_+(\mathcal{N}^*)$. ∎

PROPOSITION 2.5.10. *The cographic matroid of the graph V_8 (recall Figure 1.5.9) and the Fano matroid F_7 both have the circuit cover property.*

PROOF. We show that F_7 has the circuit cover property. For this, suppose that there exists some vector $x \in \mathbb{Z}(F_7) \cap \mathbb{R}_+(F_7) \setminus \mathbb{Z}_+(F_7)$. Choose such x with $\sum_e x_e$ minimum. This implies that $x - \chi^C \notin \mathbb{R}_+(F_7)$ for every circuit C of F_7. It is easy to check that at most one component of x is equal to 0. Let e_1, e_2 be two elements of the groundset of F_7 for which x_e is largest. Let C be the unique Fano line containing e_1 and e_2. We claim that $y := x - \chi^C \in \mathbb{R}_+(F_7)$, which yields a contradiction. Indeed, suppose there exists a cocircuit D of F_7 and $e \in D$ such that $2y_e > y(D)$. Then, $2y_e \ge y(D) + 2$, i.e., $2x_e \ge x(D) + 2|C \cap \{e\}| + 2 - |C \cap D|$, as $y(D)$ is even. Therefore, $|C \cap D| = 2$, $e \notin C$, and $x_e = x(D \setminus e)$. This implies that $x_e > x(e_2)$, i.e., $e = e_1$, a contradiction.
The proof in the case of V_8 can be found in [FG95]; we do not present it here as it has

more involved technical details. ▮

Proof of Theorem 2.5.5. Necessity of the exclusions of F_7^*, $\mathcal{M}^*(K_5)$, R_{10}, and $\mathcal{M}(P_{10})$ as minors follows by applying Corollary 2.5.8. Conversely, if $\mathcal{M}$ is a binary matroid with no F_7^*, $\mathcal{M}^*(K_5)$, R_{10}, or $\mathcal{M}(P_{10})$ minor then, by applying Theorems 1.5.8 and 2.2.4, $\mathcal{M}^*$ can be obtained by means of $1,2,3$-sums from cographic matroids with no $\mathcal{M}^*(P_{10})$ minor, copies of $\mathcal{M}(V_8)$ and copies of F_7^*. Hence, that $\mathcal{M}^*$ has the circuit cover property follows by applying Theorem 2.5.4 and Propositions 2.5.9 and 2.5.10. ▮

The next question is to study the binary matroids with the Hilbert basis property, i.e., whose cycle space forms a Hilbert basis. Easy examples of such matroids include:

(i) The binary matroid $\mathcal{P}_r^*$ (the dual of the binary projective space of rank r). (This is obvious as the incidence vectors of the circuits of $\mathcal{P}_r^*$ are linearly independent).

(ii) All uniform matroids. (The uniform matroid of rank r on E has for circuits all subsets of E of cardinality $r+1$). (Given an integer t, let $\mathcal{C}$ denote the collection of subsets of size t of a groundset E. We show that $\mathcal{C}$ is a Hilbert basis. For this, let $x \in \mathbb{Z}(\mathcal{C}) \cap \mathbb{R}_+(\mathcal{C})$. Then, t divides $x(E)$ and $0 \leq x_e \leq \frac{x(E)}{t}$ for all $e \in E$. Say, $x(E) = kt$ for $k \geq 1$ and $x \in \{0, 1, \ldots, k\}^E$. We check that $x \in \mathbb{Z}_+(\mathcal{C})$ by induction on k. This is obvious if $k = 1$. Suppose $k \geq 2$ and set $E_h := \{e \in E \mid x_e = h\}$ for $h = 0, 1, \ldots, k$. Then, $kt = x(E) = \sum_h h|E_h|$. Let I be a subset of $E \setminus E_0$ that contains E_k. Then, $y := x - \chi^I \in \{0, 1, \ldots, k-1\}^E$ with $y(E) = (k-1)t$. Hence, $y \in \mathbb{Z}_+(\mathcal{C})$ by the induction assumption, which implies that $x \in \mathbb{Z}_+(\mathcal{C})$.)

However, the characterization of the binary matroids with the Hilbert basis property turns out to be a difficult problem. One basic difficulty when dealing with this problem is that one does not know the linear description of the cone $\mathbb{R}_+(\mathcal{M})$, in general. The typical case when we know the cone $\mathbb{R}_+(\mathcal{M})$ is when $\mathcal{M}$ has the sums of circuits property. But, then, the Hilbert basis property is equivalent to the circuit cover property and this case is completely settled by Theorem 2.5.5.

One more difficulty is that the Hilbert basis property is *not* closed under taking minors. (By Proposition 2.5.6 it is closed under deletion and, by Proposition 2.5.7, it is closed under contraction only under some additional conditions.) Indeed, the binary matroid $\mathcal{P}_r^*$ has the Hilbert basis property for all r. But, every binary matroid arises as a (contraction) minor of some $\mathcal{P}_r^*$. Examples of binary matroids that do not have the Hilbert basis property include:

(i) The graphic matroid $\mathcal{M}(P_{10})$ of the Petersen graph P_{10}.

(ii) The cographic matroid $\mathcal{M}^*(K_6)$ of the complete graph on six nodes. (In the latter case, the vector x taking values 2 on all edges of K_6 except value 4 on one edge does not belong to $\mathbb{Z}_+(\mathcal{M}^*(K_6))$; recall Section **??**.)

In fact, the characterization of the Hilbert basis property is still an open problem, already within cographic matroids. Deza [Dez61, Dez82] showed that $\mathcal{M}^*(K_5)$ has the Hilbert basis property. The following further results were shown by Laurent [Lau95].

Theorem 2.5.11.

(i) *For every proper subgraph G of K_6, its cographic matroid $\mathcal{M}^*(G)$ has the Hilbert basis property.*

(ii) *If $\mathcal{M}^*(G)$ has the Hilbert basis property then G does not have K_6 as a minor.* ∎

The main part in Theorem 2.5.11 (i) consists in proving that the cographic matroid of the graph $K_6\backslash e$ (obtained by deleting one edge from K_6) has the Hilbert basis property; the proof involves many technical details. Interestingly, the part (ii) is not proved by applying a result on minors, but by constructing a concrete example of a vector with no cut cover.

One of the tools used for the proof of Theorem 2.5.11 is the following result about clique sums; the proof is analogous to that of Proposition 2.5.9.

Proposition 2.5.12. *Let $\mathcal{N}$ be the clique k-sum of two binary matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, for $k = 1, 2, 3$. If both $\mathcal{M}_1^*$ and $\mathcal{M}_2^*$ have the Hilbert basis property then so does $\mathcal{N}^*$.* ∎

No other examples of graphs (not containing K_6 as a minor) are known whose cographic matroid does not have the Hilbert basis property. It is, for instance, an interesting open question to determine whether the cographic matroid of the graph from Figure **??** (the graph obtained by replacing a triangle in K_6 by a claw $K_{1,3}$) has the Hilbert basis property. In other words, it would be interesting to understand how the Hilbert basis property behaves under the ΔY-operations for graphs. (Recall Section **??**.)

Chapter 3

Cuts, Metrics and Multicommodity Flows

Cuts and metrics play a central role in the area of multicommodity flow problems. A first fundamental result in this area is the well known max-flow min-cut theorem of Ford and Fulkerson, which gives a min-max relation for the maximum amount of flow that can be sent in a graph between one pair of a source and sink nodes, in terms of the minimum capacity of a cut separating these two nodes. Moreover, such a maximum flow and minimum cut can be found in polynomial time. Therefore, the following problem:

The minimum weight cut problem. *Given a graph $G = (V, E)$ and nonnegative edge weights $c \in \mathbb{Q}_+^E$, find a nonempty cut $\delta_G(S)$ ($S \subseteq V$, $S \neq \emptyset, V$) whose weight $c(\delta_G(S))$ is minimum.*

can be solved in polynomial time. This contrasts with the fact that the maximum cut problem is an NP-hard problem (already for unit edge weights).

The general multicommodity flow problem asks for $k \geq 1$ flows between k pairs of source-sink nodes so as to satisfy some demand requirements without violating some capacity constraints. The metric cone plays a central role here; it is indeed in some sense dual to the cone of feasible multiflows (see Theorem 3.2.5). In particular, a necessary condition for the existence of a multiflow is the so-called cut condition; see Corollary 3.2.6.

In this chapter, we present some results on multicommodity flows that are most relevant to the theme of the book, namely, cuts and metrics. In particular, we treat questions of the type: When is the cut condition sufficient for the existence of a multiflow ? When is it sufficient for the existence of an integer multiflow, when taken together with a parity condition on cuts ? We also present the notions and results in the general framework of binary matroids .

3.1 The One Commodity Case

Let us start with the most basic case of the one commodity flow problem. Let $G = (V, A)$ be a directed graph and s, t be two distinct nodes in V. An $s - t$ **flow** is a function $x : A \longrightarrow \mathbb{R}_+$ satisfying the following condition:

$$(3.1.1) \qquad \sum_{a \in \delta_G^+(v)} x_a = \sum_{a \in \delta_G^-(v)} x_a \quad \text{for all } v \in V \setminus \{s, t\};$$

this is the **flow conservation law**. (For a subset $U \subseteq V$, $\delta_G^+(U)$ denotes the set of arcs leaving U while $\delta_G^-(U)$ denotes the set of arcs entering U.) The **value** of the flow x is the quantity

$$\mathrm{val}(x) := x(\delta_G^+(s)) - x(\delta_G^-(s))$$

We are also given a **capacity** function $c \in \mathbb{R}_+^A$ and a **demand** $d_{st} > 0$. The **feasibility flow problem** asks whether there exists a flow satisfying the capacity constraints:

(3.1.2) $\qquad x_a \le c_a$ for all $a \in A$

and with value $\mathrm{val}(x) \ge d_{st}$. The **maximum flow problem** consists of finding a flow x of maximum value satisfying (3.1.2). These problems are completely understood, as shows the following result, due to Ford and Fulkerson (see, e.g., [FF62]). The proof given below shows that both problems can be solved as linear programming problems (of size bounded by the size of the graph); combinatorial algorithms are known; see, e.g., [AMO93] for more information.

THEOREM 3.1.3. (The max-flow min-cut theorem) *Let $G = (V, A)$ be a directed graph, $r, s \in V$, and $c \in \mathbb{Q}_+^A$. The maximum value of an $s-t$ flow satisfying* (3.1.2) *is equal to* $\min(c(\delta_G^+(U)) \mid U \subseteq V, s \in U, t \notin U)$. *Moreover, if c is integer valued, then there exists an integer valued optimum $s-t$ flow.*

PROOF. We can suppose that $c \in \mathbb{Z}_+^A$. Let M denote the node-arc incidence matrix of G; its rows are indexed by V, its columns by A and its (v, a)-entry is equal to 1 if a leaves v, to -1 if a enters v and to 0 otherwise. One can easily check that M is totally unimodular.[1] Let M' denote the submatrix of M where we have deleted the rows indexed by s and t and let w denote the s-row of M. The maximum flow problem can be formulated as the linear program:

$$\max(w^T x \mid M'x = 0,\ 0 \le x \le c).$$

Its dual reads:

$$\min(y^T c \mid z^T M' + y^T I \ge w^T,\ y \ge 0).$$

Let x be a flow such that $x \le c$; then, its value can be expressed as $\mathrm{val}(x) = x(\delta_G^+(U)) - x(\delta_G^-(U)) \le c(\delta_G^+(U))$ for all $U \subseteq V$ with $s \in U$, $t \notin U$. This shows that max-flow $\le$ min-cut. To prove the converse inequality, take an optimum dual solution (y, z) that is integer valued (it exists as M' is totally unimodular) and set $U := \{u \in V \setminus \{s, t\} \mid z_u \le -1\} \cup \{s\}$. We claim that $c(\delta_G^+(U)) \le y^T c$, which implies that min-cut $\le$ max-flow. For this, it suffices to check that $y_a \ge 1$ for every arc $a \in \delta_G^+(U)$; this can be easily verified using the assumption that $z^T M' + y^T I \ge w^T$, $y \ge 0$. ∎

Every $s-t$ flow can be seen as a 'fractional' system of paths. Namely, x can be decomposed as $x = \sum_P \lambda_P \chi^P$ where $\lambda_P \ge 0$ and the P's are paths from s to t in G. Moreover, if x is integer valued then the λ's can be chosen integers. In particular, an $s-t$

[1] A matrix M is **totally unimodular** if every square subdeterminant of M is equal to 0,1, or -1. We use the following fact: If M is totally unimodular and w, b, c are integer vectors, then the linear program $\max(w^T x \mid Mx \le b, 0 \le x \le c)$ and its dual both have integer optimum solutions.

flow which is $0,1$-valued corresponds to a set of arc disjoint $s-t$ paths in G. The following result, due to Menger, follows immediately from Theorem 3.1.3 by taking $c \equiv 1$.

THEOREM 3.1.4. (Menger's theorem for directed graphs) *Let $G=(V,A)$ be a directed graph and $s,t \in V$. The maximum number of arc disjoint paths from s to t in G is equal to the minimum cardinality of an $s-t$ cut, i.e., a cut $\delta^+_G(U)$ with $s \in U, t \notin U$.* ▮

The above results have analogues for undirected graphs. Let $G=(V,E)$ be an undirected graph, $s,t \in V$ and $c \in \mathbb{Q}^E_+$. Let $\tilde{G}$ denote the directed graph obtained from G by replacing every edge uv by two opposite arcs (u,v) and (v,u). We now look for an $s-t$ flow x in $\tilde{G}$ such that $x(u,v)+x(v,u) \le c_e$ for all $e=uv \in E$. Equivalently, letting $\mathcal{P}_{st}$ denote the set of paths from s to t in G, an $s-t$ **flow** in G is simply a function $f : \mathcal{P}_{st} \longrightarrow \mathbb{R}_+$ (f_P is the amount of flow sent along the path P). The capacity constraints read:

$$\sum_{P \in \mathcal{P}_{st} \mid e \in P} f_P \le c_e \text{ for all } e \in E \tag{3.1.5}$$

and the value of the flow f is $\sum_{P \in \mathcal{P}_{st}} f_P$.

