POLYNOMIAL INSTANCES OF THE POSITIVE SEMIDEFINITE AND EUCLIDEAN DISTANCE MATRIX COMPLETION PROBLEMS

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Abstract. Given an undirected graph $G = (V, E)$ with node set $V = [1, n]$, a subset $S \subseteq V$ and a rational vector $a \in \mathbf{Q}^{S \cup E}$, the positive semidefinite matrix completion problem consists of determining whether there exists a real symmetric $n \times n$ positive semidefinite matrix $X = (x_{ij})$ satisfying: $x_{ii} = a_i$ ($i \in S$) and $x_{ij} = a_{ij}$ ($ij \in E$). Similarly, the Euclidean distance matrix completion problem asks for the existence of a Euclidean distance matrix completing a partially defined given matrix. It is not known whether these problems belong to NP. We show here that they can be solved in polynomial time when restricted to the graphs having a fixed minimum fill-in; the minimum fill-in of graph G being the minimum number of edges needed to be added to G in order to obtain a chordal graph. A simple combinatorial algorithm permits to construct a completion in polynomial time in the chordal case. We also show that the completion problem is polynomially solvable for a class of graphs including wheels of fixed length (assuming all diagonal entries are specified). The running time of our algorithms is polynomially bounded in terms of n and the bitlength of the input a. We also observe that the matrix completion problem can be solved in polynomial time in the real number model for the class of graphs containing no homeomorph of K_4.

Key words. positive semidefinite matrix, Euclidean distance matrix, matrix completion, chordal graph, minimum fill-in, order of a graph, polynomial algorithm, bit model, real number model

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1. Introduction.

1.1. The matrix completion problem. This paper is concerned with the completion problem for positive semidefinite and Euclidean distance matrices. The *positive semidefinite matrix completion problem* **(P)** is defined as follows:

Given a graph $G = (V, E)$, a subset $S \subseteq V$ and a rational vector $a \in \mathbf{Q}^{S \cup E}$, determine whether there exists a real matrix $X = (x_{ij})_{i,j \in V}$ satisfying:

$$X \succeq 0 \text{ and } x_{ii} = a_i \ (i \in S), \ x_{ij} = a_{ij} \ (ij \in E). \tag{1.1}$$

(The notation $X \succeq 0$ means that X is a symmetric positive semidefinite matrix or, for short, a psd matrix.) In words, problem (P) asks whether a partially specified matrix can be completed to a psd matrix; the terminology of graphs being used as a convenient tool for encoding the positions of the specified entries. When problem (P) has a positive answer, one says that a is *completable to a psd matrix*; a matrix X satisfying (1.1) is called a *psd completion* of a and a *positive definite (pd) completion* when X is positive definite. We let $(\mathbf{P}_s)$ denote problem (P) when $S = V$, i.e., when all diagonal entries are specified. If one looks for a pd completion then one can assume without loss of generality that all diagonal entries are specified (cf. Lemma 2.5); this is however not obviously so if one looks for a psd completion (although this can be shown to be true when restricting the problem to the class of chordal graphs; cf. the proof of Theorem 3.5).

A matrix $Y = (y_{ij})_{i,j=1}^n$ is called a *Euclidean distance matrix* (a *distance matrix*, for short) if there exist vectors $u_1, \ldots, u_n \in \mathbf{R}^k$ (for some $k \geq 1$) such that

$$y_{ij} = \|u_i - u_j\|^2 \text{ for } i, j = 1, \ldots, n. \tag{1.2}$$

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(Here, $\|u\|$ denotes the Euclidean norm of vector $u \in \mathbf{R}^k$.) A set of vectors u_i satisfying (1.2) is called a *realization* of Y. Note that all diagonal entries of a distance matrix are equal to zero. The *Euclidean distance matrix completion problem* **(D)** is defined as follows:

Given a graph $G = (V, E)$ and a rational vector $d \in \mathbf{Q}^E$, determine whether there exists a real matrix $Y = (y_{ij})_{i,j \in V}$ satisfying:

$$Y \text{ is a distance matrix and } y_{ij} = d_{ij} (ij \in E). \tag{1.3}$$

Hence problem (D) asks whether a partially specified matrix can be completed to a distance matrix.

As will be recalled in Section 2.3, psd matrices and distance matrices are closely related and, thus, their associated completion problems can often be treated in an analogous manner. These matrix completion problems have many applications, e.g., to multidimensional scaling problems in statistics (cf. [28]), to the molecule conformation problem in chemistry (cf. [10], [17]), to moment problems in analysis (cf. [5]).

1.2. An excursion to semidefinite programming. The psd matrix completion problem is obviously an instance of the general *semidefinite programming feasibility problem* **(F)**:

Given integral $n \times n$ symmetric matrices $Q_0, Q_1, \ldots, Q_m$, determine whether there exist real numbers $z_1, \ldots, z_m$ satisfying:

$$Q_0 + z_1 Q_1 + \ldots + z_m Q_m \succeq 0. \tag{1.4}$$

The complexity status of problem (F) is a fundamental open question in the theory of semidefinite programming; this is true for both the Turing machine model and the real number model, the two most popular models of computation used in complexity theory. In particular, it is not known whether there exists an algorithm solving (F) whose running time is polynomial in the size L of the data, that is, the total space needed to store the entries of the matrices $Q_0, \ldots, Q_m$.

The *Turing machine model* (also called rational number model, or *bit model*; cf. [12]) works on rational numbers and, more precisely, on their binary representations; in particular, the running time of an elementary operation $(+, -, \times, \div)$ depends on the length of the binary representations of the rational numbers involved. Hence, the size L of the data of problem (F) in this model can be defined as $mn^2 L_0$, where L_0 is the maximum number of bits needed to encode an entry of a matrix Q_i. On the other hand, the *real number model* (introduced in [9]) works with real numbers and it assumes that exact real arithmetic can be performed; in particular, an elementary operation $(+, -, \times, \div)$ between any two real numbers takes unit time. Hence, the size L of the data of (F) in this model is equal to mn^2.

Semidefinite programming (SDP) deals with the decision problem (F) and its optimization version:

$$\begin{array}{ll} \max & c^T z \\ \text{s.t.} & Q_0 + z_1 Q_1 + \ldots + z_m Q_m \succeq 0 \end{array} \tag{1.5}$$

where $c \in \mathbf{Q}^m$. SDP can be seen as a generalization of linear programming (LP), obtained by replacing the nonnegativity constraints of the vector variable in LP by the semidefiniteness of the matrix variable in SDP. Information about SDP can be found in the handbook

[40]; cf. also the survey [38], and [3, 16] with an emphasis on applications to discrete optimization.

A standard result in LP is that every feasible linear system: $Ax \leq b$ with rational coefficients has a solution whose size is polynomially bounded in terms of the size of A and b (cf. [36], corollary 3.2b). This implies that the problem of testing feasibility of an LP program belongs to NP in the bit model (this fact is obvious for the real number model). Moreover, any LP optimization problem can be solved in polynomial time in the bit model using the ellipsoid algorithm of Khachiyan [22] or the interior-point method of Karmarkar [21]; it is an open question whether LP can be solved in polynomial time in the real number model (cf. p. 60 in [41]).

The feasibility problem (F) belongs to NP in the real number model (since one can test in polynomial time whether a matrix is psd, for instance, using Gaussian elimination; in fact, for a rational matrix the running time is polynomial in its bitlength (cf. p. 295 in [15])). However, it is not known whether problem (F) belongs to NP in the bit model. Indeed, in contrast with LP, it is not true that if a solution exists then one exists which is rational and has a polynomially bounded size. Consider, for instance, the following matrix:

$$X := \begin{pmatrix} 2x & 2 & 0 & 0 \\ 2 & x & 0 & 0 \\ 0 & 0 & 2 & x \\ 0 & 0 & x & 1 \end{pmatrix}. \tag{1.6}$$

Then, $x = \sqrt{2}$ is the unique real for which $X \succeq 0$; hence, this is an instance where there is a real solution but no rational solution. Consider now the following matrix (taken from [33]):

$$X = \left(\begin{array}{c|cc|c|cc|c|cc} x_1 - 2 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 1 & x_1 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & x_1 & x_2 & \dots & 0 & 0 & \dots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ \hline 0 & 0 & 0 & \dots & 1 & x_i & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & x_i & x_{i+1} & \dots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & \dots & 0 & 0 & \dots & 1 & x_{n-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & x_{n-1} & x_n \end{array}\right).$$

Then, $X \succeq 0$ if and only if $x_1 \geq 2$ and $x_{i+1} \geq x_i^2$ for $i = 1, \dots, n-1$; hence, $x_n \geq 2^{2^{n-1}}$ and thus any rational solution has exponential bitlength. More examples of 'ill-conditioned' sdp's can be found in [33].