THEOREM 3.1.6. (Menger's theorem for undirected graphs) *Let $G=(V,E)$ be an undirected graph, $s,t \in V$ and $c \in \mathbb{Q}^E_+$. The maximum value of a flow satisfying* (3.1.5) *is equal to* $\min(c(\delta_G(U)) \mid U \subseteq V, s \in U, t \notin U)$. *If $c \in \mathbb{Z}^E_+$ then there exists an integer optimum flow. In particular, the maximum number of edge disjoint $s-t$ paths is equal to* $\min(|\delta_G(U)|,\ U \subseteq V, s \in U, t \notin U)$. ▮

3.2 Multicommodity Flows: Feasibility

We now introduce general multiflow problems. We first define classical multiflows in graphs and, then, extend the definitions to the more general setting of binary matroids.

From now on, all graphs will be supposed to be undirected. Let $G=(V,E)$ be a connected graph and let $(s_1,t_1),\ldots,(s_k,t_k)$ be $k \ge 1$ pairs of nodes in V. Set $R := \{(s_1,t_1),\ldots,(s_k,t_k)\}$ and $W := \{s_1,t_1,\ldots,s_k,t_k\}$; the graph $H := (W,R)$ is called the **demand graph** while G is called the **supply graph**. Each pair (s_i,t_i) is called a **commodity pair**, or **demand edge**, the nodes s_i,t_i being its **terminals**. We are also given a **capacity** function $c \in \mathbb{Q}^E_+$ and a **demand** $d_{st} > 0$ for each demand edge $st \in R$. Let $\mathcal{P}_{st}$ denote the set of paths from s to t in G and $\mathcal{P}_R := \bigcup_{st \in R} \mathcal{P}_{st}$. A **multiflow** (or, simply, flow) is a function $f : \mathcal{P}_R \longrightarrow \mathbb{R}_+$. The multiflow f is said to be **feasible for the instance** (G,H,c,d) if it satisfies the conditions:

$$\sum_{P \in \mathcal{P}_R \mid e \in P} f_P \le c_e \quad \text{for } e \in E, \tag{3.2.1}$$

$$(3.2.2) \qquad \sum_{P\in\mathcal{P}_{st}} f_P \ge d_{st} \quad \text{for } st \in R.$$

Clearly, if there exists a feasible flow then there exists one for which equality holds in (3.2.2). It is sometimes convenient to encode an instance (G, H, c, d) in a more compact form, as $(\mathcal{G}, p)$, where $\mathcal{G} := G + H$ is the graph obtained by taking the disjoint union of G and H (it may have parallel edges) and $p \in \mathbb{Q}^{E\cup R}$ is defined by $p_e := c_e$ if $e \in E$ and $p_e := -d_e$ if $e \in R$. Hence, the demand edges in $\mathcal{G}$ are those for which $p_e < 0$, all others being supply edges. One could also encode the same instance (G, H, c, d) as $(\mathcal{G}, R, q)$ where R is the set of demand edges and $q_e = c_e$ or d_e depending whether e is a supply or a demand edge.

Let $\mathcal{C}_R$ denote the collection of circuits C in $\mathcal{G}$ such that $|C \cap R| = 1$. The paths in $\mathcal{P}_R$ correspond to the members of $\mathcal{C}_R$. Hence, a multiflow can be alternatively described as a function indexed by $\mathcal{C}_R$.

Using the latter approach, one can define more generally multiflows in matroids. For this topic, we follow essentially the seminal paper [Sey81a] by Seymour.

Let $\mathcal{M}$ be a binary matroid on E and $p \in \mathbb{Q}^E$. Set $R := \{e \in E \mid p_e < 0\}$ and let $\mathcal{C}_R$ denote the collection of circuits C in $\mathcal{M}$ for which $|C \cap R| = 1$. So, $\mathcal{C} = \bigcup_{f\in R} \mathcal{C}_f$ where $\mathcal{C}_f$ consists of the circuits C in $\mathcal{M}$ for which $C \cap R = \{f\}$. Then, a **feasible multiflow** for the instance $(\mathcal{M}, p)$ is a function $f : \mathcal{C}_R \longrightarrow \mathbb{R}_+$ satisfying:

$$(3.2.3) \qquad \sum_{C\in\mathcal{C}_R | e\in C} f_C \begin{cases} \le p(e) & \text{if } e \in E \setminus R, \\ \ge -p(e) & \text{if } e \in R. \end{cases}$$

Hence, if $\mathcal{M}$ is the graphic matroid of the graph $\mathcal{G}$ we find the usual notion of a feasible multiflow for $(\mathcal{G}, p)$. Note that no feasible flow exists for $(\mathcal{M}, p)$ if $\mathcal{C}_f = \emptyset$ for some $f \in R$. We may assume that

$$(3.2.4) \qquad \mathcal{C}_f \ne \emptyset \quad \text{for every } f \in R$$

(or, equivalently, that the set R contains no cocircuit of $\mathcal{M}$). In the graphic case, (3.2.4) holds whenever the supply graph is connected.

The next result gives a criterion for the existence of a feasible multiflow. In the graphic case, it is known as the Japanese theorem as it is due to Iri [Iri71] and Onaga and Kakusho [OK71].

THEOREM 3.2.5. **(Japanese theorem)** *Let $\mathcal{M}$ be a binary matroid on E and $p \in \mathbb{Q}^E$. The following assertions are equivalent.*

(i) *There exists a feasible multiflow for $(\mathcal{M}, p)$.*

(ii) *$p^T x \ge 0$ for all $x \in \mathbb{R}^E_+$ such that $x(C\setminus\{f\}) - x_f \ge 0$ for all $C \in \mathcal{C}_R$ with $C \cap R = \{f\}$.*

(iii) **(The distance condition)** *$p^T x \ge 0$ for all $x \in \mathbb{R}^E_+$ such that $x(C \setminus \{e\}) - x_e \ge 0$ for all circuits C in $\mathcal{M}$ and $e \in C$, i.e., for all $x \in \mathrm{MET}(\mathcal{M}^*)$.*

PROOF. The equivalence (i) $\Longleftrightarrow$ (ii) follows by applying Farkas' lemma.[2] The implication (ii) $\Longrightarrow$ (iii) is obvious. We check (iii) $\Longrightarrow$ (ii). For this, let $x \in \mathbb{R}^E_+$ such that $x_f \le$

[2] We use the following version of Farkas' lemma (cf. [Sch86]): Given a matrix A and a vector c, there exists $x \ge 0$ such that $Ax \le c$ if and only if $y^T c \ge 0$ for all $y \ge 0$ satisfying $y^T A \ge 0$.

$x(C \setminus \{f\})$ for all circuits C with $C \cap R = \{f\}$. Define $y \in \mathbb{R}_+^E$ by setting

$$y_f := \min\{x(C \setminus \{f\}) \mid C \in \mathcal{C}_f\} \quad \text{for } f \in R,$$

$$y_e := \min\{x_e, x(C \setminus \{e\}) \mid C \text{ circuit in } \mathcal{M}, C \cap R = \emptyset, e \in C\} \quad \text{for } e \in E \setminus R.$$

We claim that $y \in \text{MET}(\mathcal{M}^*)$. For this, let $C := \{e_1, \ldots, e_k\}$ be a circuit in $\mathcal{M}$; we show that $y(e_1) \leq y(C \setminus \{e_1\})$. Suppose that $y(e_i) = x(e_i)$ for $i = 2, \ldots, s$ (where $1 \leq s \leq k$) and $y(e_i) = x(C_i \setminus \{e_i\})$ for $i = s+1, \ldots, k$, where C_i is a circuit contained in $E \setminus R$ and containing e_i. So, $e_2, \ldots, e_s \in E \setminus R$. If e_1 belongs to some C_i then $y(C \setminus \{e_1\}) \geq y(e_i) = x(C_i \setminus \{e_i\}) \geq x(e_1) \geq y(e_1)$. Hence, we can suppose that $e_1 \in C^* := C \triangle C_{s+1} \triangle \ldots \triangle C_k$. As $C^* \setminus \{e_1\} \subseteq \{e_2, \ldots, e_s\} \cup \bigcup_{s+1 \leq i \leq k} C_i \setminus \{e_i\}$, we obtain that

$$\begin{aligned} y(C \setminus \{e_1\}) &= x(e_2) + \ldots + x(e_s) + x(C_{s+1} \setminus \{e_{s+1}\}) + \ldots + x(C_k \setminus \{e_k\}) \\ &\geq x(C^* \setminus \{e_1\}) \geq y(e_1); \end{aligned}$$

the latter inequality follows from the definition of $y(e_1)$ as $C^* \setminus \{e_1\} \subseteq E \setminus R$. By (iii) applied to the element $y \in \text{MET}(\mathcal{M}^*)$, $p^T y \geq 0$. This implies that $p^T x \geq 0$, as $p^T x \geq p^T y$ because $x_e \geq y_e$ for $e \in E \setminus R$ and $x_f \leq y_f$ for $f \in R$. ∎

Corollary 3.2.6. *Let $\mathcal{M}$ be a binary matroid and $p \in \mathbb{Q}^E$. If there exists a feasible multiflow for the instance $(\mathcal{M}, p)$ then the following condition holds:*

(3.2.7) $\quad p(D) \geq 0$ *for all cocircuits (or cocycles) D in $\mathcal{M}$.*

Proof. Apply Theorem 3.2.5 (iii) with the vector $x = \chi^D$. ∎

In the case when $\mathcal{M}$ is the graphic matroid of the graph $\mathcal{G} = (V, E)$, then the cocircuits in $\mathcal{M}$ are the cuts of $\mathcal{G}$ and (3.2.7) is known as the **cut condition**; we keep the same terminology for arbitrary matroids. Moreover, the condition (iii) in Theorem 3.2.5 reads: $p^T x \geq 0$ for all $x \in \text{MET}(\mathcal{G})$ or, equivalently, for all metrics x on V (extending p to the complete graph on V by zero components). By analogy, the elements of the cone $\text{MET}(\mathcal{M})^*$ are also called **metrics on** $\mathcal{M}$; hence, the terminology used for the 'distance condition'.

When the multiflow instance consists of $(\mathcal{M}, R, p)$, where $R \subseteq E$ is the demand set and $p \in \mathbb{Z}_+^E$ (with p_e being viewed as a demand for $e \in R$ and as a capacity for $e \in E \setminus R$), the cut condition can be reformulated as

(3.2.8) $\quad p(D \setminus R) - p(D \cap R) \geq 0$ for all cocycles D in $\mathcal{M}$.

Another condition which will play an essential role for the existence of *integer* multiflows is the following **parity condition**:

(3.2.9) $\quad p(D) \in 2\mathbb{Z}$ for all cocircuits D in $\mathcal{M}$.

We now formulate a result about metrics on $\mathcal{M}$ that we will need later; it was already proved within the proof of Theorem 3.2.5. Given $\ell \in \mathbb{Z}_+^E$, define $m := dist(\mathcal{M}, \ell) \in \mathbb{Z}_+^E$ by

$$m_e := \min\{\ell_e, \ell(C \setminus \{e\}) \text{ for } C \text{ circuit in } \mathcal{M} \text{ with } e \in C\} \text{ for } e \in E.$$

(This generalizes the shortest path metrics in graphs.)

LEMMA 3.2.10. *For a matroid $\mathcal{M}$ and $\ell \in \mathbb{Z}_+^E$, $dist(\mathcal{M}, \ell) \in \mathrm{MET}(\mathcal{M}^*)$.* ∎

EXAMPLE 3.2.11. Consider the graphs K_4 and K_5 with the demand edges as indicated in Figure 3.2.12; capacities and demands are all taken equal to 1. Then, there exists a fractional flow for the given instance on K_4, but no integer flow exists (as one cannot find two disjoint paths joining the endnodes of each of the two commodity pairs). On the other hand, there exists *no* flow for the instance on K_5. Indeed, the distance condition from the Japanese theorem is violated (by the metric taking value 2 on the demand edges and value 1 on the supply edges; this is the path metric of $K_{2,3}$). Note, however, that the cut condition is satisfied. (Indeed, setting $p_e = -1$ on the demand edges and $p_e = 1$ on the supply edges, the inequality $-p^T x \leq 0$ is nothing but the pentagonal inequality ! Hence, it is satisfied by all cuts of K_5.) ∎

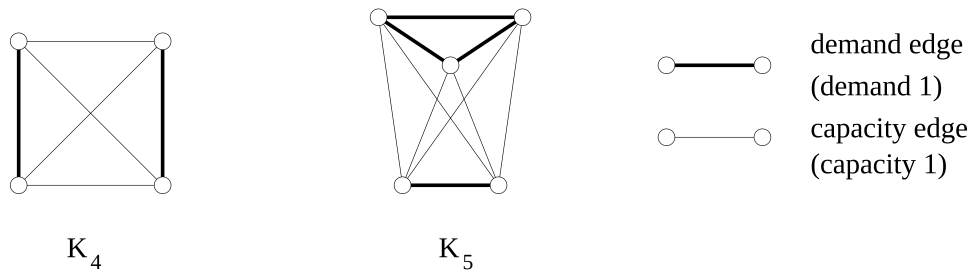


Figure 3.2.12

By the Japanese theorem, checking the existence of a feasible multiflow for the instance $(\mathcal{M}, p)$ amounts to verifying whether

$$\min\{p^T x \mid x \in \mathrm{MET}(\mathcal{M}^*)\} \geq 0.$$

As mentioned in Section 2.2, this can be done in polynomial time for several classes of binary matroids, in particular, for the following classes: $\mathcal{M}$ with no F_7 minor, or $\mathcal{M}$ with no F_7^* minor. When $\mathcal{M}$ is the graphic matroid of the graph $\mathcal{G}$ on n nodes, the condition (iii) of the Japanese theorem can be rewritten as: $\min(p^T x \mid x \in \mathrm{MET}_n) \geq 0$, after extending p to the complete graph K_n by zero components; hence, this is a simple linear programming problem. When $\mathcal{M}$ is cographic, then checking (iii) amounts to solving a minimum weight T-join problem; see Section 3.3.

The distance condition requires checking whether $p^T x \geq 0$ holds for *all* metrics. What happens if we suppose that this inequality holds only when x belongs to a restricted set of metrics ? Can this sometimes be already sufficient for ensuring the existence of a flow ? If we restrict ourselves to considering only (0,1)-valued metrics (i.e., to the metrics $x = \chi^D$ for some cocircuit in $\mathcal{M}$, in view of Lemma 2.2.2) then we have the question of determining for which instances the cut condition is sufficient; it will be treated in detail in the next subsections. Schwärzler and Sebö [SS93] consider the case when the distance condition holds only for all $(0, 1, 2)$-valued metrics; see Theorem **??**.

3.3 Multicommodity Flows: The Cographic Case

We consider here multicommodity flow problems in cographic matroids. They are, in a sense, completely understood and, in fact, equivalent to T-join problems. This was first observed by Seymour [Sey81b].

Let $\mathcal{G} = (V, E)$ be a graph, $R \subseteq E$, and $p \in \mathbb{Z}_+^E$. A multiflow for $\mathcal{M}^*(\mathcal{G})$ is a function $f = (f_C)_{C \in \mathcal{C}_R}$ indexed by $\mathcal{C}_R$, the set of cuts in $\mathcal{G}$ that contain exactly one edge from R. The distance condition reads:

$$\sum_{e \in E \setminus R} p_e x_e - \sum_{e \in R} p_e x_e \geq 0$$

for all $x \in \mathrm{MET}(\mathcal{M}(\mathcal{G})) = \{x \in \mathbb{R}_+^E \mid x(\delta_{\mathcal{G}}(S) \setminus \{e\}) - x_e \geq 0 \text{ for all } S \subseteq V, e \in C\}$. As $\mathrm{MET}(\mathcal{M}(\mathcal{G})) = \mathrm{CYC}(\mathcal{M}(\mathcal{G}))(= \mathrm{EUL}(\mathcal{G}))$ (by Theorem 1.4.5), the distance condition is, in fact, equivalent to the 'cut' condition:

(3.3.1) $\quad x(C \setminus R) - x(C \cap R) \geq 0$ for all cycles C in $\mathcal{G}$.

In other words - in the terminology of the next Section 3.5 - all cographic matroids are ∞-flowing. In fact, they are also ∞-cycling, which means the following: If p satisfies (3.3.1) and the parity condition:

(3.3.2) $\quad p(C) \in 2\mathbb{Z}$ for all cycles C in $\mathcal{G}$,

then there exists an integer flow for $(\mathcal{M}^*(\mathcal{G}), R, p)$. This fact follows from Theorem 1.3.5 (which states that, in a bipartite graph, the minimum cardinality of a T-join is equal to the maximum number of disjoint T-cuts).

PROPOSITION 3.3.3. *Let $\mathcal{G} = (V, E)$ be a bipartite graph, $R \subseteq E$, and T be the set of nodes from V having an odd degree in R. The following assertions are equivalent.*

(i) *There exists a list $\{D_f \mid f \in R\}$ where, for each $f \in R$, D_f is a T-cut in $\mathcal{G}$ such that $D_f \cap R = \{f\}$, and the D_f's are pairwise disjoint.*

(ii) *$|C \setminus R| \geq |C \cap R|$ for all cycles C in $\mathcal{G}$.*

PROOF. (i) $\Longrightarrow$ (ii) is obvious. We now check (ii) $\Longrightarrow$ (i). By definition of T, the set R is a T-join. Moreover, R is a T-join of minimum cardinality as, by (ii), $|R \triangle C| \geq |R|$ for all cycles C. Since $\mathcal{G}$ is bipartite, we know from Theorem 1.3.5 that there exists $|R|$ pairwise disjoint T-cuts: $D_1, \ldots, D_{|R|}$ in $\mathcal{G}$. Hence, $|R| \geq \sum_{1 \leq i \leq |R|} |D_i \cap R| \geq |R|$, as $|D_i \cap R| \geq 1$ for all i since a T-join and a T-cut have a nonempty intersection. Therefore, equality holds throughout, which shows that $|D_i \cap R| = 1$ for all i, thus proving (i). ∎

Theorem 3.3.4. *Let $\mathcal{G} = (V, E)$ be a graph, $R \subseteq E$, and $p \in \mathbb{Z}_+^E$ satisfying* (3.3.1) *and* (3.3.2). *Then, there exists an integer feasible multiflow for $(\mathcal{M}^*(\mathcal{G}), R, p)$. Therefore, if $p \in \mathbb{Z}_+^E$ satisfies* (3.3.1), *then there exists a half-integral feasible multiflow for $(\mathcal{M}^*(\mathcal{G}), R, p)$.*

Proof. Let T denote the set of nodes in V having an odd degree in R. Suppose $p \in \mathbb{Z}_+^E$ satifies (3.3.1) and (3.3.2). Let $\mathcal{G}'$ denote the graph obtained from $\mathcal{G}$ by replacing every edge e by a path P_e consisting of p_e edges (and contracting an edge with $p_e = 0$). Then, $\mathcal{G}'$ is bipartite. Set $R' := \cup_{e \in R} P_e$; then R' is a T-join in $\mathcal{G}'$. Applying Proposition 3.3.3 to $(\mathcal{G}', R')$, we can find a list $D_1, \ldots, D_{p(R)}$ of T-cuts in $\mathcal{G}$ covering every edge $e \in E \setminus R$ at most p_e times and every edge $e \in R$ exactly p_e times. Hence, $p(R) = \sum_i |D_i \cap R| \geq p(R)$ (as a T-join and a T-cut meet) which implies that $|D_i \cap R| = 1$ for all i. Therefore, the list $D_1, \ldots, D_{p(R)}$ provides a feasible integer multiflow for $(\mathcal{M}^*(\mathcal{G}), R, p)$. ∎

From an algorithmic point of view, one can find such a flow using any primal-dual algorithm for the minimum weight T-join problem. Indeed, consider the following pair of dual linear programming problems:

$$\begin{array}{lll} \min & p^T x & = \\ \text{s.t.} & x(D) \geq 1 \ (D \ T\text{-cut}) & \text{s.t.} \\ & x_e \geq 0 \ (e \in E) & \end{array} \quad \begin{array}{ll} \max & \sum_{D \ T\text{-cut}} y_D \\ & \sum_{D | e \in D} y_D \leq p_e \ (e \in E) \\ & y_D \geq 0 \ (D \ T\text{-cut}) \end{array}$$

The minimization program has an integer optimum solution for every $p \in \mathbb{Z}_+^E$ (by Theorem 1.4.1); under (3.3.1), χ^R is such an optimum solution. Moreover, under (3.3.2), the maximization program too has an integer optimum solution (by Theorem 1.4.4). If (3.3.1) holds and $y = (y_D)$ is an optimum solution to the maximization program then, by the complementary slackness conditions,

$$e \in R \Longrightarrow \sum_{D | e \in D} y_D = p_e, \ y_D > 0 \Longrightarrow |D \cap R| = 1,$$

i.e., y provides a feasible multiflow for $(\mathcal{M}^*(\mathcal{G}), R, p)$.

3.4 Multicommodity Flows in Graphs

We consider in this section the classical multicommodity flow problems in graphs (i.e., for graphic matroids). We are given a supply graph $G = (V, E)$ with capacities c_e $(e \in E)$ on its edges and a demand graph $H = (W, R)$ with demands d_{st} $(st \in R)$ on its edges. A feasible flow for the instance (G, H, c, d) is a function $f : \mathcal{P}_R = \bigcup_{st \in R} \mathcal{P}_{st} \longrightarrow \mathbb{Q}_+$ such that

$$\begin{array}{ll} \sum_{P \in \mathcal{P}_R | e \in P} f_P \leq c_e & \text{for } e \in E, \\ \sum_{P \in \mathcal{P}_{st}} f_P = d_{st} & \text{for } st \in R. \end{array}$$

(We remind that $\mathcal{P}_{st}$ denotes the set of paths from s to t in G.)

As was seen above, a necessary condition for the existence of a feasible multiflow for the instance (G, H, c, d) is the following cut condition:

(3.4.1) $$c(\delta_G(U)) \geq d(\delta_H(U)) \quad \text{for all } U \subseteq V.$$

The following parity condition plays an essential role for the existence of an integer fractional flow:

(3.4.2) $$c(\delta_G(U)) + d(\delta_H(U)) \in 2\mathbb{Z} \quad \text{for all } U \subseteq V.$$

We present the following results.

THEOREM 3.4.3. **(Lomonosov's theorem)**[Lom79] *Suppose that H is K_4, C_5, or a union of two stars. If (c,d) satisfies the cut condition* (3.4.1) *and the parity condition* (3.4.2)*, then there exists an integer feasible multiflow for (G,H,c,d).*

THEOREM 3.4.4. **(Karzanov's theorem)**[Kar87] *Suppose that H is K_5. If (c,d) satisfies the cut condition* (3.4.1) *and the parity condition* (3.4.2)*, then there exists an integer feasible multiflow for (G,H,c,d).*

THEOREM 3.4.5. **(Okamura's theorem)** [Oka83] *Suppose that G is planar and that there exist two faces I and O of G such that, for each pair $(s,t) \in R$, either $s,t \in I$, or $s,t \in O$. If (c,d) satisfies the cut condition* (3.4.1) *and the parity condition* (3.4.2)*, then there exists an integer feasible multiflow for (G,H,c,d).*

This extends an earlier result by Okamura and Seymour [OS81], where the terminal pairs were restricted to lie within *one* face of G. The next result follows as an immediate consequence (using graph duality) from Theorem 3.3.4.

THEOREM 3.4.6. **(Seymour's theorem)** *Suppose that $G + H$ is planar. If (c,d) satisfies the cut condition* (3.4.1) *and the parity condition* (3.4.2)*, then there exists an integer feasible multiflow for (G,H,c,d).*

For each of the above results, a dual theorem has been proved, in terms of packings of cuts.

THEOREM 3.4.7. **(Karzanov's theorem)**[Kar85] *Suppose that $H = K_4$, or C_5, or a union of two stars. Let $\ell \in \mathbb{Z}_+^E$ such that $\ell(C) \in 2\mathbb{Z}$ for all circuits C in G. Then, there exist $X_1, \ldots, X_t \subseteq V$ such that*

$$\begin{array}{ll} \sum_{1 \leq j \leq t} \chi_e^{\delta_G(X_j)} \leq \ell_e & \text{for all } e \in E, \\ \sum_{1 \leq j \leq t} \chi_{st}^{\delta_H(X_j)} = d_{(G,\ell)}(s,t) & \text{for all } st \in R. \end{array}$$

The same result holds under the assumptions of Okamura's theorem (as shown by Schrijver [Sch89]) as well as under those of Seymour's theorem (as an easy corollary of Theorem 2.5.2).

3.5 Multicommodity Flows in Matroids

We investigate now multicommodity flows in general binary matroids. We will consider, in particular, the following questions: When is the cut condition sufficient for ensuring the existence of a (fractional) flow ? When are the cut condition together with the parity condition sufficient for the existence of an integer flow ? When does the existence of a fractional flow imply the existence of an integer one (assuming or not the parity condition) ? (This will lead us, respectively, to the notions of flowing, cycling, and routing matroids; see below.) We have seen in the previous subsections several concrete examples of binary matroids (graphic or cographic) that have a positive answer for some of these questions. We present here the precise definitions for these several matroidal properties; we show the interconnections between the various notions, together with several results. Our exposition here follows essentially the papers by Seymour [Sey81a] and Sebö [Seb90].

3.5.1 When is the Cut Condition Sufficient ?

We introduce the notions of R-flowing and R-cycling matroids; they basically correspond to the case when the cut condition (resp. the cut and parity conditions taken together) are sufficient for ensuring the existence of a feasible multiflow (resp. of an integer multiflow), for a given demand set R. For convenience, we now consider an instance $(\mathcal{M}, R, p)$, where $\mathcal{M}$ is a binary matroid on E, $R \subseteq E$ is the 'demand set', $p \in \mathbb{Z}_+^E$ with p_e being viewed as a 'capacity' for $e \in E \setminus R$ and as a 'demand' for $e \in R$. If there exists a feasible (fractional) multiflow for the instance $(\mathcal{M}, R, p)$ then the following cut condition holds:

(3.5.1) $$p(D \setminus R) - p(D \cap R) \geq 0 \text{ for all cocircuits } D \text{ in } \mathcal{M}.$$

Following Seymour [Sey81a], we say that $\mathcal{M}$ is **R-flowing** if, for all $p \in \mathbb{Z}_+^E$ satisfying the cut condition (3.5.1), there exists a feasible multiflow for $(\mathcal{M}, R, p)$; moreover, $\mathcal{M}$ is **R-flowing in $\mathbb{Z}_+$** if the flow can be chosen to be integer valued. $\mathcal{M}$ is said to be **R-cycling** if, for all $p \in \mathbb{Z}_+^E$ satisfying the cut condition (3.5.1) and the parity condition (3.2.9), there exists a feasible integer multiflow for $(\mathcal{M}, R, p)$. Finally, given an integer k, $0 \leq k \leq \infty$, $\mathcal{M}$ is **k-flowing** (resp. **k-cycling**) if $\mathcal{M}$ is R-flowing (resp. R-cycling) for all choices of $R \subseteq E$ with $|R| \leq k$. The next result shows how the ∞-flowing and ∞-cycling properties relate with the sums of circuits property, studied earlier in Section 2.2. (We remind that $\mathcal{M}$ is said to have the sums of circuits property if $\mathrm{CYC}(\mathcal{M}) = \mathrm{MET}(\mathcal{M})$.)