However, Ramana [33] has developed an exact duality theory for SDP which enables him to show the following results: Problem (F) belongs to NP $\cap$ co-NP in the real number model. In the bit model, (F) belongs to NP if and only if it belongs to co-NP; hence, (F) is not NP-complete nor co-NP complete unless NP=co-NP.

Algorithms have been found that permit to solve the optimization problem (1.5) *approximatively* in polynomial time; they are based on the ellipsoid method (cf. [15]) and interior-point methods (cf. [31], [3]).

More precisely, set $K := \{z \in \mathbf{R}^m \mid Q_0 + \sum_{i=1}^m z_i Q_i \succeq 0\}$ and, given $\epsilon > 0$, set $S(K, \epsilon) := \{y \mid \exists z \in K \text{ with } \|z - y\| < \epsilon\}$ ('the points that are in the ϵ-neighborhood of K') and $S(K, -\epsilon) := \mathbf{R}^m \setminus S(\mathbf{R}^m \setminus K, \epsilon)$ ('the points that are at distance at least ϵ from the border of K'). let L denote the maximum bit size of the entries of the matrices $Q_0, \ldots, Q_m$. Assume that we know a constant $R > 0$ such that either $K = \emptyset$ or $\exists\ z \in K$ with $\|z\| \leq R$. Then, the ellipsoid based algorithm, given rational $\epsilon > 0$, either finds $y \in S(K, \epsilon)$ for which $c^T z \leq c^T y + \epsilon$ for all $z \in S(K, -\epsilon)$, or asserts that $S(K, -\epsilon) = \emptyset$. Its running time is polynomial in n, m, L and $\log \epsilon$ and this algorithm is polynomial in the bit model.

Assume that we know a constant $R > 0$ such that $\|z\| \leq R$ for all $z \in K$ and a point $z^* \in K$ for which $Q_0 + \sum_{i=1}^m z_i^* Q_i \succ 0$ (z^* is 'strictly feasible'). There is an interior-point algorithm which finds $y \in K$ strictly feasible such that $c^T y \geq \max_{z \in K} c^T z - \epsilon$ in time polynomial in $n, m, L, \log \epsilon, \log R$ and the bitlength of z^*. Note, however, that no polynomial bound has been established for the bitlengths of the intermediate numbers occurring in the algorithm.

Khachiyan and Porkolab have shown that problem (F) and its analogue in rational numbers can be solved in polynomial time in the bit model for a fixed number m of variables.

THEOREM 1.1.

(i) [32] *Problem (F) can be solved in polynomial time for any fixed m.*

(ii) [23] *The following problem can be solved in polynomial time for any fixed m:*
Given $n \times n$ integral symmetric matrices $Q_0, Q_1, \ldots, Q_m$, find rational numbers $z_1, \ldots, z_m$ satisfying (1.4) *or determine that no such numbers exist.* □

The result from Theorem 1.1 (ii) extends to the context of semidefinite programming the result of Lenstra [29] on the polynomial solvability of integer LP in fixed dimension.

1.3. Back to the matrix completion problem. As the matrix completion problem is a special instance of SDP, it can be solved *approximatively* in polynomial time; specific interior-point algorithms for finding approximate psd and distance matrix completions have been developed, e.g., in [19],[10],[2],[30]. However, such algorithms are not garanteed to find *exact* completions in polynomial time. This motivates our study in this paper of some classes of matrix completion problems that can be solved exactly in polynomial time.

As mentioned earlier, one of the difficulties in the complexity analysis of SDP arises from the fact that a rational SDP problem might have no rational solution (recall the example from (1.6)). This raises the following question in the context of matrix completion:

If a rational partial matrix has a psd completion, does a rational completion always exist ?

We do not know of a counterexample to this statement. On the other hand, we will show that the answer is positive, e.g., when the graph of specified entries is chordal or has minimum fill-in 1 (cf. Lemma 4.2). (Note that the answer is obviously positive if a pd completion exists.)

Motivated by the above discussion, let us define for each of the problems (P) and (D) its rational analogue (P^{Q}) and (D^{Q}). Problem **($\mathbf{P}^{\mathrm{Q}}$)** is defined as follows:

Given a graph $G = (V, E)$, a subset $S \subseteq V$ and a rational vector $a \in \mathbf{Q}^{S \cup E}$, find a rational matrix X satisfying (1.1) *or determine that no such matrix exists.*

When $S = V$ (i.e., all diagonal entries are specified), we denote the problem as $(\mathbf{P}_s^{\mathrm{Q}})$. Problem $(\mathbf{D}^{\mathrm{Q}})$ is defined as follows:

Given a graph $G = (V, E)$ and a rational vector $d \in \mathbf{Q}^E$, find a rational matrix Y satisfying (1.3) or determine that no such matrix exists.

The complexity of the problems (P), (D), $(\mathrm{P}^{\mathrm{Q}})$, and $(\mathrm{D}^{\mathrm{Q}})$ is not known; in particular, it is not known whether they belong to NP in the bit model (they do trivially in the real number model). In this paper, we present some instances of graphs for which the completion problems can be solved in polynomial time. All our complexity results apply for the bit model (unless otherwise specified, as in Section 5.3).

Recall that a graph is said to be *chordal* if it does not contain a circuit of length ≥ 4 as an induced subgraph. Then, the *minimum fill-in* of graph G is defined as the minimum number of edges needed to be added to G in order to obtain a chordal graph. Note that computing the minimum fill-in of a graph is an NP-hard problem [42]. The following is the main result of Sections 3 and 4.

THEOREM 1.2. *For any integer $m \geq 0$, problems (P), (P^{Q}), (D) and (D^{Q}) can be solved in polynomial time (in the bit model) when restricted to the class of graphs whose minimum fill-in is equal to m.*

The essential ingredients in the proof of Theorem 1.2 are the subcase $m = 0$ (chordal case), Theorem 1.1, and the link (exposed in Section 2.3) between psd matrices and distance matrices. In the chordal case, a simple combinatorial algorithm permits to solve the completion problem in polynomial time.

The psd matrix completion problem for chordal graphs has been extensively studied in the literature (cf. the survey of Johnson [18] for detailed references). In some sense, this problem has been solved by Grone, Johnson, Sá and Wolkowicz [14] who, building upon a result of Dym and Gohberg [11], have characterized when a vector a indexed by the nodes and edges of a chordal graph admits a psd completion; cf. Theorem 3.1. From this follows the polynomial time solvability of problem (P_s) for chordal graphs. In fact, the result from Theorem 3.1 is proved in [14] in a constructive manner and, thus, yields an algorithm permitting to solve problem $(\mathrm{P}_s^{\mathrm{Q}})$ for chordal graphs. This algorithm has a polynomial running time in the real number model; however, it has to be modified in order to achieve a polynomial running time in the bit model.

To summarize, the result from Theorem 1.2 also holds in the real number model for chordal graphs ($m = 0$); it would hold for all graphs having fixed minimum fill-in $m \geq 1$ if the result from Theorem 1.1 would remain valid in the real number model[1].

We present in Section 5.1 another class of graphs for which the matrix completion problem (P_s) can be solved in polynomial time (in the bit model). This class contains (generalized) circuits and wheels having a fixed length (and fatness); these graphs arise naturally when considering the polar approach to the psd matrix completion problem. Then, Section 5.2 contains a brief description of this polar approach, together with some open questions and remarks. In the final Section 5.3, we consider the matrix completion problem for the class of graphs containing no homeomorph of K_4 (it contains circuits). Then a condition characterizing existence of a psd or distance matrix completion exists which permits to obtain a simple combinatorial algorithm solving the existence and construction problems in polynomial time in the real number model.

[1] L. Porkolab (private communication, 2000) claims to have a proof of this fact.

2. Preliminaries. We recall here some basic facts about Schur complements and Euclidean distance matrices that will be needed in the paper and we make a few observations about psd completions.

2.1. Schur complements. For a symmetric matrix M, set $\mathrm{In}(M) := (p, q, r)$, where p (resp. q, r) denotes the number of positive (resp. negative, zero) eigenvalues of M. When $M \succeq 0$, a maximal nonsingular principal submatrix of M is a nonsingular principal submatrix of M of largest possible order, thus equal to the rank of M.