THEOREM 3.5.2. [Sey81a] *The following assertions are equivalent for a binary matroid $\mathcal{M}$.*

(i) *$\mathcal{M}$ is ∞-cycling.*

(ii) *$\mathcal{M}$ is ∞-flowing.*

(iii) *$\mathcal{M}$ is 4-flowing.*

(iv) *$\mathcal{M}$ has no F_7, R_{10} or $\mathcal{M}(K_5)$ minor.*

(v) *The dual matroid $\mathcal{M}^*$ has the sums of circuits property.*

PROOF. The assertions (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are trivial and (iv) $\Longleftrightarrow$ (v) holds by Theorem 2.2.3. (ii) $\Longleftrightarrow$ (v) follows by applying the Japanese theorem. (iii) $\Longrightarrow$ (iv) will follow from Propositions 3.5.3 and 3.5.6. Finally, If $\mathcal{M}$ is a matroid as in (iv) then, by Theorems 1.5.8 and 2.2.4, $\mathcal{M}$ can be obtained by means of $1, 2, 3$-sums from graphic matroids of planar graphs, cographic matroids, and copies of $\mathcal{M}(V_8)$ and F_7^*. Therefore, (iv) $\Longrightarrow$ (i) will follow from Propositions 3.5.4, 3.5.5, 3.5.7 and Theorem 3.3.4. ∎

PROPOSITION 3.5.3. *Let $\mathcal{M}$ be a binary matroid on E and $R, Z_1, Z_2 \subseteq E$ such that $Z_2 \cap (Z_1 \cup R) = \emptyset$. If $\mathcal{M}$ is R-flowing (resp. R-cycling), then $\mathcal{M} \backslash Z_1 / Z_2$ is $(R \backslash Z_1)$-flowing (resp. $(R \backslash Z_1)$-cycling).*

PROOF. It suffices to show the result for $|Z_1 \cup Z_2| = 1$.
(i) Suppose that $Z_1 = \{e_0\}$, $Z_2 = \emptyset$ and that $\mathcal{M}$ is R-flowing; we show that $\mathcal{M} \backslash e_0$ is $(R \setminus \{e_0\})$-flowing. Let $p' \in \mathbb{Z}_+^{E \backslash \{e_0\}}$ such that

$$p'(D' \setminus (R \setminus \{e_0\})) \geq p'(D' \cap (R \setminus \{e_0\}))$$

for all cocycles D' in $\mathcal{M} \backslash e_0$. Extend p' to $p \in \mathbb{Z}_+^E$ by setting $p(e_0) := 0$. Then,

$$p(D \setminus R) \geq p(D \cap R)$$

holds for all cocircuits D in $\mathcal{M}$, as $D \backslash \{e_0\}$ is a cocycle in $\mathcal{M} \backslash e_0$, $p(D \backslash R) = p'((D \backslash \{e_0\}) \backslash (R \backslash \{e_0\}))$ and $p(D \cap R) = p'((D \setminus \{e_0\}) \cap (R \setminus \{e_0\}))$. Hence, there exists a feasible flow $f = (f_C)_{C \in \mathcal{L}}$ for $(\mathcal{M}, R, p)$ where $\mathcal{L} \subseteq \mathcal{C}_R$ and $f_C > 0$ for $C \in \mathcal{L}$. As $p(e_0) = 0$ no circuit from $\mathcal{L}$ contains e_0. Therefore, f yields a feasible multiflow for $(\mathcal{M} \backslash e_0, R \setminus \{e_0\})$.

Suppose now that $\mathcal{M}$ is R-cycling; we show that $\mathcal{M} \backslash e_0$ is $(R \backslash \{e_0\})$-cycling. For this, it suffices to observe that, in the above, if $p'(D')$ is even for all cocycles D' in $\mathcal{M} \backslash e_0$, then $p(D)$ is even for all cocycles D in $\mathcal{M}$.

(ii) Suppose now that $Z_2 := \{e_0\}$, $e_0 \notin R$, $Z_1 = \emptyset$ and that $\mathcal{M}$ is R-flowing; we show that $\mathcal{M}/e_0$ is R-flowing. let $p' \in \mathbb{Z}_+^{E \backslash \{e_0\}}$ such that

$$p'(D' \setminus R) \geq p'(D' \cap R)$$

for all cocycles D' in $\mathcal{M}/e_0$. Let k be an integer such that $k \geq p'(R)$. Extend p' to $p \in \mathbb{Z}_+^E$ by setting $p(e_0) := k$. Let D be a cocycle in $\mathcal{M}$. If $e_0 \notin D$ then D is a cocycle in $\mathcal{M}/e_0$ and, thus, $p(D \setminus R) = p'(D \setminus R) \geq p'(D \cap R) = p(D \cap R)$. If $e_0 \in D$ then $p(D \setminus R) \geq p(e_0) \geq p'(R) \geq p'(D \cap R) = p(D \cap R)$. Therefore, there exists a feasible multiflow $f = (f_C)_{C \in \mathcal{C}_R}$ for $(\mathcal{M}, R, p)$. If $C \in \mathcal{C}_R$ does not contain e_0 then C is a cycle in $\mathcal{M}/e_0$; hence, there exists a circuit $\theta(C)$ in $\mathcal{M}/e_0$ such that $C \cap R \subseteq \theta(C) \subseteq C$; if C contains e_0 set $\theta(C) := C \setminus \{e_0\}$. We can define a flow g for $(\mathcal{M}/e_0, R, p')$ as follows: For each circuit C' in $\mathcal{M}/e_0$ such that $|C' \cap R| = 1$, set

$$g_{C'} := \sum_{C \in \mathcal{C}_R | \theta(C) = C'} f_C.$$

This shows that $\mathcal{M}/e_0$ is R-flowing.

In order to show that $\mathcal{M}/e_0$ is R-cycling if $\mathcal{M}$ is R-cycling, it suffices to check that $p(D)$ is even for all cocycles D in $\mathcal{M}$ whenever $p'(D')$ is even for all cocycles D' in $\mathcal{M}/e_0$. This holds if we choose $p(e_0) = k$ where k is an integer such that $k \geq p'(R)$ and k has the same parity as $p'(D \setminus \{e_0\})$ for some cocycle D in $\mathcal{M}$ containing e_0 (this parity does not depend on the choice of D; indeed, if D_1 and D_2 are two cocycles in $\mathcal{M}$ containing e_0 then $D_1 \triangle D_2$ is a cocycle in $\mathcal{M}/e_0$, which implies that $0 \equiv p'(D_1 \triangle D_2) \equiv p(D_1 \setminus \{e_0\}) + p(D_2 \setminus \{e_0\})$ (modulo 2)). ∎

PROPOSITION 3.5.4. *If $\mathcal{M}$ is the $1,2$-sum of $\mathcal{M}_1$ and $\mathcal{M}_2$ and $\mathcal{M}_1$, $\mathcal{M}_2$ are both k-cycling, then* *so is $\mathcal{M}$.*

PROOF. The case of the 1-sum is trivial; we consider the case of the 2-sum. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be defined on E_1 and E_2 with $E_1 \cap E_2 = \{e_0\}$, and let $\mathcal{M} := \mathcal{M}_1 \triangle \mathcal{M}_2$ be their 2-sum. Let $\emptyset \neq R \subseteq E := E_1 \triangle E_2$ with $|R| \leq k$. Set $R_i := R \cap E_i$ for $i = 1, 2$. Let $p \in \mathbb{Z}_+^E$ satisfying:

$$(a) \qquad p(D \setminus R) - p(D \cap R) \geq 0 \text{ for all cocycles } D \text{ in } \mathcal{M},$$

$$(b) \qquad p(D) \in 2\mathbb{Z} \text{ for all cocycles } D \text{ in } \mathcal{M}.$$

For $i = 1, 2$, choose a cocircuit D_i in $\mathcal{M}_i$ containing e_0 for which the quantity

$$p(D_i \setminus (R_i \cup \{e_0\})) - p(D_i \cap R_i)$$

is minimum; denote it by n_i. From (a) applied to the cocycle $D_1 \triangle D_2$, we obtain that $n_1 + n_2 \geq 0$. We can suppose without loss of generality that $n_1 \geq n_2$; then, $n_1 \geq 0$. As $R \neq \emptyset$ at least one of R_1, R_2 is nonempty. We can suppose that $R_2 \neq \emptyset$ (as $n_2 \geq 0$ if $R_2 = \emptyset$). For $i = 1, 2$ define $p_i \in \mathbb{Z}_+^{E_i}$ by

$$p_i(e) = p(e) \text{ if } e \in E_i \setminus \{e_0\},$$
$$p_i(e_0) = n_1.$$

We consider as demand set for $\mathcal{M}_1$ the set $R_1 \cup \{e_0\}$ (whose size is $\leq k$ since $R_2 \neq \emptyset$) and as demand set for $\mathcal{M}_2$ the set R_2. Let D be a cocircuit in $\mathcal{M}_1$; we claim that

$$p_1(D \setminus (R_1 \cup \{e_0\}) \geq p_1(D \cap (R_1 \cup \{e_0\}).$$

This is obvious if $e_0 \notin D$ as D is then a cocycle in $\mathcal{M}$. If $e_0 \in D$ this follows from the definition of n_1. Let D be a cocircuit in $\mathcal{M}_2$; we claim that

$$p_2(D \setminus R_2) \geq p_2(D \cap R_2).$$

This is obvious if $e_0 \notin D$ and, if $e_0 \in D$, then this follows using (a) applied to $D_1 \triangle D$ and the definition of n_1.
Moreover, for $i = 1, 2$, $p_i(D)$ is even for all cocycles D in $\mathcal{M}_i$. This follows from (b) if $e_0 \notin D$. If $e_0 \in D$ then $D \triangle D_1$ is a cocycle in $\mathcal{M}$, which implies that $p(D \triangle D_1) \equiv p_i(D) + p_1(D_1)$ (modulo 2) is even. But $p_1(D_1)$ is even by the definition of n_1. Therefore, $p_i(D)$ is even.
By the assumption, $\mathcal{M}_1$ is $R_1 \cup \{e_0\}$-cycling and $\mathcal{M}_2$ is R_2-cycling. Hence, there exists a list $\mathcal{L}_1$ of circuits in $\mathcal{M}_1$ each containing exactly one element from $R_1 \cup \{e_0\}$ and providing a feasible multiflow for $(\mathcal{M}_1, R_1 \cup \{e_0\}, p_1)$. Similarly, there exists a list $\mathcal{L}_2$ of circuits in $\mathcal{M}_2$ each containing exactly one element from R_2 and providing a feasible multiflow for $(\mathcal{M}_2, R_2, p_2)$. Let $\mathcal{L}_i(e_0)$ consist of the members of $\mathcal{L}_i$ that contain e_0, for $i = 1, 2$. Then, $\mathcal{L}_1(e_0)$ counts exactly n_1 members and $\mathcal{L}_2(e_0)$ counts at most n_1 members. For $C_1 \in \mathcal{L}_1(e_0)$ and $C_2 \in \mathcal{L}_2(e_0)$, $C_1 \triangle C_2$ is a cycle in $\mathcal{M}$ and, thus, contains a circuit C in $\mathcal{M}$ such that $C_2 \cap R \subseteq C$. By pairing each member $C_2 \in \mathcal{L}_2(e_0)$ with a (distinct) member $C_1 \in \mathcal{L}_1(e_0)$, we obtain a list of circuits in $\mathcal{M}$ which, together with $\mathcal{L}_1 \setminus \mathcal{L}_1(e_0)$ and $\mathcal{L}_2 \setminus \mathcal{L}_2(e_0)$, produces an integer feasible flow for $(\mathcal{M}, R, p)$. ∎

PROPOSITION 3.5.5. *If $\mathcal{M}$ is the 3-sum of $\mathcal{M}_1$ and $\mathcal{M}_2$ and $\mathcal{M}_1$, $\mathcal{M}_2$ are both ∞-cycling, then so is $\mathcal{M}$.*

PROOF. [3] Let $R \subseteq E := E_1 \triangle E_2$ and $p \in \mathbb{Z}_+^E$ satisfying:

$$(a) \qquad p(D \setminus R) - p(D \cap R) \text{ for all cocycles } D \text{ in } \mathcal{M},$$

[3] The proof is more complicated than in the case of the sums of circuits property (in Proposition 2.2.7); in particular, working with the clique 3-sum rather than with the 3-sum does not seem to help here, as the values on the common circuit may have to be redefined in order to meet the parity condition.

$$(b) \qquad p(D) \in 2\mathbb{Z} \text{ for all cocycles } D \text{ in } \mathcal{M}.$$

Set $Z := E_1 \cap E_2 := \{e_1, e_2, e_3\}$, which is a circuit in both $\mathcal{M}_1$ and $\mathcal{M}_2$. Set $R_i := R \cap E_i$ for $i = 1, 2$. For $1 \leq i < j \leq 3$, let d_{ij} denote the minimum value of the quantity

$$p(D \setminus (R_1 \cup Z)) - p(D \cap R_1)$$

taken over all cocycles D in $\mathcal{M}_1$ such that $D \cap Z = \{e_i, e_j\}$; let D_{ij} be such a cocircuit realizing the minimum. Then, $d_{ij} \equiv p(D_{ij} \setminus Z)$ (modulo 2). As $D_{12} \triangle D_{13} \triangle D_{23}$ is a cocycle in $\mathcal{M}$ we obtain from (b) that $p(D_{12} \triangle D_{13} \triangle D_{23})$ is even. Therefore,

$$d_{12} + d_{13} + d_{23} \in 2\mathbb{Z}.$$

Set

$$n_1 := \frac{1}{2}(d_{12} + d_{13} - d_{23}),$$
$$n_2 := \frac{1}{2}(d_{12} + d_{23} - d_{13}),$$
$$n_3 := \frac{1}{2}(d_{13} + d_{23} - d_{12}).$$

Then, $n_1, n_2, n_3 \in \mathbb{Z}$. Set

$$X := \{e_i \in Z \mid n_i > 0\}, \; Y := \{e_i \in Z \mid n_i \leq 0\}.$$

Define $p_i \in \mathbb{Z}_+^{E_i}$ by

$$\begin{array}{ll} p_i(e) = p(e) & \text{if } e \in E_i \setminus Z, \\ p_i(e_j) = |n_j| & \text{for } j = 1, 2, 3. \end{array}$$

We take $R_1 \cup X$ as demand set for $\mathcal{M}_1$ and $R_2 \cup Y$ as demand set for $\mathcal{M}_2$. Let D be a cocircuit in $\mathcal{M}_1$; we claim that

$$p_1(D \setminus (R_1 \cup X)) \geq p_1(D \cap (R_1 \cup X)) \text{ and } p_1(D) \in 2\mathbb{Z}.$$

If $D \cap Z = \emptyset$ then this is obvious from (a) and (b), as D is a cocycle in $\mathcal{M}$. If, say, $D \cap Z = \{e_1, e_2\}$ then

$$p_1(D \setminus (R_1 \cup Z)) - p_1(D \cap R_1) \geq d_{12} = n_1 + n_2,$$
$$p_1(D \cap X) - p_1(D \cap Y) = n_1 + n_2,$$

yielding the required inequality. Moreover, $p_1(D)$ is even (because $p_1(D_{12}) \equiv d_{12} + |n_1| + |n_2| \equiv 2d_{12}$ is even and $p(D \triangle D_{12})$ is even as $D \triangle D_{12}$ is a cocycle in $\mathcal{M}$).
Let D be a cocircuit in $\mathcal{M}_2$; we claim that

$$p_2(D \setminus (R_2 \cup Y)) \geq p_2(D \cap (R_2 \cup Y)) \text{ and } p_2(D) \in 2\mathbb{Z}.$$

This is clear if $D \cap Z = \emptyset$. Suppose, say, that $D \cap Z = \{e_1, e_2\}$. Then, $D \triangle D_{12}$ is a cocycle in $\mathcal{M}$. Hence, $p(D \triangle D_{12})$ is even, $p_2(D \cap X) \geq n_1 + n_2$, and

$$p((D \triangle D_{12}) \setminus R) - p((D \triangle D_{12}) \cap R) \geq 0, \text{ i.e.,}$$

$$p_2(D \setminus (R_2 \cup Y)) - p_2(D \cap X) + p_1(D_{12} \setminus (R_1 \cup Z)) - p_2(D \cap R_2) - p_1(D_{12} \cap R_1) \geq 0,$$

which yields the required inequality.
As $\mathcal{M}_1$ and $\mathcal{M}_2$ are ∞-cycling, there exist a list $\mathcal{L}_1$ of circuits in $\mathcal{M}_1$, each containing exactly one element from $R_1 \cup X$, covering each element $e \in E_1 \setminus (R_1 \cup X)$ (resp. $e \in R_1 \cup X$) at most (resp. exactly) $p_1(e)$ times; and a list $\mathcal{L}_2$ of circuits in $\mathcal{M}_2$, each containing one element from $R_2 \cup Y$ and covering each $e \in E_2 \setminus (R_2 \cup Y)$ (resp. $e \in R_2 \cup Y$) at most (resp. exactly) $p_2(e)$ times. With each pair (C, y) where $C \in \mathcal{L}_1$ and $y \in C \cap Y$, we can associate a (distinct) circuit $C' \in \mathcal{L}_2$ with $y \in C'$;

with each pair (C, x) where $C \in \mathcal{L}_2$ and $x \in C \cap X$, we can associate a circuit $C' \in \mathcal{L}_1$ such that $x \in C'$. We do this in such a way that no C' is associated to more than one pair.

For $C \in \mathcal{L}_1$ containing the element $f \in R_1$ we construct a circuit C^* in $\mathcal{M}$ containing f in the following way. Clearly, $C \cap X = \emptyset$. If $C \cap Y = \emptyset$ then $C^* := C$. If $C \cap Y \neq \emptyset$, for each $y \in C \cap Y$ let $C_y \in \mathcal{L}_2$ be associated with (C, y) (as indicated above). Set $C_1 := \triangle(C_y \mid y \in C \cap Y)$. Then, $(C \triangle C_1) \cap Y = \emptyset$. If $(C \triangle C_1) \cap X = \emptyset$ then take $C^* := C \triangle C_1$. Otherwise, each $x \in (C \triangle C_1) \cap X$ belongs to C_y for some $y \in C \cap Y$; let C_x be the circuit of $\mathcal{L}_1$ associated with the pair (C_y, x). Set $C_2 := \triangle(C_x \mid x \in (C \triangle C_1) \cap X)$. Then, $(C \triangle C_1 \triangle C_2) \cap X = \emptyset$. If $C \triangle C_1 \triangle C_2$ meets Y continue as before. The process must stop as the circuits invoked from associations with pairs all correspond to distinct members in $\mathcal{L}_1$ and $\mathcal{L}_2$. Let C^* be the circuit obtained at the end of the process; it contains f. We repeat this construction for each $f \in R_1 \cup R_2$ and $C \in \mathcal{L}_1 \cup \mathcal{L}_2$ containing it. This produces a list of circuits yielding an integer multiflow for $(\mathcal{M}, R, p)$. ▮

PROPOSITION 3.5.6. *None of the three matroids F_7, R_{10} and $\mathcal{M}(K_5)$ is 4-flowing.*

PROOF. In order to show that $\mathcal{M}$ is not 4-flowing, it suffices to exhibit a set $R \subseteq E$ with $|R| \leq 4$, a vector $p \in \mathbb{Z}^E$ with $p_e < 0$ for $e \in R$, $p_e \geq 0$ for $e \in E \setminus R$, and a metric $m \in \mathrm{MET}(\mathcal{M}^*)$ such that $p^T m < 0$, while $p(D) \geq 0$ for all cocircuits D in $\mathcal{M}$. Let $\mathcal{M}$ be any of F_7, R_{10} or $\mathcal{M}(K_5)$. We take $|p_e| = 1$ for all $e \in E$. Define $\ell \in \mathbb{Z}^E$ by $\ell(e) := 1$ for $e \in E \setminus R$ and $\ell(e) := |E|$ for $e \in R$. Set $m := dist(\mathcal{M}, \ell)$, defined by $m(e) := \min(\ell(e), \ell(C \setminus \{e\})$ for C circuit in $\mathcal{M}, e \in C)$; hence, $m(e) = 1$ for $e \in E \setminus R$. Then, $m \in \mathrm{MET}(\mathcal{M}^*)$ by Lemma 3.2.10. In the case of $\mathcal{M}(K_5)$, R consists of one edge and a disjoint triangle in K_5; then, $m(e) = 2$ for $e \in R$. In the case of F_7, R is a circuit in F_7 of size 3 (a Fano line); then, $m(e) = 2$ for $e \in R$. Finally, in the case of R_{10}, R is a subset of size 3 in a circuit of size 4; then, $m(e) = 3$ for $e \in R$. Then, $p^T m < 0$ in all three cases. One checks easily that $p(D) \geq 0$ for all cocircuits, in all three cases. ▮

PROPOSITION 3.5.7. *The matroids F_7^* and $\mathcal{M}(V_8)$ are both ∞-cycling.*

PROOF. Details can be found in [Sey81a]. ▮

Seymour [Sey81a] presents several other results on flowing and cycling matroids. Let us mention some of them (without proofs):

(i) $\mathcal{M}$ is ∞-flowing in $\mathbb{Z}_+ \iff \mathcal{M}$ is 2-flowing in $\mathbb{Z}_+ \iff \mathcal{M}$ has no $\mathcal{M}(K_4)$ minor.

(ii) $\mathcal{M}$ is 1-flowing in $\mathbb{Z}_+ \iff \mathcal{M}$ has no F_7^* minor. (We know from Theorem 3.1.6 that graphic matroids are 1-flowing in $\mathbb{Z}_+$.)

(iii) $\mathcal{M}$ is 3-flowing $\iff \mathcal{M}$ is 3-cycling $\iff \mathcal{M}$ has no F_7, R_{10} or $\mathcal{M}(H_6)$ minor. (H_6 is the graph shown in Figure 3.5.8.)

(iv) $\mathcal{M}$ is 2-flowing $\iff \mathcal{M}$ is 2-cycling $\iff \mathcal{M}$ has no $AG(2,3)$ or S_8 minor. (Representation matrices for $AG(2,3)$ and S_8 are the respective matrices $M_{AG(2,3)}$ and M_{S_8} shown below.)

Note that the characterization of the 1-flowing (1-cycling, $\{f\}$-flowing, $\{f\}$-cycling) matroids remains an open problem; see the conjectures (9.1),(9.2),(9.4) in [Sey81a] about it. On the other hand, the $\{f\}$-flowing matroids in $\mathbb{Z}_+$ are known. Indeed, it is easy to see that a matroid $\mathcal{M}$ is $\{f\}$-flowing in $\mathbb{Z}_+$ if and only if the collection $\{C \setminus \{f\} \mid$

C circuit in $\mathcal{M}, f \in C\}$ has the $\mathbb{Z}_+$-max-flow min-cut property. By Theorem 2.3.6 (from [Sey77]), this is the case if and only if $\mathcal{M}$ has no F_7^* minor using f.

The following class of R-cycling (graphic) matroids is provided by Theorem **??**: Let $\mathcal{G} := (V, E)$ be a graph and $R \subseteq E$. If R is a circuit on five nodes, or K_4 or a union of two stars, then $\mathcal{M}(\mathcal{G})$ is R-cycling.

$$M_{AG(2,3)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad M_{S_8} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

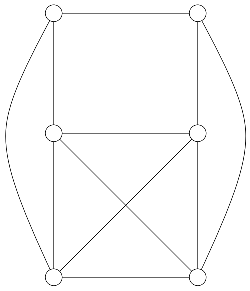


Figure 3.5.8: The graph H_6

We now see how the R-flowing and R-cycling properties can be expressed in terms of the $\mathbb{Q}_+$- and $\mathbb{Z}_+$-MFMC properties, considered earlier in Section 2.3. Given a binary matroid $\mathcal{M}$ on E and $R \subseteq E$, let $\mathcal{O}(\mathcal{M}, R)$ denote the collection of R-odd circuits in $\mathcal{M}$ (we remind that a set C is R-odd if $|C \cap R|$ is odd). Clearly, given $p \in \mathbb{Z}_+^E$, $(\mathcal{M}, R, p)$ satisfies the cut condition (3.5.1) if and only if

$$p(R) = \min\{p(D \triangle R) \mid D \text{ cycocle in } \mathcal{M}\}.$$

Consider the following pair of dual linear programs:[4]

$$(3.5.9) \qquad \begin{array}{llcll} \min & p^T x & = & \max & \sum_{C \in \mathcal{O}(\mathcal{M},R)} y_C \\ & x(C) \geq 1 \ (C \in \mathcal{O}(\mathcal{M}, R)) & & & \sum_{C \in \mathcal{O}(\mathcal{M},R) | e \in C} y_C \leq p_e \ (e \in E) \\ & x_e \geq 0 \ (e \in E) & & & y_C \geq 0 \ (C \in \mathcal{O}(\mathcal{M}, R)) \end{array}$$

The integer solutions to the minimization program are precisely the incidence vectors of the sets $D \triangle R$ for D cocycle in $\mathcal{M}$ (as one can check that a set meets all members of $\mathcal{O}(\mathcal{M}, R)$ if and only if it contains some $D \triangle R$). Suppose that the cut condition holds. Then, $p(R)$ is an upper bound for the optimum value. Clearly, a feasible (integer) flow exists for $(\mathcal{M}, R, p)$ if and only if the maximization program has a (integer) solution with value $p(R)$ (which implies that χ^R is an integer optimum solution to the minimization program).

Lemma 3.5.10. *The following are equivalent for a binary matroid $\mathcal{M}$.*

[4] We are considering the matroidal analogues for the notions of T-joins and T-cuts, considered in Sections 1.3 and 3.3.

(i) $\mathcal{M}$ *is* ∞*-flowing (resp.* ∞*-flowing in* $\mathbb{Z}_+$*).*

(ii) $\mathcal{O}(\mathcal{M}, R)$ *has the* $\mathbb{Q}_+$*-MFMC (resp.* $\mathbb{Z}_+$*-MFMC) property for all* $R \subseteq E$*.*

PROOF. (ii) $\Longrightarrow$ (i) is clear by the above observations. We check (i) $\Longrightarrow$ (ii). Let $p \in \mathbb{Z}_+^E$ and $R \subseteq E$. If p satisfies the cut condition with respect to R, then the result holds by the above observations. Otherwise, let D_0 be a cocycle such that $p(D_0 \triangle R) = \min\{p(D \triangle R) \mid D \text{ cocycle in } \mathcal{M}\}$. Now, p satisfies the cut condition with respect to the set $D_0 \triangle R$. The result follows after noting that

$$\min\{p(D \triangle D_0 \triangle R) \mid D \text{ cocycle in } \mathcal{M}\} = \min\{p(D \triangle R) \mid D \text{ cocycle in } \mathcal{M}\}$$

and that $\mathcal{O}(\mathcal{M}, R) = \mathcal{O}(\mathcal{M}, D_0 \triangle R)$. ∎

Following Seymour [Sey81a] we say that $\mathcal{M}$ is **R-packing** if, for all $p \in \mathbb{Z}_+^E$ satisfying the parity condition (3.2.9), both programs in (3.5.9) have integer optimum solutions. $\mathcal{M}$ is **k-packing** if it is R-packing for all $R \subseteq E$ with $|R| \leq k$. The analogue of Lemma 3.5.10 reads:

LEMMA 3.5.11. *A binary matroid* $\mathcal{M}$ *is* ∞*-packing if and only if it is* ∞*-cycling.* ∎

Note that R-cyclingness does not imply, in general, R-packingness. For example, taking for R the edge set of a K_4 subgraph in K_5, $\mathcal{M}(K_5)$ is R-cycling but not R-packing. But, it is easy to see that: k-packing $\Longrightarrow$ k-cycling. Seymour [Sey81a] shows that

(i) $\mathcal{M}$ is k-cycling $\Longleftrightarrow$ $\mathcal{M}$ is k-packing, for $k = 1, 2, 3$.

(ii) $\mathcal{M}$ is k-packing $\Longleftrightarrow$ $\mathcal{M}$ is k-cycling $\Longleftrightarrow$ $\mathcal{M}$ is ∞-cycling, for $k \geq 4$.

We conclude with an application for the cycle polytope $\mathrm{CYC}^{\square}(\mathcal{M})$ of a matroid $\mathcal{M}$.

PROPOSITION 3.5.12. *Let* $\mathcal{M}$ *be a binary matroid with no* F_7*,* R_{10} *or* $\mathcal{M}(K_5)$ *minor. For every* $w \in \mathbb{Z}^E$ *such that* $w(D) \in 2\mathbb{Z}$ *for all cocircuits* D *in* $\mathcal{M}$*, both the linear program:*

$$(3.5.13) \qquad \begin{array}{lll} \max & w^T x & \\ & x(F) - x(C \setminus F) \leq |F| - 1 & (F \subseteq C, |F| \text{ odd}, C \text{ cycle in } \mathcal{M}) \\ & 0 \leq x_e \leq 1 & (e \in E) \end{array}$$

and its dual linear program have integer optimum solutions.