LEMMA 2.1. *Let* $M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ *be a symmetric matrix, where* A *is nonsingular. Then,*

$$\mathrm{In}(M) = \mathrm{In}(A) + \mathrm{In}(C - B^T A^{-1} B);$$

the matrix $C - B^T A^{-1} B$ *is known as the* Schur complement *of* A *in* M*. In particular,* $M \succeq 0$ *if and only if* $A \succeq 0$ *and* $C - B^T A^{-1} B \succeq 0$*. Moreover, if* $M \succeq 0$ *and if* A *is a maximal nonsingular principal submatrix of* M*, then* $C = B^T A^{-1} B$. $\square$

As a direct application, we have the following results which will be used at several occasions in the paper.

LEMMA 2.2. *Let* X *be a symmetric matrix having the following block decomposition:*

$$X = \begin{array}{c} \\ \ell \\ n \\ m \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \ell & n & m \end{array} \\ \begin{pmatrix} T & R^T & Z^T \\ R & A & S \\ Z & S^T & D \end{pmatrix} \end{array}, \tag{2.1}$$

where T, R, Z, A, S, D *are rational matrices of suitable orders; all entries of* X *being specified except those of* Z *that have to be determined in order to obtain* $X \succeq 0$*. Assume that*

$$X_1 := \begin{pmatrix} T & R^T \\ R & A \end{pmatrix} \succeq 0, \; X_2 := \begin{pmatrix} A & S \\ S^T & D \end{pmatrix} \succeq 0.$$

In the case when $n \geq 1$ *and* $A \neq 0$*, let* A_0 *be a maximal nonsingular principal submatrix of* A*, let*

$$A = \begin{pmatrix} A_0 & B \\ B^T & C \end{pmatrix}, \; X = \begin{pmatrix} T & R_0^T & R_1^T & Z^T \\ R_0 & A_0 & B & S_0 \\ R_1 & B^T & C & S_1 \\ Z & S_0^T & S_1^T & D \end{pmatrix}$$

denote the corresponding block decompositions of A *and* X*. Then,* $X \succeq 0$ *if we set*

$$Z := S_0^T A_0^{-1} R_0 \tag{2.2}$$

when $n \geq 1$ *and* $A \neq 0$*, and* $Z := 0$ *otherwise.*

Proof. The result follows using Lemma 2.1 after noting that the Schur complement of A_0 in X is given by

$$\begin{pmatrix} T & R_1^T & Z^T \\ R_1 & C & S_1 \\ Z & S_1^T & D \end{pmatrix} - \begin{pmatrix} R_0^T \\ B^T \\ S_0^T \end{pmatrix} A_0^{-1} \begin{pmatrix} R_0 & B & S_0 \end{pmatrix}$$

$$= \begin{pmatrix} T - R_0^T A_0^{-1} R_0 & R_1^T - R_0^T A_0^{-1} B & Z^T - R_0^T A_0^{-1} S_0 \\ R_1 - B^T A_0^{-1} R_0 & C - B^T A_0^{-1} B & S_1 - B^T A_0^{-1} S_0 \\ Z - S_0^T A_0^{-1} R_0 & S_1^T - S_0^T A_0^{-1} B & D - S_0^T A_0^{-1} S_0 \end{pmatrix}$$

$$= \begin{pmatrix} T - R_0^T A_0^{-1} R_0 & 0 & Z^T - R_0^T A_0^{-1} S_0 \\ 0 & 0 & 0 \\ Z - S_0^T A_0^{-1} R_0 & 0 & D - S_0^T A_0^{-1} S_0 \end{pmatrix}.$$

Indeed, the Schur complement $C - B^T A_0^{-1} B$ of A_0 in A is equal to 0 since $A \succeq 0$ and A_0 is a maximal nonsingular principal submatrix of A; as $X_1, X_2 \succeq 0$ this implies that $R_1 - B^T A_0^{-1} R_0 = S_1 - B^T A_0^{-1} S_0 = 0$. □

LEMMA 2.3. *Let X be a symmetric matrix of the form*

$$X = \begin{pmatrix} T & R^T \\ R & A \end{pmatrix}$$

where $A \succeq 0$ and T is a symmetric matrix of order ℓ whose diagonal entries are all equal to some scalar N. Let A_0 be a maximal nonsingular principal submatrix of A and let

$$A = \begin{pmatrix} A_0 & B \\ B^T & B^T A_0^{-1} B \end{pmatrix}, \quad X = \begin{pmatrix} T & R_0^T & R_1^T \\ R_0 & A_0 & B \\ R_1 & B^T & B^T A_0^{-1} B \end{pmatrix}$$

denote the corresponding block decompositions of A and X. Then, $X \succeq 0$ if and only if (i) $R_1 = B^T A_0^{-1} R_0$ and (ii) $T - R_0^T A_0^{-1} R_0 \succeq 0$. In particular, X is pd if and only if A and $T - R^T A^{-1} R$ are pd. Moreover, $T - R_0^T A_0^{-1} R_0$ is psd for N large enough (namely, for N greater or equal to the largest eigenvalue of $R_0^T A_0^{-1} R_0 - T_0$, where T_0 has zero diagonal entries and as off-diagonal entries those of T). □

2.2. Some observations about positive semidefinite completions. Given a graph $G = (V, E)$, a subset $S \subseteq V$, a vector $a \in \mathbf{Q}^{S \cup E}$, and a scalar $N > 0$, let $a^N \in \mathbf{Q}^{V \cup E}$ denote the extension of a obtained by setting $a_i := N$ for all $i \in V \setminus S$. Obviously,

LEMMA 2.4. *a is completable to a psd matrix if and only if a^N is completable to a psd matrix for some $N > 0$ (and then for all $N' \geq N$).* □

Therefore, if one can "guess" a value N to assign to the unspecified diagonal entries, then one can reduce the problem to the case when all diagonal entries are specified. This can be done when the graph G of specified off-diagonal entries is chordal as we see later or if we look for a pd completion as the next result shows.

LEMMA 2.5. *Given $a \in \mathbf{Q}^{S \cup E}$, let $b := (a_i (i \in S), a_{ij} (ij \in E,\ i, j \in S))$ denote its restriction to the subgraph induced by S. Then, a has a pd completion if and only if b has a pd completion.*

Proof. Apply Lemma 2.3. □

This result does not extend to psd completions (which contradicts a claim from [14] (psd case in Prop. 1)). Indeed, the following partial matrix

$$\begin{pmatrix} ? & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

has no psd completion while its lower principal 2×2 submatrix is psd.

A final observation is that if a partial matrix contains a fully specified row, then the completion problem can be reduced to considering a matrix of smaller order. Indeed, suppose that $A = (a_{ij})$ is a partial symmetric matrix whose first row is fully specified. If $a_{11} < 0$ then A is not completable. If $a_{11} = 0$ then A is completable if and only if its first row is identically zero and its lower principal submatrix of order $n-1$ is completable. If $a_{11} > 0$ then one can reduce to a problem of order $n-1$ by considering the Schur complement of a_{11} in A.

2.3. Euclidean distance matrices. The following connection (2.4) between psd and distance matrices has been established by Schoenberg [35]. Let $Y = (y_{ij})_{i,j\in V}$ be a square symmetric matrix with zeros on its main diagonal and whose rows and columns are indexed by a set V, and let i_0 be a given element of V. Then, $\varphi_{i_0}(Y)$ denotes the square symmetric matrix $X = (x_{ij})_{i,j\in V\setminus\{i_0\}}$ whose rows and columns are indexed by set $V \setminus \{i_0\}$ and whose entries are given by

$$x_{ij} = \frac{1}{2}(y_{i_0 i} + y_{i_0 j} - y_{ij}) \text{ for } i,j \in V \setminus \{i_0\}. \tag{2.3}$$

Then, one can easily verify that

$$Y \text{ is a distance matrix } \Longleftrightarrow \varphi_{i_0}(Y) \succeq 0. \tag{2.4}$$

(Indeed, a set of vectors u_i $(i \in V)$ forms a realization of the matrix Y if and only if $\varphi_{i_0}(Y)$ is the Gram matrix of the vectors $u_i - u_{i_0}$ $(i \in V \setminus \{i_0\})$, which means that its (i,j)-th entry is equal to $(u_i - u_{i_0})^T(u_j - u_{i_0})$.) Thus, φ_{i_0} establishes a linear bijection between the set of distance matrices of order $|V|$ and the set of psd matrices of order $|V|-1$. Relation (2.4) has a direct consequence for the corresponding matrix completion problems. Let $G = (V,E)$ be a graph and assume that $i_0 \in V$ is a universal node, i.e., that i_0 is adjacent to all other nodes of G. Then, an algorithm permitting to solve the psd matrix completion problem for graph $G\backslash i_0$ can be used for solving the distance matrix completion problem for graph G and vice versa. Indeed,

$$\begin{aligned} &Y \text{ is a distance matrix completion of } d \in \mathbf{R}^E \\ &\Longleftrightarrow \varphi_{i_0}(Y) \text{ is a psd completion of } \varphi_{i_0}(d) \end{aligned} \tag{2.5}$$

(For the definition of $\varphi_{i_0}(d)$ use (2.3) restricted to the pairs ij with $i,j \in V \setminus \{i_0\}$, $i = j$, or $i \neq j$ with ij edge of G.) For more information about connections between the two problems, see [20],[26].