PROOF. By the assumption, we know that $\mathcal{M}$ is ∞-packing. Set $R := \{e \in E \mid w_e \geq 0\}$ and $p_e := |w_e|$ for $e \in E$. Then,

$$\max(w^T x \mid x \in \mathrm{MET}^{\square}(\mathcal{M}^*)) = w(R) - \min(p^T x \mid x(C) \geq 1 \text{ for all } C \in \mathcal{O}(\mathcal{M}, R), x \geq 0).$$

Moreover, both the above minimization program and its dual (expressed in (3.5.9)) have integer optimum solutions. Let D be a cocycle in $\mathcal{M}$ such that $\chi^{D \triangle R}$ is an optimum solution for the minimization program and let $y = (y_C)_{C \in \mathcal{O}(\mathcal{M}, R)}$ be an integer optimum solution for its dual program. Then, χ^D is an optimum solution for (3.5.13). Let $\mathcal{C}$ consist of the members $C \in \mathcal{O}(\mathcal{M}, R)$ such that $y_C > 0$. Then,

$$C \in \mathcal{C} \Longrightarrow |C \cap (D \triangle R)| = 1, \;\; e \in D \triangle R \Longrightarrow \sum_{C \in \mathcal{C} | e \in C} y_C = p_e.$$

Let $u_{C,F}$ and z_e be the dual variables associated with (3.5.13). Set

$$\begin{array}{lll} z_e := & p_e - \sum_{C \in \mathcal{C} | e \in C} y_C & \text{for } e \in R, \\ z_e := & 0 & \text{for } e \in E \setminus R, \\ u_{C,F} := & y_C & \text{if } C \in \mathcal{C} \text{ and } F = (C \cap D) \triangle (C \cap (D \triangle R)), \\ u_{C,F} := & 0 & \text{otherwise.} \end{array}$$

Then, $(u_{C,F}), (z_e)$ are feasible for the dual program of (refLPsigma) as

$$\sum_{(C,F) | e \in F} u_{C,F} - \sum_{(C,F) | e \in C \setminus F} u_{C,F} + z_e \geq w_e$$

with equality if $e \in D \cup R$; optimality follows from the fact that the complementary slackness conditions are satisfied. ∎

3.5.2 Routing Matroids

We have investigated above the matroids for which the cut condition together with the parity condition are sufficient for ensuring the existence of an integer multiflow (these are the cycling matroids). In a sense, the most drastic way of strenghtening the cut condition is to require the existence of a fractional flow. Hence arises the question of studying the matroids for which the existence of a fractional flow implies the existence of an integer flow, provided the parity condition is satisfied.

Note that this question has an easy answer if we do not require the parity condition. Indeed, a matroid $\mathcal{M}$ satisfies: the existence of a fractional flow for $(\mathcal{M}, R, p)$ implies the existence of an integer flow for all demand sets R and all $p \in \mathbb{Z}_+^E$, if and only if $\mathcal{M}$ has no $\mathcal{M}(K_4)$ minor ([Seb90]). (Recall Example 3.2.11 for the exclusion of K_4.)

So we shall now deal with instances satisfying the parity condition. We will see in the next subsection that assuming the parity condition is in some sense not really a restriction, as every instance can be modified to a new instance in which the parity condition automatically holds.

Let M be a binary matroid on E and $R \subseteq E$. Following Sebö [Seb90], we say that $\mathcal{M}$ is R-**routing** if, for every $p \in \mathbb{Z}_+^E$ satisfying the parity condition (3.2.9), the existence of a fractional flow for $(\mathcal{M}, R, p)$ implies the existence of an integer flow. Call $\mathcal{M}$ k-**routing** if $\mathcal{M}$ is R-routing for all $R \subseteq E$ with $|R| \leq k$. Obviously, R-cycling matroids are R-routing (as the existence of a fractional flow implies the cut condition). On the other hand, the three matroids F_7, R_{10} and $\mathcal{M}(K_5)$ (which are not ∞-cycling) are ∞-routing [Seb90].

A large class of R-routing (graphic) matroids is provided by Theorem **??**. Namely, if $\mathcal{G} = (V, E)$ is a graph and $R \subseteq E$ is the edge set of a subgraph on at most five nodes, then $\mathcal{M}(\mathcal{G})$ is R-routing.

The characterization of R-routing matroids seems to be a hard problem (it can be reformulated as a question about Hilbert bases; see Corollary 3.5.20 below). We refer to [Seb90] for a tour d'horizon on open questions, results, and links with other notions. Some tools are given there for studying routing matroids, in particular, relative to taking minors and k-sums. Let us mention some of them (without proofs):

(i) Every minor of a ∞-routing matroid is ∞-routing.

(ii) If $\mathcal{M}_1$ and $\mathcal{M}_2$ are ∞-routing then their 1-sum is ∞-routing. If, moreover, $\mathcal{M}_2$ is ∞-cycling, then their 2- and 3-sums are ∞-routing.

(iii) If $\mathcal{M}_1$ and $\mathcal{M}_2$ are minimal non ∞-flowing then their 2-sum is minimal non ∞-routing.

As F_7, R_{10} and $\mathcal{M}(K_5)$ are minimal non ∞-flowing, by taking their 2-sums we obtain six nonisomorphic minimal non ∞-routing matroids. These six matroids are called the **bi-nonflowing matroids** in [Seb90] and denoted as $B_{i,j}$ where i and j are the indices of the members in the 2-sum (for instance, $B_{5,7}$ is the 2-sum of $\mathcal{M}(K_5)$ and F_7; $B_{10,10}$ is the 2-sum of two copies of R_{10}, etc.) The only graphic matroid among them is $B_{5,5}$, which is the graphic matroid of the graph shown in Figure 3.5.15. Sebö [Seb90] gives a class of matroids for which the exclusion of the $B_{i,j}$'s as a minor suffices to characterize ∞-routingness.

THEOREM 3.5.14. [Seb90] *Let $\mathcal{M}$ be a binary matroid with no $AG(2,3)$, S_8, or $\mathcal{M}(H_6)$ minor. Then, $\mathcal{M}$ is ∞-routing if and only if $\mathcal{M}$ does not have any of the bi-nonflowing matroids $B_{i,j}$ as a minor.*

PROOF. We only sketch the proof. Let $\mathcal{M}$ have no $AG(2,3)$, S_8, or $\mathcal{M}(H_6)$ minor. Based on results from [Sey80] (see [Sey81a]), we conclude that $\mathcal{M}$ can be obtained by means of proper 1,2-sums from matroids with no F_7, R_{10} or $\mathcal{M}(K_5)$ minor, and copies of F_7, R_{10} and $\mathcal{M}(K_5)$. Suppose that $\mathcal{M}$ is a counterexample to the theorem with smallest $|E|$. Then, $\mathcal{M}$ cannot be decomposed as a 1-sum of two matroids (because ∞-routingness is preserved by the 1-sum operation). If $\mathcal{M}$ is 3-connected (i.e., cannot be decomposed as a 2-sum) then $\mathcal{M}$ is F_7, R_{10}, $\mathcal{M}(K_5)$, or a matroid without these three minors and, thus, $\mathcal{M}$ is ∞-routing. Hence, we can decompose $\mathcal{M}$ by the 2-sum operation until arriving at a list $\mathcal{L}$ of 3-connected matroids. Each member in $\mathcal{L}$ is F_7, R_{10}, $\mathcal{M}(K_5)$, or a matroid without these three minors. Moreover, as the bi-nonflowing matroids are excluded minors of $\mathcal{M}$, at most one member from $\mathcal{L}$ belongs to $\{F_7, R_{10}, \mathcal{M}(K_5)\}$. Therefore, $\mathcal{M}$ is routing (as the 2-sum of an ∞-routing matroid and an ∞-cycling matroid is ∞-routing). ∎

Note that the class of matroids with no $AG(2,3)$, S_8, or $\mathcal{M}(H_6)$ minor is precisely the minimal class closed under taking minors and 1,2-sums that contains all ∞-flowing matroids (i.e., with no F_7, R_{10} or $\mathcal{M}(K_5)$ minor) as well as F_7, R_{10} and $\mathcal{M}(K_5)$.

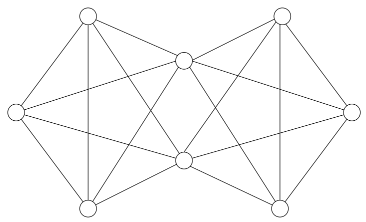


Figure 3.5.15: $B_{5,5}$

3.5.3 From Cyclingness to Routingness

An intermediate step between cyclingness (when the cut + parity condition are sufficient) and routingness (when the distance + parity conditions are sufficient) is to consider matroids for which the distance condition - assumed only for a restricted set of metrics - and the parity condition are sufficient for the existence of an integer multiflow.

Let $\mathcal{M}$ be a matroid on E and let $\Delta(\mathcal{M})$ or, simply, Δ, be a set of metrics on $\mathcal{M}$, i.e., $\Delta \subseteq \mathrm{MET}(\mathcal{M}^*)$. One says that $\mathcal{M}$ is **∞-flowing with respect to Δ** (resp. **∞-cycling with respect to Δ**) if, for all $p \in \mathbb{Z}^E$ satisfying the parity condition (3.2.9) and the following restricted distance condition:

(3.5.16) $$p^T m \geq 0 \text{ for all } m \in \Delta(\mathcal{M}),$$

there exists a feasible fractionnal (resp. integer) multiflow for $(\mathcal{M}, p)$. If $\Delta(\mathcal{M})$ is the set of the incidence vectors of the cocircuits in $\mathcal{M}$ then (3.5.16) is the usual cut condition (3.2.7); if we take for Δ the set of metrics on $\mathcal{M}$ that are 0,1-valued then we obtain the same condition (as every 0,1-valued metric on $\mathcal{M}$ is a conic combination of cocircuits). Schwärzler and Sebö [SS93] consider the case when Δ is the set $\Delta_{\{0,1,2\}}$ of the metrics with values in $\{0,1,2\}$. Examples of $(0,1,2)$-valued metrics can be constructed as follows: Let D_1, D_2, D_3 be three cocycles in $\mathcal{M}$; define $m \in \{0,1,2\}^E$ by

$$m(e) := \begin{cases} 1 & \text{if } e \in D_1, \\ 2 & \text{if } e \in (D_2 \cup D_3) \setminus D_1, \\ 0 & \text{if } e \in E \setminus (D_1 \cup D_2 \cup D_3). \end{cases}$$

Then, $m \in \mathrm{MET}(\mathcal{M}^*)$. Let $\Delta_{(CC3)}$ be the class of metrics constructed in this way; so, $\Delta_{(CC3)} \subseteq D_{\{0,1,2\}}$.

THEOREM 3.5.17. [SS93] *The following assertions are equivalent for a matroid $\mathcal{M}$.*

(i) *$\mathcal{M}$ is ∞-cycling with respect to $\Delta_{(CC3)}$.*

(ii) *$\mathcal{M}$ is ∞-flowing with respect to $\Delta_{\{0,1,2\}}$.*

(iii) *$\mathcal{M}$ has no $AG(2,3)$, S_8, R_{10}, $\mathcal{M}(H_6)$, $B_{5,5}$, $B_{5,7}$, or $B_{7,7}$ minor.* ∎

The proof of this result is along the same lines as that of Theorem 3.5.14. For graphs, the metrics in $\Delta_{(CC3)}$ can be further specified. Given a partition of V into $r+s$ (possibly empty) classes $A_1, \ldots, A_r, B_1, \ldots, B_s$ such that $\cup_i A_i, \cup_j B_j \neq \emptyset$, define the semimetric m on V by

$$m(uv) := \begin{cases} 1 & \text{if } u \in A_i, v \in B_j, \\ 2 & \text{if } u \in A_i, v \in A_j (i \neq j), \text{ or } u \in B_i, v \in B_j (i \neq j), \\ 0 & \text{if } u, v \in A_i \text{ or } u, v \in B_i. \end{cases}$$

So, m is the path metric of the complete bipartite graph $K_{r,s}$ if all A_i, B_j's are singletons. Let $\Delta_{bip(r,s)}$ denote the class of the semimetrics on V that are defined in the above way. Hence, $\Delta_{bip(r,s)} \subseteq \Delta_{(CC3)}$.

THEOREM 3.5.18. *Let $\mathcal{G} = (V, E)$ be a graph. The following assertions are equivalent.*

(i) *$\mathcal{M}(\mathcal{G})$ is ∞-cycling with respect to $\Delta_{bip(2,3)}$.*

(ii) *$\mathcal{M}(\mathcal{G})$ is ∞-flowing with respect to $\Delta_{\{0,1,2\}}$.*

(iii) *$\mathcal{G}$ has no H_6 or $B_{5,5}$ minor. ($B_{5,5}$ is the graph from Figure 3.5.15.)* ∎

In this context, Theorem **??** can be reformulated as follows: Let $\mathcal{G} = (V, E)$ be a graph and $p \in \mathbb{Z}^E$ such that $p(\delta_\mathcal{G}(u)) \in 2\mathbb{Z}$ for all $u \in V$. If the edge set $R := \{e \in E \mid p_e < 0\}$ is a subgraph of $\mathcal{G}$ on at most five vertices then there exists an integer flow for $(\mathcal{M}(\mathcal{G}), p)$ if and only if $m^T p \geq 0$ for all $m \in \Delta_{bip(2,3)}$.

3.5.4 Flowing, Cycling, Routing Reformulated via Polarity

We give here some new interpretations for the R-flowing, R-cycling, and R-routing properties; they are basically obtained by applying the polarity[5] for cones. Let $\mathcal{M}$ be a binary matroid on E and $R \subseteq E$. Set

$$\mathcal{X}_R := \{2\chi^e (e \in E \setminus R),\ \chi^C (C \text{ circuit in } \mathcal{M}, |C \cap R| \leq 1)\},$$

$$\mathcal{Y}_R := \{\chi^{D\setminus R} - \chi^{D\cap R} (D \text{ cocircuit in } \mathcal{M}), 2\chi^e (e \in E)\}.$$

(The notation χ^e stands for $\chi^{\{e\}}$.) Obviously, the cone $\mathbb{R}_+(\mathcal{X}_R)$ consists of the vectors $p \in \mathbb{Z}_+^E$ such that a flow exists for $(\mathcal{M}, R, p)$, and the polar of the cone $\mathbb{R}_+(\mathcal{Y}_R)$ consists of the nonnegative vectors p satisfying the cut condition (3.5.1). Hence, $\mathbb{R}_+(\mathcal{X}_R) \subseteq (\mathbb{R}_+(\mathcal{Y}_R))^*$, with equality precisely when $\mathcal{M}$ is R-flowing. The polar equality: $(\mathbb{R}_+(\mathcal{X}_R))^* = \mathbb{R}_+(\mathcal{Y}_R)$ will be reformulated in Lemma 3.5.21.

LEMMA 3.5.19. *Let $p \in \mathbb{Z}_+^E$.*

(i) *Suppose that $\mathcal{M}$ has no F_7^* minor, no coloops, no coparallel elements, and that $\mathcal{C}_f \neq \emptyset$ for all $f \in R$ (i.e., R contains no cocircuit). Then, $\mathbb{Z}(\mathcal{X}_R) = \mathbb{Z}(\mathcal{M})$.*

(ii) *$p \in \mathbb{R}_+(\mathcal{X}_R) \iff$ there exists a fractional multiflow for $(\mathcal{M}, R, p)$.*

(iii) *$p \in \mathbb{Z}_+(\mathcal{X}_R) \iff$ there exists an integer multiflow for $(\mathcal{M}, R, p)$ and p satisfies the parity condition (3.2.9).*

PROOF. (i) By the assumption, $\mathcal{M}$ has the lattice property, i.e., its cycle lattice $\mathbb{Z}(\mathcal{M})$ is characterized by the parity condition. The inclusion $\mathbb{Z}(\mathcal{X}_R) \subseteq \mathbb{Z}(\mathcal{M})$ follows using Lemma 2.4.7. Conversely, let C be a circuit in $\mathcal{M}$ such that $|C \cap R| \geq 2$. We show that $\chi^C \in \mathbb{Z}(\mathcal{X}_R)$. For each $f \in R$ let C_f be a circuit such that $C_f \cap R = \{f\}$. Then, $2\chi^f = 2\chi^{C_f} - 2\chi^{C_f \setminus \{f\}} \in \mathbb{Z}(\mathcal{X}_R)$. Moreover, $C^* := C \triangle (\triangle_{f \in C\cap R} C_f)$ is a cycle contained in $E \setminus R$. This implies that $\chi^C \in \mathbb{Z}(\mathcal{X}_R)$ as $\chi^C \equiv \chi^{C^*} + \sum_{f \in C \cap R} \chi^{C_f}$ (modulo 2). Assertion (ii) and the only if part in (iii) are obvious. Conversely, let $f = (f_C)_{C \in \mathcal{C}_R}$ be an integer feasible flow for $(\mathcal{M}, R, p)$ and set $q := \sum_C f_C \chi^C$; so, $q \in \mathbb{Z}_+(\mathcal{X}_R)$. Write $p - q$ as $2u + v$, where u is nonnegative integer valued and v is 0,1-valued. As $p - q$ satisfies the parity

[5] For a cone $C \subseteq \mathbb{R}^n$, its **polar** C^* consists of the vectors $y \in \mathbb{R}^n$ such that $y^T x \geq 0$ for all $x \in C$. It is well known that $(C^*)^* = C$.

condition, we deduce that v is the incidence vector of a cycle in $\mathcal{M}$ contained in $E \setminus R$. Hence, $p - q \in \mathbb{Z}_+(\mathcal{X}_R)$, implying that $p = (p-q) + q \in \mathbb{Z}_+(\mathcal{X}_R)$. ∎

COROLLARY 3.5.20. *Let $\mathcal{M}$ be a binary matroid on E with no F_7^* minor and with no loops nor coloops. Let $R \subseteq E$ such that $\mathcal{C}_f \neq \emptyset$ for all $f \in R$. Then, $\mathcal{M}$ is R-routing if and only if the set $\mathcal{X}_R$ is a Hilbert basis, i.e., if $\mathbb{Z}_+(\mathcal{X}_R) = \mathbb{Z}(\mathcal{X}_R) \cap \mathbb{R}_+(\mathcal{X}_R)$.* ∎

LEMMA 3.5.21. *Let $\mathcal{M}$ be a binary matroid on E and $R \subseteq E$. The following are equivalent.*

(i) $(\mathbb{R}_+(\mathcal{X}_R))^* = \mathbb{R}_+(\mathcal{Y}_R)$.

(ii) *For every $x \in \mathrm{MET}(\mathcal{M}^*)$ (metric on $\mathcal{M}$), there exists $\alpha_D \geq 0$ (D cocycle in $\mathcal{M}$) such that $x_e \geq \sum_{D|e\in D} \alpha_D$ for all $e \in E$, with equality for $e \in R$.*

(iii) *$\mathcal{M}$ is R-flowing.*

PROOF. (i) $\iff$ (ii). Changing the signs on R, (i) can be reformulated as saying that $C_1 = C_2$, where

$$C_1 := \{x \mid x_e \geq 0 (e \in E \setminus R),\ x(C \setminus \{f\}) \geq x_f (C \in \mathcal{C}_f, f \in R)\},$$

$$C_2 := \{x \mid x_e \geq \sum_{D|e\in D} \alpha_D (e \in E \setminus R),\ x_f \leq \sum_{D|f\in D} \alpha_D (f \in R) \text{ for } \alpha_D \geq 0 (D \text{ cocircuit})\}.$$

Note that, for $x \in C_2$, there exist some α_D's for which equality: $x_f = \sum_{D|f\in D} \alpha_D$ holds for all $f \in R$. Hence, (i) $\Longrightarrow$ (ii) is obvious. Let us suppose that (ii) holds; we show that $C_1 = C_2$. The inclusion $C_2 \subseteq C_1$ is obvious. We check the converse inclusion. Let $x \in C_1$. Define $x' \in \mathbb{R}^E$ by $x'_e := x_e$ for all $e \in E$ except $x'_f = 0$ if $x_f < 0$ for $f \in R$. Clearly, $x' \in C_1$ and $x' \geq 0$. Let $y \in \mathrm{MET}(\mathcal{M}^*)$ such that $y_e \leq x'_e$ for $e \in E \setminus R$ and $y_f \geq x'_f$ for $f \in R$ (that such y exists was proved within the proof of the Japanese theorem). By (ii) applied to y, there exist $\alpha_D \geq 0$ such that $y_e \geq \sum_{D|e\in D} \alpha_D$ $(e \in E \setminus R)$ and $y_f = \sum_{D|f\in D} \alpha_D$ $(f \in R)$. Hence, $x_e = x'_e \geq y_e \geq \sum_{D|e\in D} \alpha_D$ $(e \in E \setminus R)$ and $x_f \leq x'_f \leq y_f = \sum_{D|f\in D} \alpha_D$ $(f \in R)$. This shows that $x \in C_2$. ∎

Following [Seb90], a matroid satisfying Lemma 3.5.21 (ii) is said to have the **R-metric property**. Hence, R-flowingness is equivalent to the R-metric property. We say that $\mathcal{M}$ has the **even R-property** if, for every metric $x \in \mathrm{MET}(\mathcal{M}^*)$ such that $x(C)$ is even for all cycles C in $\mathcal{M}$, there exist nonnegative integer scalars α_D (D cocircuit in $\mathcal{M}$) such that $x_e \geq \sum_{D|e\in D} \alpha_D$ for all $e \in E$, with equality for all $e \in R$.

For $x \in \mathrm{MET}(\mathcal{M}^*)$, let y be defined from x by changing the signs of the components of x on R. Then, x satisfies Lemma 3.5.21 (ii) (resp. with integer valued α_D) if and only if $y \in \mathbb{R}_+(\mathcal{Y}_R)$ (resp. $\mathbb{Z}_+(\mathcal{Y}_R)$). This implies the following result.

LEMMA 3.5.22. *Let $\mathcal{M}$ be a binary matroid on E with no F_7 minor, no loops and no parallel elements. Then, $\mathcal{M}$ has the even R-metric property if and only if the set $\mathcal{Y}_R$ is a Hilbert basis, i.e., if $\mathbb{Z}_+(\mathcal{Y}_R) = \mathbb{Z}(\mathcal{Y}_Y) \cap \mathbb{R}_+(\mathcal{Y}_R)$.* ∎

Hence, the R-routingness property and the even R-metric property are, in a sense, dual of one another. It is easy to see that

$$\begin{array}{l}\mathcal{M} \text{ is } R\text{-routing and has the even } R\text{-metric property} \\ \text{(i.e., both } \mathcal{X}_R, \mathcal{Y}_R \text{ are Hilbert bases)} \\ \Longrightarrow \mathcal{M} \text{ is } R\text{-cycling.}\end{array}$$

The converse implication does not hold automatically. There are several cases, however, when it has been shown to hold. See the results from Theorems ?????.

3.5.5 How to Avoid the Parity Condition ?

Most of the results presented so far about the existence of integer flows assume the parity condition. What we want now to point out is that this assumption is not really a loss of generality. Indeed, as is shown by Sebö [Seb90], if the parity condition does not hold for the instance $(\mathcal{M}, R, p)$ then one can construct a new instance $(\mathcal{M}', R', p')$ - in which the parity condition holds - in such a way that an integer flow exists for $(\mathcal{M}, R, p)$ if and only if one exists for $(\mathcal{M}', R', p')$. This construction will yield, moreover, a new necessary condition for the existence of an integer flow - stronger than the cut condition - and first discovered by Frank [Fra90].

let us now explain the construction. Consider the instance $(\mathcal{M}, R, p)$ where $R \subseteq E$ and $p \in \mathbb{Z}_+^E$. Set

$$\Sigma := \{e \in E \mid p_e \text{ is odd}\}.$$

let M be a representation matrix for $\mathcal{M}$, with columns u_e $(e \in E)$. Set $\tau := \sum_{e \in \Sigma} u_e$, $E' := E \cup \{t\}$ (where t is a new element not in E) and

$$M' := \begin{array}{c} \begin{matrix} E & t \end{matrix} \\ \begin{pmatrix} M & \tau \end{pmatrix} \end{array}.$$

If $\tau = 0$ then Σ is a cycle in $\mathcal{M}$ and p satisfies the parity condition; we set $\mathcal{M}' := \mathcal{M}$. Otherwise, let $\mathcal{M}'$ be the binary matroid on E' with representation matrix M'; its cycles are the sets: C and $(C \triangle \Sigma) \cup \{t\}$, for C cycle in $\mathcal{M}$. (Note that this is the same construction as the one given for graft matroids in the context of T-joins/T-cuts in Section 2.3.) We consider the new multiflow instance $(\mathcal{M}', R', p')$, where $R' := R \cup \{t\}$, $p' \in \mathbb{Z}_+^{E'}$ with $p'(t) := 1$ and $p'(e) := p(e)$ for $e \in E$. It is called the **Eulerian extension** of $(\mathcal{M}, R, p)$. Clearly, p' satisfies the parity condition (it suffices to check that $p'(D')$ is even for every D' whose incidence vector is a row of M').

LEMMA 3.5.23. *The following are equivalent.*

(i) *There exists an integer feasible multiflow for* $(\mathcal{M}, R, p)$.

(ii) *There exists an integer feasible multiflow for* $(\mathcal{M}', R', p')$.

PROOF. (ii) $\Longrightarrow$ (i) is clear. Conversely, let $(f_C)_{C \in \mathcal{C}_R}$ be an integer multiflow for $(\mathcal{M}, R, p)$. Then, $x := \sum_C f_C \chi^C \in \mathbb{Z}_+(\mathcal{M})$ and $x_e \leq p_e$ for all e with equality for $e \in R$. It suffices to show the existence of a circuit C' in $\mathcal{M}'$ such that $t \in C'$ and $C' \setminus \{t\} \subseteq \{e \in E \mid p_e - x_e \geq 1\}$ (as, then, $x + \chi^{C'} \leq p'$ with equality for $e \in R'$). Indeed, one can easily check that the set $\{t\} \cup \{e \in E \mid p_e - x_e \text{ is odd}\}$ is a cycle in $\mathcal{M}'$. ∎

Therefore, a necessary condition for the existence of an integer flow for $(\mathcal{M}, R, p)$ is the existence of a fractional flow for $(\mathcal{M}', R', p')$. Consider the following condition:

$$\text{(3.5.24)} \qquad \begin{array}{lcl} & \text{For every } \ w \in \mathrm{CYC}(\mathcal{M}^*), & \\ \max & \sum_{C \in \mathcal{O}(\mathcal{M}^*, \Sigma)} y_C & \leq \quad \sum_{e \in E \setminus R} w_e p_e - \sum_{e \in R} w_e p_e. \\ & \sum_{C \mid e \in C} y_C \leq w_e \ (e \in E) & \\ & y_C \geq 0 \ (C \in \mathcal{O}(\mathcal{M}^*, \Sigma)) & \end{array}$$

We remind that $\mathcal{O}(\mathcal{M}^*, \Sigma)$ consists of the cocircuits in $\mathcal{M}$ having an odd intersection with Σ. Note that (3.5.24) implies the cut condition (obtained by taking $w := \chi^D$ for a cocycle D in $\mathcal{M}$). (The left hand side in the inequality from (3.5.24) is the 'size' of a w-packing in $\mathcal{O}(\mathcal{M}^*, \Sigma)$.)