3. The Matrix Completion Problem for Chordal Graphs. We consider here the matrix completion problems for chordal graphs. First, we recall results from [14] and [4] yielding a good characterization for the existence of a completion; then, we see how they can be used for constructing a completion in polynomial time.

3.1. Characterizing existence of a completion. Let $G = (V,E)$ be a graph and let $a \in \mathbf{Q}^{V\cup E}$ be a vector; in the distance matrix case, the entries of a indexed by V (corresponding to the diagonal entries of a matrix completion) are assumed to be equal to zero. If $K \subseteq V$ is a clique in G (i.e., any two distinct nodes in K are joined by an edge in G), the entries a_{ij} of vector a are well-defined for all nodes $i,j \in K$; then, we let $a(K)$ denote the $|K| \times |K|$ symmetric matrix whose rows and columns are indexed by K and with ij-th entry a_{ij} for $i,j \in K$. Obviously, if a is completable to a psd matrix, then a satisfies:

$$a(K) \succeq 0 \text{ for every maximal clique } K \text{ in } G. \tag{3.1}$$

Similarly, if a is a completable to a distance matrix, then a satisfies:

$$a(K) \text{ is a distance matrix for every maximal clique } K \text{ in } G. \tag{3.2}$$

The conditions (3.1) and (3.2) are not sufficient in general for ensuring the existence of a completion. For instance, if $G = (V, E)$ is a circuit and $a \in \mathbf{Q}^{V \cup E}$ has all its entries equal to 1 except one entry on an edge equal to -1, then a satisfies (3.1) but a is not completable to a psd matrix. However, if G is a chordal graph, then (3.1) and (3.2) suffice for ensuring the existence of a completion.

THEOREM 3.1. *Let $G = (V, E)$ be a chordal graph and let $a \in \mathbf{R}^{V \cup E}$. If a satisfies* (3.1)*, then a is completable to a psd matrix* [14]*; if a satisfies* (3.2)*, then a is completable to a distance matrix* [4]*; moreover, if a is rational valued, then a admits a rational completion.* □

As the maximal cliques in a chordal graph can be enumerated in polynomial time [37] (cf. below) and as one can check positive semidefiniteness of a rational matrix in polynomial time (cf. [15], p. 295), one can verify whether (3.1) holds in polynomial time when G is chordal; in view of (2.4), one can also verify whether (3.2) holds in polynomial time when G is chordal. This implies:

THEOREM 3.2. *Problems (P_s) and (D) can be solved in polynomial time for chordal graphs.* □

The proof given in [14, 4] for Theorem 3.1 is constructive; thus, it provides an algorithm for constructing a completion and, as we see below, a variant of it can be shown to have a polynomial running time. The proof is based on the following properties of chordal graphs. Let $G = (V, E)$ be a graph.

Then, G is chordal if and only if it has a perfect elimination ordering; moreover, such an ordering can be found in polynomial time [34]. An ordering $v_1, \ldots, v_n$ of the nodes of a graph $G = (V, E)$ being called a *perfect elimination ordering* if, for every $j = 1, \ldots, n-1$, the set of nodes v_k with $k > j$ that are adjacent to v_j induces a clique in G. For $j = 1, \ldots, n-1$, let K_j denote the clique consisting of node v_j together with the nodes v_k ($k > j$) that are adjacent to v_j; then the cliques $K_1, \ldots, K_{n-1}$ comprise all maximal cliques of a chordal graph G.

Hence, if G is chordal and not a clique, then one can find (in polynomial time) an edge $e \notin E$ for which the graph $H := G + e$ (obtained by adding e to G) is chordal. (Indeed, let i be the largest index in $[1, n]$ for which there exists $j > i$ such that v_i and v_j are not adjacent in G; then we can choose for e the pair ij as $v_1, \ldots, v_n$ remains a perfect elimination ordering for H.)

Moreover, if G is chordal then, for any $e \notin E$, there exists a unique maximal clique in $G + e$ containing edge e [14] (easy to check).

Therefore, if G is complete and not a clique, we can order the missing edges in G as $e_1, \ldots, e_p$ in such a way that the graph $G_q := (V, E \cup \{e_1, \ldots, e_q\})$ is chordal for every $q = 1, \ldots, p$. For $q = 1, \ldots, p$, let K_q be the unique maximal clique in G_q containing edge e_q. Given $a \in \mathbf{Q}^{V \cup E}$ satisfying (3.1), set $G_0 := G$ and $x_0 := a$. We execute the following step for $q = 1, \ldots, p$.

Find $z_q \in \mathbf{Q}$ for which the vector $x_q := (x_{q-1}, z_q)$ of $\mathbf{Q}^{V \cup E(G_q)}$ satisfies:

$$x_q(K_q) \succeq 0. \tag{3.3}$$

This can be done in view of Lemma 2.2 (case $\ell = m = 1$) applied to the matrix $X := x_q(K_q)$ and one can choose for z_q the rational value given by (2.2). Then, the final vector

$x_p = (a, z_1, \ldots, z_p)$ provides a rational psd completion of a. This shows Theorem 3.1 in the psd case (the Euclidean distance matrix case being similar).

As mentioned earlier, the preprocessing step (find the suitable ordering $e_1, \ldots, e_p$ of the missing edges and the cliques K_q) can be done in polynomial time. Then, one can construct the values $z_1, \ldots, z_p$ yielding a psd completion of a in $p \leq n^2$ steps. Therefore, the algorithm is polynomial in the real number model. In order to show polynomiality in the bit model, one has to verify that the encoding sizes of $z_1, \ldots, z_p$ remain polynomially bounded in terms of n and the encoding size of a. This is, however, not clear. Indeed, both R_0 and S_0 in the definition of z_q via (2.2) may involve some previously defined z_h for $h < q$ (in fact, the same may hold for A_0); then, we have a quadratic dependence between z_q and the previously defined $z_1, \ldots, z_{q-1}$ which may cause a problem when trying to prove that the encoding size of z_q remains polynomially bounded. However, as we see below, the above algorithm can be modified to obtain a polynomial running time. The basic idea is that, instead of adding the missing edges one at a time, one adds them by 'packets' consisting of edges sharing a common endnode. Then, in view of Lemma 2.2, one can specify simultaneously all the entries on these edges, which permits to achieve a linear dependency among the z_q's.

3.2. Constructing a psd completion in polynomial time. Let $G = (V, E)$ be a chordal graph and let $1, \ldots, n$ denote a perfect elimination ordering of its nodes. For $i \in [1, n]$, set

$$J(i) := \{j \in [1, n] : \ j > i \ \text{ and } \ ij \notin E\}$$

and let $i_1 > \ldots > i_L$ denote the elements $i \in [1, n]$ for which $J(i) \neq \emptyset$. For $\ell = 1, \ldots, L$, set $F_\ell := \{i_\ell j \mid j \in J(i_\ell)\}$ and let G_ℓ denote the graph with node set V and edge set $E \cup F_1 \cup \ldots \cup F_\ell$. Hence, we have a sequence of graphs:

$$G_0 := G \subseteq G_1 \subseteq \ldots \subseteq G_\ell \subseteq \ldots \subseteq G_L \tag{3.4}$$

where each G_ℓ is chordal (since $1 \ldots n$ remains a perfect elimination ordering of its nodes) and G_L is the complete graph. We now show that G_ℓ has only one maximal clique which is not a clique in $G_{\ell-1}$.

LEMMA 3.3. *For $\ell = 1, \ldots, L$, there is a unique maximal clique K_ℓ in G_ℓ which is not a clique in $G_{\ell-1}$. Moreover, $J(i_\ell) \cup \{i_\ell\} \subseteq K_\ell$, the set $K_\ell \setminus \{i_\ell\}$ is a clique in $G_{\ell-1}$, and the set $K_\ell \setminus J(i_\ell)$ is a clique in G.*

Proof. Let K be a maximal clique in G_ℓ which is not a clique in $G_{\ell-1}$; then, $i_\ell \in K$ and $K \cap J(i_\ell) \neq \emptyset$; we first show that $J(i_\ell) \subseteq K$. For this, assume that $j, j' \in J(i_\ell)$ with $j \in K$ and $j' \notin K$. By maximality of K, there exists an element $i \in K$ such that i and j' are not adjacent in G_ℓ. Then, $i < i_\ell$ since the set $[i_\ell, n]$ is a clique in G_ℓ. Therefore, the pairs ij and ii_ℓ are edges of G_ℓ and, thus, of G. Since the ordering of the nodes is a perfect elimination ordering for G, this implies that i_ℓ and j must be adjacent in G, yielding a contradiction.

Suppose now that K, K' are two distinct maximal cliques in G_ℓ such that $i_\ell \in K \cap K'$ and $J(i_\ell) \subseteq K \cap K'$. Then, there exist nodes $i \in K \setminus K'$, $i' \in K' \setminus K$ that are not adjacent in G_ℓ. Given a node $j \in J(i_\ell)$, one can easily verify that (i, i_ℓ, i', j) is an induced circuit in $G_{\ell-1}$, which contradicts the fact that $G_{\ell-1}$ is chordal and, thus, shows unicity of the clique K_ℓ. It is obvious that $K_\ell \setminus \{i_\ell\}$ is a clique in $G_{\ell-1}$. We now verify that $K_\ell \setminus J(i_\ell)$ is a clique in G. For this, note first that i_ℓ is adjacent to every node of $K_\ell \setminus (J(i_\ell) \cup \{i_\ell\})$ in

G_ℓ and, thus, in G. Suppose now that $x \neq y$ are two nodes in $K_\ell \setminus (J(i_\ell) \cup \{i_\ell\})$ that are not adjacent in G. Then, as xy is an edge of $G_{\ell-1}$, we have: $x = i_h$, $y \in J(i_h)$ for some $h \leq \ell - 1$ and, thus, $i_\ell < x, y$. As i_ℓ is adjacent to both x and y in G this implies that x and y must be adjacent in G, yielding a contradiction. □

We now describe the modified algorithm. Let $G = (V, E)$ be a chordal graph and let $a \in \mathbf{Q}^{V \cup E}$ satisfying (3.1). Setting $x_0 := a$, we execute the following step for $\ell = 1, \ldots, L$.

Find $z_\ell \in \mathbf{Q}^{F_\ell}$ for which the vector $x_\ell := (x_{\ell-1}, z_\ell) \in \mathbf{Q}^{V \cup E(G_\ell)}$ satisfies:

$$x_\ell(K_\ell) \succeq 0. \tag{3.5}$$

Then, the final vector $x_L = (a, z_1, \ldots, z_L)$ provides a rational psd completion of a. For instance, we can choose for z_ℓ the value given by relation (2.2), applying Lemma 2.2 to the matrix $X := x_\ell(K_\ell)$. (Indeed, in view of Lemma 3.3, $X_1 = a(K_\ell \setminus J(i_\ell)) \succeq 0$ and $X_2 = x_{\ell-1}(K_\ell \setminus \{i_\ell\})$; thus, $X_2 \succeq 0$ can be verified by induction.)

We verify that the encoding sizes of $z^1, \ldots, z^L$ are polynomially bounded in terms of n and the encoding size of a. For this, we note that $z^1, \ldots, z^L$ are determined by a recurrence of the form:

$$z_\ell = S_\ell^T A_\ell^{-1} R_\ell \text{ for } \ell = 1, \ldots, L \tag{3.6}$$

where R_ℓ, A_ℓ, S_ℓ are matrices of (appropriate) orders $\leq n$. A crucial observation is that all entries of R_ℓ and A_ℓ belong to the set, denoted as $\mathcal{A}$, of entries of a (as $K_\ell \setminus J(i_\ell)$ is a clique in G, by Lemma 3.3), while the entries of S_ℓ belong to the set $\mathcal{A} \cup \mathcal{Z}_{\ell-1}$, where $\mathcal{Z}_{\ell-1}$ denotes the set of entries of $(z^1, \ldots, z^{\ell-1})$.

For $r \in \mathbf{Q}$, let $\langle r \rangle$ denote the encoding size of r, i.e., the number of bits needed to encode r in binary notation and, for a vector $x = (x_1, \ldots, x_p) \in \mathbf{Q}^p$, set $s(x) := \max(\langle x_1 \rangle, \ldots, \langle x_p \rangle)$. One can verify that, for two vectors $x, y \in \mathbf{Q}^p$, $\langle x^T y \rangle \leq \langle n \rangle + s(x) + s(y)$. Let S_a denote the maximum encoding length of the entries of vector a and, for $\ell = 1, \ldots, L$, set $S_\ell := \max(\langle z \rangle \mid z \in \mathcal{Z}_\ell)$. We derive from (3.6) that

$$S_\ell \leq \langle n \rangle + s(A_\ell^{-1} R_\ell) + S_a + S_{\ell-1}$$

for all ℓ (setting $S_0 := 0$). This implies that

$$S_L \leq L(S_a + \langle n \rangle) + \sum_{\ell=1}^{L} s(A_\ell^{-1} R_\ell).$$

As $L \leq n$, we obtain that all encoding sizes of $z^1, \ldots, z^L$ are polynomially bounded in terms of n and the encoding size of a. (We also use here the fact that the entries of A_ℓ^{-1} are polynomially bounded in the input size; cf. chap. 1.3 in [15].) Thus, we have shown:

THEOREM 3.4. *Problem ($P_s^{\mathbf{Q}}$) can be solved in polynomial time for chordal graphs.* □

We finally indicate how to solve the general problem when some diagonal entries are unspecified.

THEOREM 3.5. *Problems (P) and ($P^{\mathbf{Q}}$) can be solved in polynomial time for chordal graphs.*

Proof. Let $G = (V, E)$ be a chordal graph, let $S \subseteq V$ and let $a \in \mathbf{Q}^{S \cup E}$ satisfying: $a(K) \succeq 0$ for each maximal clique $K \subseteq S$ (else, we can conclude that a is not completable).

Following Lemma 2.4, we search for a scalar $N > 0$ such that a is completable if and only if its extension $a^N \in \mathbf{Q}^{V \cup E}$ (assigning value N to the unspecified diagonal entries) is completable or, equivalently, $a^N(K) \succeq 0$ for all maximal cliques K in G. Note that each matrix $a^N(K)$ has the same form as matrix X from Lemma 2.3. Therefore, such N exists if and only if the linear condition (i) from Lemma 2.3 holds for each clique K and an explicit value for N can be constructed as indicated in Lemma 2.3. Once N has been determined, we proceed with completing a^N by applying the algorithm presented above. □

To conclude note that the algorithm presented in this section outputs a pd completion if one exists.

3.3. Constructing a distance matrix completion. The distance matrix completion problem for chordal graphs can be solved in an analogous manner. Namely, let $G = (V, E)$ be a chordal graph, let

$$G_0 := G \subseteq \ldots G_\ell \subseteq \ldots \subseteq G_L$$

be the sequence of chordal graphs from (3.4), let K_ℓ ($\ell = 1, \ldots, L$) be the cliques constructed in Lemma 3.3, and let $a \in \mathbf{Q}^E$ satisfying (3.2). Setting $a_0 := a$, we execute the following step for $\ell = 1, \ldots, L$:

Find $z_\ell \in \mathbf{Q}^{F_\ell}$ for which the vector $x_\ell := (a_{\ell-1}, z_\ell) \in \mathbf{Q}^{E(G_\ell)}$ satisfies:

$$x_\ell(K_\ell) \text{ is a distance matrix.} \tag{3.7}$$

Then, the final vector $x_L = (a, z_1, \ldots, z_L)$ provides a distance matrix completion of a. The above step can be performed as follows. If $K_\ell = J(i_\ell) \cup \{i_\ell\}$, then we let z_ℓ be defined by $z_\ell(j) := x_{\ell-1}(j_0, j)$ for $j \in J(i_\ell)$, where j_0 is a given element of $J(i_\ell)$. Otherwise, let $j_0 \in K_\ell \setminus (J(i_\ell) \cup \{i_\ell\})$; then j_0 is a universal node in $G[K_\ell]$, the subgraph of G induced by K_ℓ. Therefore, in view of relation (2.5), we can find z_ℓ satisfying (3.7) by applying Lemma 2.2. The polynomial running time of the above algorithm follows from the polynomial running time of the corresponding algorithm in the psd case. Thus, we have shown:

THEOREM 3.6. *Problem ($D^{\mathbf{Q}}$) can be solved in polynomial time for chordal graphs.*

4. The Matrix Completion Problem for Graphs with Fixed Minimum Fill-In. In this section we describe an algorithm permitting to solve problems (P), ($P^{\mathbf{Q}}$), (D), and ($D^{\mathbf{Q}}$) in polynomial time for the graphs having minimum fill-in m, where $m \geq 1$ is a given integer. This algorithm is based on Theorems 1.1, 3.1, 3.2, 3.4 and 3.6.