Proposition 3.5.25. [Seb90]

(i) *If there exists a fractional multiflow for $(\mathcal{M}', R', p')$, then* (3.5.24) *holds.*

(ii) *Suppose that $\mathcal{M}$ and $\mathcal{M}^*$ are both ∞-flowing. Then, there exists a fractional flow for $(\mathcal{M}', R', p')$ if and only if* (3.5.24) *holds.*

Proof. (i) Let $w \in \mathrm{CYC}(\mathcal{M}^*)$. Extend w to $w' \in \mathbb{Z}^{E'}$ by setting $w'(t) := \min(w(C \triangle \Sigma) \mid C$ cycle in $\mathcal{M})$. One can easily check that $w' \in \mathrm{MET}((\mathcal{M}')^*)$. As there exists a fractional flow for $(\mathcal{M}', R', p')$ we obtain that $\sum_{e \in E \setminus R} p_w w_e \geq \sum_{e \in R} p_e w_e + w'(t)$, by by applying the Japanese theorem This shows that (3.5.24) holds, as $w'(t) \geq$ max-size of a w-packing in $\mathcal{O}(\mathcal{M}^*, \Sigma)$.
(ii) As $\mathcal{M}^*$ is ∞-flowing we know from Lemma 3.5.10 that $\mathcal{O}(\mathcal{M}^*, \Sigma)$ has the $\mathbb{Z}_+$-MFMC property, i.e., the left hand side in the inequality from (3.5.24) is equal to $\min(w(C \triangle \Sigma) \mid C$ cycle in $\mathcal{M})$. As $\mathcal{M}$ is ∞-flowing we also have that $\mathrm{CYC}(\mathcal{M}^*) = \mathrm{MET}(\mathcal{M}^*)$. Let $p \in \mathbb{Z}_+^E$ for which (3.5.24) holds. We show the existence of a fractional flow by checking that the distance condition from the Japanese theorem holds. For this, let $w' \in \mathrm{MET}((\mathcal{M}')^*)$; hence, its projection w on E belongs to $\mathrm{MET}(\mathcal{M}^*)$. Using (3.5.24) with w we obtain that

$$\sum_{e \in E \setminus R} w_e p_e - \sum_{e \in R} w_e p_e \geq \min(w(C \triangle \Sigma) \mid C \text{ cycle in } \mathcal{M}) \geq w'(t),$$

which yields that $\sum_{e \in E \setminus R} w'_e p'_e - \sum_{e \in R} w'_e p'_e \geq 0$. ∎

It is an open question to determine the matroids for which the existence of a fractional flow in their Eulerian extension implies the existence of an integer flow. Sebö [Seb90] conjectures that this holds for the graphic matroids of planar graphs (for them, (3.5.24) ensures the existence of a fractional flow in the Eulerian extension).

3.6 Approximating Multicommodity Flows Via Cuts

Let $G = (V, E)$ be a graph on n nodes. We supose that G is undirected. For each edge $e \in E$, we are given a capacity $C_e \geq 0$. There are k commodity pairs (s_h, t_h) of specified source-sink nodes. For each commodity pair (s_h, t_h), a demand D_h is given. A classical problem is to find what is the maximum flow that can be sent so as not to exceed the capacities and to satisfy at best the demands.

More precisely, we consider the following multiflow problem, also known as the **concurrent flow problem**: *Determine what is the maximum scalar λ^* such that it is possible to satisfy simultaneously the fraction $\lambda^* D_h$ of demand for each commodity pair (s_h, t_h).* This maximum scalar λ^* is called the **max-flow**. It can be computed by solving a linear programming problem. Indeed, the concurrent flow problem can be formulated as a linear program in the following way. For each commodity pair (s_h, t_h), let $\mathcal{P}_h$ denote the set of paths connecting s_h to t_h in G. We introduce variables f_P for the paths $P \in \bigcup_{1 \leq h \leq k} \mathcal{P}_h$. Then, the max-flow λ^* is the optimum value of the following linear program:

$$
\begin{array}{lll}
\lambda^* = \max & \lambda & \\
& \sum_{P \in \mathcal{P}_h} f_P \geq \lambda D_h & (h = 1, \ldots, k) \\
(3.6.1) & \sum_{1 \leq h \leq k} \sum_{P \in \mathcal{P}_h | e \in P} f_P \leq C_e & (e \in E) \\
& f_P \geq 0 & (P \in \bigcup_{1 \leq h \leq k} \mathcal{P}_h)
\end{array}
$$

An upper bound for the max-flow can be easily formulated using cuts. For a subset $S \subseteq V$, let

$$cap(S) := \sum_{e \in \delta_G(S)} C_e$$

denote the **capacity of the cut** $\delta(S)$ and let

$$dem(S) := \sum D_h$$

denote its **demand**, where the sum is taken over all commodity pairs (s_h, t_h) that are separated by the partition $(S, V \setminus S)$.

LEMMA 3.6.2. $\lambda^* \leq \min_{S \subseteq V} \dfrac{cap(S)}{dem(S)}$.

PROOF. We use the formulation of λ^* from (3.6.1). let S be a subset of V. Then,

$$
\begin{aligned}
cap(S) = \textstyle\sum_{e \in \delta_G(S)} C_e & \geq \textstyle\sum_{e \in \delta_G(S)} \sum_{1 \leq h \leq k} \sum_{P \in \mathcal{P}_h | e \in P} f_P \\
& = \textstyle\sum_{1 \leq h \leq k} \sum_{P \in \mathcal{P}_h} \sum_{e \in \delta_G(S) \cap P} f_P \\
& \geq \textstyle\sum_{h | (s_h, t_h) \text{ is separated by } (S, V \setminus S)} \sum_{P \in \mathcal{P}_h} f_P \\
& \geq \textstyle\sum_{h | (s_h, t_h) \text{ is separated by } (S, V \setminus S)} \lambda^* D_h \\
& = \lambda^* dem(S).
\end{aligned}
$$

∎

If S realizes the minimum of $\min_{S \subseteq V} \frac{cap(S)}{dem(S)}$, then the quantity $\frac{cap(S)}{dem(S)}$ is called the **min-cut**. The inequality from Lemma 3.6.2 is, in general, a strict inequality. However, it is an equality in some special cases. In particular, it is an equality for the cases $k = 1, 2$ of one or two commodity pairs. In the case $k = 1$, this follows from the well known max-flow min-cut theorem by Ford and Fulkerson [FF62]. In the case $k = 2$ this is the max-flow min-cut theorem for two commodities, proved by Hu [Hu63]. These two results will be derived from some properties of ℓ_1-embeddings; see Corollary 3.6.7 below.

For an arbitrary number k of commodity pairs, the max-flow is in general less than the min-cut. However, the gap between the max-flow and the min-cut cannot be too big. The first result in this direction is due to Leighton and Rao [LR88], who proved that the ratio

$$\rho := \frac{\min_{S\subseteq V} \frac{cap(S)}{dem(S)}}{\lambda^*}$$

is less than or equal to $O(\log n)$ in the case when all demands are one and there is a commodity between any pair of vertices, i.e., $k = \binom{n}{2}$. Several improvements of this result have been proposed in [KARR90], [PT93], [GVY93]. The best result is due to Linial, London and Rabinovitch [LLR94] (and, independently, to Aumann and Rabani [AR94]); they proved that $\rho \leq O(\log k)$ for arbitrary demands and capacities and an arbitrary number k of commodities. Moreover, the bound is tight (see [LR88]). This bound is based on the results of Section **??** (more precisely, on Theorems **??** and **??**), which state that every semimetric on n points can be embedded into some ℓ_1-space with distortion $O(\log n)$.

THEOREM 3.6.3. [LLR94] *There exists a subset $S \subseteq V$ such that $\frac{cap(S)}{dem(S)} \leq O(\log k) \cdot \lambda^*$. Moreover, such a set can be found in random polynomial time.*

PROOF. We remind that λ^* is the optimum value of the linear program (3.6.1). Applying the linear programming duality theorem, we obtain that

$$\begin{array}{lll} \lambda^* = \min & \sum_{e\in E} C_e z_e & \\ & \sum_{e\in P} z_e \geq y_h & (P \in \mathcal{P}_h, h = 1, \ldots, k) \\ & \sum_{1\leq h\leq k} D_h y_h \geq 1 & \\ & y_h \geq 0 & (h = 1, \ldots, k) \\ & z_e \geq 0 & (e \in E) \end{array} \tag{3.6.4}$$

Let (y, z) be an optimum solution to the program (3.6.4). We can suppose that equality holds in the constraint $\sum_{1\leq h\leq k} D_h y_h \geq 1$ (if not, replace y by $y' := (\sum_{1\leq h\leq k} D_h y_h)^{-1} y$). Moreover, we can suppose that $y_h = \min(\sum_{e\in P} z_e \mid P \in \mathcal{P}_h)$ for every $h = 1, \ldots, k$ and that $z_e \leq \sum_{e'\in P} z_{e'}$ for every edge $e = (i, j)$ of G and every path P from i to j in G. Let d_z denote the path metric of the weighted graph (G, z) where, for $i, j \in V$, $d_z(i, j)$ is defined as the smallest value of $\sum_{e\in P} z_e$ for P path from i to j in G. Any semimetric on V of the form d_z for some $z \in \mathbb{R}^E_+$ is called a **semimetric based on** G. Then, λ^* can be reformulated as

$$\lambda^* = \min_{d \text{ semimetric based on } G} \frac{\sum_{e=(i,j)\in E} C_e d(i, j)}{\sum_{1\leq h\leq k} D_h d(s_h, t_h)}. \tag{3.6.5}$$

Let d be a semimetric based on G. If d happens to be a cut semimetric $\delta(S)$, then

$$\frac{\sum_{e=(i,j)\in E} C_e d(i, j)}{\sum_{1\leq h\leq k} D_h d(s_h, t_h)} = \frac{cap(S)}{dem(S)}.$$

More generally, if d is isometrically ℓ_1-embeddable, then $d = \sum_S \lambda_S \delta(S)$ for some $\lambda_S \geq 0$, which implies that

$$\frac{\sum_{e=(i,j)\in E} C_e d(i,j)}{\sum_{1\leq h\leq k} D_h d(s_h,t_h)} = \frac{\sum_S \lambda_S cap(S)}{\sum_S \lambda_S dem(S)} \geq \min_S \frac{cap(S)}{dem(S)}.$$

In general, d is not isometrically ℓ_1-embeddable but, by Theorem **??**, there exist vectors $w_i \in \mathbb{R}^K$ $(i \in V)$ such that

$$\text{(3.6.6)} \qquad \begin{cases} \| w_i - w_j \|_1 \leq d(i,j) & \text{for all } i,j \in V, \\ d(s_h,t_h) \leq O(\log k) \| w_{s_h} - w_{t_h} \|_1 & \text{for } h = 1,\ldots,k. \end{cases}$$

(We apply Theorem **??** with $X = V$ and $Y = \{s_h, t_h \mid h = 1,\ldots,k\}$.) Therefore,

$$\frac{\sum_{e=(i,j)\in E} C_e d(i,j)}{\sum_{1\leq h\leq k} D_h d(s_h,t_h)} \geq \frac{\sum_{e=(i,j)\in E} C_e \| w_i - w_j \|_1}{O(\log k) \sum_{1\leq h\leq k} D_h \| w_{s_h} - w_{t_h} \|_1} \geq \frac{1}{O(\log k)} \min_S \frac{cap(S)}{dem(S)}$$

(using the argument above). This shows that $\lambda^* \geq \frac{1}{O(\log k)} \min_S \frac{cap(S)}{dem(S)}$.

A subset S realizing $\frac{cap(S)}{dem(S)} \leq O(\log k)\lambda^*$ can be found in the following way. Solve the linear program: $\lambda^* = \min(\sum_{e\in E} C_e z_e \mid \sum_{1\leq h\leq k} D_h d_z(s_h,t_h) = 1, z \geq 0)$, let z be an optimum solution and $d := d_z$. Find vectors $w_i \in \mathbb{R}^K$ $(i \in V)$ satisfying (3.6.6) (this can be done in random polynomial time; see [LLR94]). Determine the coordinate index $r \in K$ for which $\frac{\sum_{e=(i,j)\in E} C_e |w_i(r) - w_j(r)|}{\sum_{1\leq h\leq k} D_h |w_{s_h}(r) - w_{t_h}(r)|}$ is minimum. The semimetric $d^{(r)}$ on V defined by $d^{(r)}(i,j) := |w_i(r) - w_j(r)|$ for $i,j \in V$ is ℓ_1-embeddable by construction. Hence, $d^{(r)} = \sum_{A\in\mathcal{A}} \lambda_A \delta(A)$ where $\lambda_A > 0$ for $A \in \mathcal{A}$. Such a decomposition can be easily found; recall, e.g., Proposition **??** and its proof. Then, a set A realizing $\min_{A\in\mathcal{A}} \frac{cap(A)}{dem(A)}$ satisfies: $\frac{cap(A)}{dem(A)} \leq O(\log k)\lambda^*$. ∎

COROLLARY 3.6.7. *For the case of $k \leq 2$ commodity pairs, max-flow = min-cut, i.e.,* $\lambda^* = \min_{S\subseteq V} \frac{cap(S)}{dem(S)}$.

PROOF. We use the proof of Theorem 3.6.3. Let d be a semimetric based on G which is an optimum solution for the program (3.6.5). Suppose first that $k = 1$. Define the semimetric d' on V by setting $d'(i,j) := |d(i,s_1) - d(j,s_1)|$ for $i,j \in V$. Then, $d(i,j) \geq d'(i,j)$ for all $i,j \in V$ and $d'(s_1,t_1) = d(s_1,t_1)$. Hence, $\lambda^* = \frac{\sum_{e=(i,j)\in E} C_e d(i,j)}{\sum_{1\leq h\leq k} D_h d(s_h,t_h)} \geq \frac{\sum_{e=(i,j)\in E} C_e d'(i,j)}{\sum_{1\leq h\leq k} D_h d'(s_h,t_h)}$. As d' is isometrically ℓ_1-embeddable (by construction), the latter quantity is greater than or equal to $\min_S \frac{cap(S)}{dem(S)}$. This shows that $\lambda^* = \min_S \frac{cap(S)}{dem(S)}$. Suppose now that $k = 2$. We define a semimetric d' on V by setting $d'(i,j) := \max(|d(i,s_1) - d(j,s_1)|, |d(i,s_2) - d(j,s_2)|)$ for $i,j \in V$. Hence, d' is isometrically ℓ_∞^2-embeddable (by construction), $d'(i,j) \leq d(i,j)$ for all $i,j \in V$, and $d'(s_h,t_h) = d(s_h,t_h)$ for $h = 1,2$. We can now conclude the proof as in the previous case, because the semimetric d' is again ℓ_1-embeddable. To see it, observe that ℓ_∞^2 is an isometric subspace of ℓ_1^2; indeed, one can easily check that the mapping $(x_1,x_2) \mapsto (\frac{x_1+x_2}{2}, \frac{x_1-x_2}{2})$ provides an isometric embedding of $(\mathbb{R}^2, d_{\ell_\infty})$ into $(\mathbb{R}^2, d_{\ell_1})$. ∎

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[1]Boldface numbers refer to the pages where items are introduced.