Let $G = (V, E)$ be a graph with minimum fill-in m, let $S \subseteq V$ and let $a \in \mathbf{Q}^{S \cup E}$ be given. (Again we assume that $a_i = 0$ for $i \in V$ in the distance matrix case.) We first execute the following step.

Step 0. *Find edges $e_1, \ldots, e_m \notin E$ for which the graph $H := (V, E \cup \{e_1, \ldots, e_m\})$ is chordal and find the maximal cliques $K_1, \ldots, K_p$ in H.* (Such edges exist since G has minimum fill-in m and they can be found in polynomial time, simply by enumeration as m is fixed. The maximal cliques in H can also be enumerated in polynomial time since H is chordal and, moreover, $p \leq n$.)

Then, we perform step x in order to solve problem (x) for x=P,$P^{\mathbf{Q}}$,D,$D^{\mathbf{Q}}$.

Step P. *Determine whether there exist real numbers* $z_1, \ldots, z_m, z_{m+1}$ *for which the vector* $x \in \mathbf{Q}^{V \cup E(H)}$ *defined by* $x_i := a_i$ $(i \in S)$, $x_i := z_{m+1}$ $(i \in V \setminus S)$, $x_{ij} = a_{ij}$ $(ij \in E)$, *and* $x_{e_h} := z_h$ $(h = 1, \ldots, m)$, *satisfies:*

$$x(K_1) \succeq 0, \ldots, x(K_p) \succeq 0. \tag{4.1}$$

Step D. *Determine whether there exist real numbers* $z_1, \ldots, z_m$ *for which the vector* $x \in \mathbf{Q}^{E(H)}$ *defined by* $x_{ij} = a_{ij}$ $(ij \in E)$, *and* $x_{e_h} := z_h$ $(h = 1, \ldots, m)$, *satisfies:*

$$x(K_1), \ldots, x(K_p) \text{ are distance matrices.} \tag{4.2}$$

Then, a has a completion if and only if the answer in step P or D is positive.

Step $\mathbf{P^Q}$. *Find rational numbers* $z_1, \ldots, z_m, z_{m+1}$ *for which (4.1) holds or determine that no such numbers exist; if they exist, find a rational psd completion of* x.

Step $\mathbf{D^Q}$. *Find rational numbers* $z_1, \ldots, z_m$ *for which* (4.2) *holds or determine that no such numbers exist; if they exist, find a rational distance matrix completion of* x.

Steps P and $\mathrm{P^Q}$ can be executed in the following manner. Let M denote the block diagonal matrix with the p matrices $x(K_1), \ldots, x(K_p)$ as diagonal blocks (and zeros elsewhere). Hence, M has order $|K_1| + \ldots + |K_p| \leq n^2$ and (4.1) holds if and only if $M \succeq 0$. Clearly, the matrix M can be written under the form:

$$M = Q_0 + z_1 Q_1 + \ldots + z_{m+1} Q_{m+1}$$

where $Q_1, \ldots, Q_{m+1}$ are symmetric matrices with (0,1)-entries and Q_0 is a symmetric matrix whose nonzero entries belong to the set of entries of a. Therefore, in view of Theorem 1.1, one can determine the existence of $z_1, \ldots, z_{m+1}$ satisfying (4.1) in polynomial time. Then, finding a rational psd completion of x in step $\mathrm{P^Q}$ can be done in polynomial time in view of Theorem 3.4.

In the distance matrix case, we use the following construction for distance matrices. For $a = 1, \ldots, p$, let D_a be a square symmetric matrix whose rows and columns are indexed by set V_a and let i_a be a given element of V_a. We construct a new matrix D, denoted as $D_1 \oplus \ldots \oplus D_p$, whose rows and columns are indexed by set $V_1 \cup \ldots \cup V_p$ and whose entries are given by

$$D(i,j) = \begin{cases} D_a(i,j) & \text{if } i,j \in V_a,\ a \in [1,p], \\ D_a(i,i_a) + D_b(j,i_b) & \text{if } i \in V_a,\ j \in V_b,\ a \neq b \in [1,p]. \end{cases} \tag{4.3}$$

LEMMA 4.1. *$D_1 \oplus \ldots \oplus D_p$ is a distance matrix if and only if $D_1, \ldots, D_p$ are distance matrices.*

Proof. The 'only if' part is obvious. Conversely, assume that $D_1, \ldots, D_p$ are distance matrices; we show that $D := D_1 \oplus \ldots \oplus D_p$ is a distance matrix. For $a \in [1,p]$, let $u_i^a \in \mathbf{R}^{n_a}$ $(i \in V_a)$ be vectors providing a realization of D_a; we can assume without loss of generality that $u_{i_a}^a = 0$. Then, we construct a sequence of vectors $w_i \in \mathbf{R}^{n_1 + \ldots + n_p}$ $(i \in \bigcup_{a=1}^p V_a)$ by setting $w_i := (0_{n_1}, \ldots, 0_{n_{a-1}}, u_i^a, 0_{n_{a+1}}, \ldots, 0_{n_p})$ for $i \in V_a$ (0_n denotes the zero vector in $\mathbf{R}^n$). One can easily verify that the vectors w_i provide a realization of D. ∎

Steps D and $\mathrm{D^Q}$ can be performed as follows. Let $M := x(K_1) \oplus \ldots \oplus x(K_p)$ denote the matrix indexed by $K_1 \cup \ldots \cup K_p$ constructed as indicated in relation (4.3). Clearly, M can be written under the form:

$$M = Q_0 + z_1 Q_1 + \ldots + z_m Q_m$$

where $Q_1, \ldots, Q_m$ are symmetric matrices with entries in $\{0,1\}$ and Q_0 is a symmetric matrix whose nonzero entries are sums of at most two entries of a. Let i_0 be a given element of $K_1 \cup \ldots \cup K_p$. Then,

$$\varphi_{i_0}(M) = \varphi_{i_0}(Q_0) + z_1 \varphi_{i_0}(Q_1) + \ldots + z_m \varphi_{i_0}(Q_m).$$

Hence, (4.2) holds if and only if matrix M is a distance matrix (by Lemma 4.1) or, equivalently, if and only if $\varphi_{i_0}(M)$ is positive semidefinite (by relation (2.4)). Therefore, in view of Theorems 3.2 and 3.6, steps D and $\mathrm{D}^{\mathbf{Q}}$ can be executed in polynomial time. This completes the proof of Theorem 1.2.

LEMMA 4.2. *When the minimum fill-in m is equal to 1, existence of a completion implies existence of a rational one.*

Proof. To see it, suppose first that all diagonal entries are specified; then, steps P and $\mathrm{P}^{\mathbf{Q}}$ can be executed in an elementary manner. Indeed, each matrix $x(K_i)$ $(i = 1, \ldots, p)$ has at most one unspecified entry z_1. Hence, the set of scalars z_1 for which $x(K_i) \succeq 0$ is an interval of the form $I_i = [\beta_i - \sqrt{\alpha_i}, \beta_i + \sqrt{\alpha_i}]$ where $\alpha_i, \beta_i \in \mathbf{Q}$ (easy to see from Lemma 2.2). Therefore, (4.1) holds if and only if $z_1 \in \bigcap_{i=1}^{p} I_i = [u, v]$, where $u := \max_i(\beta_i - \sqrt{\alpha_i})$ and $v := \min_i(\beta_i + \sqrt{\alpha_i})$. Moreover, if there is a completion (i.e., if $u \leq v$), then one can find one with z_1 rational. This is obvious if $u < v$ and, if $u = v$, this follows from the fact (easy to verify) that

$$\beta - \sqrt{\alpha} = \beta' + \sqrt{\alpha'},\ \alpha, \alpha', \beta, \beta' \in \mathbf{Q} \Longrightarrow \sqrt{\alpha}, \sqrt{\alpha'} \in \mathbf{Q}.$$

Suppose now some diagonal entries are unspecified. If there is a completion with value z_2 at the unspecified diagonal entries, then we can assume that z_2 is rational (replacing if necessary z_2 by a larger rational number). Then, by the above discussion, the off-diagonal unspecified entry z_1 can also be chosen to be rational. ◻

5. Further Results and Open Questions. We present in Section 5.1 another class of graphs for which the completion problem can be solved in polynomial time (in the bit model). Then, we discuss in Section 5.2 some open questions arising when considering a polar approach to the positive semidefinite completion problem. Finally, we describe in Section 5.3 a simple combinatorial algorithm permitting to solve the completion problem in polynomial time (in the real number model) for the class of graphs containing no homeomorph of K_4.

5.1. Another class of polynomial instances. We present here another class of graphs for which the positive semidefinite matrix completion problem (P_s) can be solved in polynomial time. Given two integers $p, q \geq 1$, let $\mathcal{G}_{p,q}$ be the class consisting of the graphs $G = (V, E)$ satisfying the following properties: There exist two disjoint subsets V_1, V_2 of V such that $\min(|V_1|, |V_2|) = p$, the set $F := \{ij \mid i \in V_1, j \in V_2\}$ is disjoint from E, the graph

$$H := (V, E \cup F)$$

is chordal, and H has q maximal cliques that are not cliques in G. We show:

THEOREM 5.1. *Given integers $p, q \geq 1$, the positive semidefinite completion problem (P_s) can be solved in polynomial time (in the bit model) over the class $\mathcal{G}_{p,q}$.*

Examples of graphs belonging to class $\mathcal{G}_{p,q}$ arise from circuits, wheels and some generalizations. A *generalized circuit* of length n is defined in the following manner: its node set is $U_1 \cup \ldots \cup U_n$ with two nodes $u \in U_i$, $v \in U_j$ being adjacent if and only if $i = j$ or $j = i + 1$ (modulo n); a *generalized wheel* of length n is obtained by adding a set U_0 (the *center* of the wheel) of pairwise adjacent nodes to a generalized circuit of length n and making each node in U_0 adjacent to each node in $U_1 \cup \ldots \cup U_n$. Call a generalized circuit or wheel *p-fat* if $\min(|U_i| : i = 1, \ldots, n) = p$. Then, any p-fat generalized circuit or wheel of length $q + 2$ belongs to $\mathcal{G}_{p,q}$. We will see in Section 5.2 that generalized circuits and wheels arise as basic objects when studying the matrix completion problem on graphs of small order.

figwheel.eps

FIG. 5.1. *(a) The wheel of length 4; (b) a 2-fat generalized wheel of length 4*

The proof of Theorem 5.1 is based on the following result of Barvinok [8], which shows that one can test feasibility of a system of quadratic equations in polynomial time for any fixed number of equations[2].

THEOREM 5.2. *For $i = 1, \ldots, m$, let $f_i(x) = x^T A_i x + b_i^T x + c_i$ be a quadratic polynomial in $x \in \mathbf{R}^n$, where A_i is an $n \times n$ symmetric matrix, $b_i \in \mathbf{R}^n$ and $c_i \in \mathbf{R}$. One can test feasibility of the system: $f_i(x) = 0$ for $i = 1, \ldots, m$ in polynomial time (in the bit model) for any given m.* $\square$

Proof of Theorem 5.1. Let $G = (V, E)$ be a graph in class $\mathcal{G}_{p,q}$ and let $a \in \mathbf{R}^{V \cup E}$ be given. We are also given the sets V_1 and V_2 for which, say, $p = |V_1| \leq |V_2|$ and adding to G all edges in $F := \{ij \mid i \in V_1, j \in V_2\}$ creates a chordal graph H. We show that deciding whether a can be completed to a psd matrix amounts to testing the feasibility of a system of m quadratic polynomials where m depends only on p and q. As H is chordal, a is completable to a psd matrix if and only if there exists a matrix Z of order $V_2 \times V_1$ for which $x := (a, Z) \in \mathbf{R}^{V \cup E \cup F}$ satisfies: $x(K) \succeq 0$ for each maximal clique K in H. We assume that $x(K) = a(K) \succeq 0$ for each maximal clique K of H contained in G (else, we can conclude that a is not completable). Consider now a maximal clique K of H which is not contained in G. Then, $x(K)$ has the following form:

$$x(K) = \begin{array}{c c} & \begin{array}{ccc} V_1 \cap K & V_0 \cap K & V_2 \cap K \end{array} \\ \begin{array}{c} V_1 \cap K \\ V_0 \cap K \\ V_2 \cap K \end{array} & \begin{pmatrix} T & R^T & Z_K^T \\ R & A & S \\ Z_K & S^T & D \end{pmatrix} \end{array},$$

setting $V_0 := V \setminus (V_1 \cup V_2)$ and $Z_K := Z[V_2 \cap K, V_1 \cap K]$, the submatrix of Z with row indices in $V_2 \cap K$ and column indices in $V_1 \cap K$. With the notation of Lemma 2.2, we obtain that $x(K) \succeq 0$ if and only if the following matrix

$$M_K := \begin{pmatrix} T - R_0^T A_0^{-1} R_0 & Z_K^T - R_0^T A_0^{-1} S_0 \\ Z_K - S_0^T A_0^{-1} R_0 & D - S_0 A_0^{-1} S_0 \end{pmatrix}$$

is positive semidefinite (we have assumed that $A \succeq 0$). We can apply again a Schur decomposition to matrix M_K in order to reformulate the condition on Z. Setting $T_K :=$

[2] In [8] Barvinok considers the homogeneous case, where each equation is of the form: $f_i(x) = x^T A_i x = 0$ for some symmetric matrix A_i. However, the general nonhomogeneous case can be derived from it (Barvinok, personal communication, 1998).

$T - R_0^T A_0^{-1} R_0$, $Z' := Z_K - S_0^T A_0^{-1} R_0$, and $D' := D - S_0 A_0^{-1} S_0$, we have that $M_K = \begin{pmatrix} T_K & Z'^T \\ Z' & D' \end{pmatrix}$. Let D'_0 be a largest nonsingular submatrix of D' and let

$$D' = \begin{pmatrix} D'_0 & E \\ E^T & F \end{pmatrix}, \; M_K = \begin{pmatrix} T_K & Z_0'^T & Z_1'^T \\ Z'_0 & D'_0 & E \\ Z'_1 & E^T & F \end{pmatrix}$$

denote the corresponding block decompositions of D' and M_K. Taking the Schur complement of D'_0 in M_K, we obtain that $M_K \succeq 0$ if and only if

$$D' \succeq 0, \; T_K - Z_0'^T D_0'^{-1} Z'_0 \succeq 0, \; \text{ and } Z'_1 - E^T D_0'^{-1} Z'_0 = 0.$$

Let $Y_K := Z[V_2, V_1 \cap K]$ denote the column submatrix of Z with column indices in $V_1 \cap K$ and set

$$V_K := \begin{array}{c} \\ V_2 \cap K \\ V_2 \setminus K \end{array} \begin{array}{c} V_1 \cap K \\ \begin{pmatrix} S_0^T A_0^{-1} R_0 \\ 0 \end{pmatrix} \end{array}, \; Q_K := \begin{pmatrix} D_0^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \; G_K := (-E^T D_0'^{-1} \quad I \quad 0).$$

Then,

$$T_K - Z_0'^T D_0'^{-1} Z'_0 = T_K - (Y_K - V_K)^T Q_K (Y_K - V_K),$$

$$Z'_1 - E^T D_0'^{-1} Z'_0 = G_K (Y_K - V_K).$$

Therefore, the condition: $x(K) \succeq 0$ can be rewritten as the sytem:

$$\begin{cases} \text{(1K) } T_K - (Y_K - V_K)^T Q_K (Y_K - V_K) \succeq 0, \\ \text{(2K) } G_K (Y_K - V_K) = 0, \end{cases}$$

where T_K, V_K, Q_K are matrices depending on input data a. We can reformulate condition (1K) as an equation by introducing a new square matrix S_K of order $V_1 \cap K$ as 'slack variable'; namely, rewrite (1K) as

$$\text{(1'K) } T_K - (Y_K - V_K)^T Q_K (Y_K - V_K) - S_K^T S_K = 0.$$

Now, let $z_1, \ldots, z_p \in \mathbf{R}^{V_2}$ denote the columns of matrix Z, and let s_i^K (for $i \in V_1 \cap K$) denote the columns of matrix S_K for each clique K. Then, condition (1'K) can be expressed as a system of $\binom{|V_1 \cap K|+1}{2}$ equations of the form:

$$f(z_1, \ldots, z_p, s_i^K \; (i \in V_1 \cap K)) = 0,$$

where f is a quadratic polynomial; similarly for condition (2K). The total number of quadratic equations obtained in this manner depends only on p and q. Therefore, in view of Theorem 5.2, one can check feasibility of this system in polynomial time when p and q are fixed. □

Let $\mathcal{G}'_{p,q}$ denote the subclass of $\mathcal{G}_{p,q}$ consisting of the graphs G for which every maximal clique of H (the chordal extension of G) which is not a clique of G is not contained in $V_1 \cup V_2$. Then, the Euclidean distance matrix completion problem can be solved in figX.eps

FIG. 5.2. *The matrix completion problem for generalized circuits of length 4*

polynomial time over the class $\mathcal{G}'_{p,q}$ for any fixed p and q. The proof is similar to that of Theorem 5.1, since we can get back to the psd case using relation (2.4) (a matrix and its image under φ_{i_0} having same pattern of unknown entries if i_0 belongs to $V \setminus (V_1 \cup V_2)$). In particular, the Euclidean distance matrix completion problem can be solved in polynomial time for generalized circuits of length 4 and fixed fatness, or for generalized wheels (with a nonempty center) of fixed length and fatness.

The complexity of the psd completion problem for generalized wheels and circuits is not known; in fact, in view of the remark made at the end of Section 2.2, it suffices to consider circuits. In view of Theorem 5.1, the problem is polynomial if we fix the length and the fatness of the circuit. It would be particularly interesting to determine the complexity of the completion problem for generalized circuits of length 4 and unrestricted fatness. This problem can be reformulated as follows: Determine whether and how one can fill the unspecified entries in the blocks marked '?' of the matrix X shown in Figure 5.2, so as to obtain $X \succeq 0$ (all entries are assumed to be specified in the grey blocks). Indeed, as will be seen in Section 5.2, these graphs constitute in some sense the next case to consider after chordal graphs.

5.2. A polar approach to the completion problem. Given a graph $G = (V, E)$, consider the cone C_G consisting of the matrices $X = (x_{ij})_{i,j \in V}$ satisfying $X \succeq 0$ and $x_{ij} = 0$ for all $i \neq j$ such that $ij \notin E$. Call $X \in C_G$ *extremal* if X lies on an extremal ray of the cone C_G (i.e., $X = Y + Z$ with $Y, Z \in C_G$ implies that $Y = \alpha X$ for some $\alpha \geq 0$) and define the *order* of G as the maximum rank of an extremal matrix $X \in C_G$. It is shown in [1] that $a \in \mathbf{R}^{V \cup E}$ is completable to a psd matrix if and only if a satisfies:

$$\sum_{ij \in E} a_{ij} x_{ij} + \sum_{i \in V} a_i x_{ii} \geq 0 \tag{5.1}$$

for every extremal matrix $X = (x_{ij}) \in C_G$. One might suspect that the psd matrix completion problem is somewhat easier to solve for graphs having a small order since the extremal matrices in C_G have then a small rank. Indeed, the graphs of order 1 are precisely the chordal graphs, for which the problem is polynomially solvable. On the other hand, a circuit of length n has order $n - 2$ which is the highest possible order for a graph on n nodes. Moreover, if i_0 is a universal node in a graph G, then both graphs G and $G \setminus i_0$ have the same order, which corroborates the observation made at the end of Section 2.2. A natural question concerns the complexity of the problem for graphs of order 2.

The graphs of order 2 have been characterized in [27]. It is shown there that, up to a simple graph operation (clique-sum), they belong to two basic classes $\mathcal{G}_1$ and $\mathcal{G}_2$. All the graphs in $\mathcal{G}_1$ have minimum fill-in at most 3; hence, the problem is polynomially solvable for them (by Theorem 1.2). The graphs in class $\mathcal{G}_2$ are the generalized wheels of length 4 (and unrestricted fatness). Hence, if the psd matrix completion problem is polynomially solvable for generalized wheels of length 4, then the same holds for all graphs of order 2.

5.3. The matrix completion problem for graphs with no homeomorph of K_4. We now discuss the matrix completion problem for the class $\mathcal{H}$ consisting of the graphs figK4.eps

FIG. 5.3. *A homeomorph of K_4*

containing no homeomorph of K_4 as a subgraph; a *homeomorph* of K_4 being obtained from K_4 by repacing its edges by paths. (Graphs in $\mathcal{H}$ are also known as *series parallel graphs.*) Clearly, $\mathcal{H}$ contains all circuits. The case of circuits is certainly interesting to understand since circuits are the most simple nonchordal graphs.

Similarly to the chordal case, a condition characterizing existence of a psd completion is known for the graphs in $\mathcal{H}$. Namely, the following is shown in [24] (using a result of [7]). Given a graph $G = (V, E)$ in $\mathcal{H}$ and $a \in \mathbf{R}^{V \cup E}$ satisfying: $a_i = 1$ for all $i \in V$, then a has a psd completion if and only if the scalars $x_e := \frac{1}{\pi} \arccos a_e$ $(e \in E)$ satisfy the inequalities:

$$\sum_{e \in F} x_e - \sum_{e \in C \setminus F} x_e \leq |F| - 1 \quad \text{for all } F \subseteq C \text{ with } C \text{ circuit in } G, |F| \text{ odd.} \tag{5.2}$$

PROPOSITION 5.3. [6] *Given $x \in [0,1]^E$, one can test in polynomial time whether x satisfies the linear system (5.2).*

Proof. Consider the graph $\tilde{G} := (V \cup V', \tilde{E})$ where $V' := \{i' \mid i \in V\}$ and $\tilde{E}$ consists of the pairs ij, $i'j'$, ij', $i'j$ for $ij \in E$. Define $z \in \mathbf{R}^{\tilde{E}}$ by $z_{ij} = z_{i'j'} = x_{ij}$ and $z_{i'j} = z_{ij'} = 1 - x_{ij}$ for $ij \in E$. Then, it is easy to see that x satisfies (5.2) if and only if $z(P) \geq 1$ for every path P from i to i' in $\tilde{G}$ and every $i \in V$. The result now follows as one can compute shortest paths in polynomial time. □

Therefore, problem (P_s) is polynomial time solvable in the real number model for graphs in $\mathcal{H}$. It is not clear how to extend this result to the bit model since the scalars $x_e := \frac{1}{\pi} \arccos a_e$ are in general irrational and, thus, one encounters problems of numerical stability when trying to check whether (5.2) holds.

Moreover, there is a simple combinatorial algorithm (already briefly mentioned in [25]) permitting to construct a psd completion in polynomial time in the real number model. Let $G = (V, E)$ be a graph in $\mathcal{H}$ and let $a \in \mathbf{R}^{V \cup E}$ be given satisfying $a_i = 1$ for all $i \in V$. The algorithm performs the following steps.

1. Set $x_e := \frac{1}{\pi} \arccos a_e$ for $e \in E$ and test whether x satisfies (5.2). If not, one can conclude that a has no psd completion. Otherwise, go to 2.

2. Find a set F of edges disjoint from E for which the graph $H := (V, E \cup F)$ is chordal and contains no homeomorph of K_4.

3. Find an extension $y \in [0,1]^{E \cup F}$ of x satisfying the linear system (5.2) with respect to graph H.

4. Set $b_e := \cos(\pi y_e)$ for $e \in E \cup F$ and $b_i := 1$ for $i \in V$. Then, b is completable to a psd matrix (since y satisfies (5.2) and H has no homeomorph of K_4) and one can compute a psd completion X of b with the algorithm of Section 3.2 (since H is chordal). Then, X is a completion of a.

All steps can be executed in polynomial time. This follows from earlier results for steps 1 and 4; for step 2 use a result of [39] and, for step 3, one can use an argument similar to the proof of Proposition 5.3. Namely, given $x \in [0,1]^E$ satisfying (5.2), in order to extend x to $[0,1]^{E\cup\{e\}}$ in such a way that (5.2) remains valid with respect to $G+e$, one has to find a scalar $\alpha \in [0,1]$ satisfying $L_1 \ \leq\ \alpha\ \leq\ L_2$, where

$$L_1 := \max_{C,F | e\in C\setminus F} (x(F) - x(C\setminus (F\cup \{e\})) - |F| + 1),$$

$$L_2 := \min_{C,F | e\in F} (x(C\setminus F) - x(F\setminus e) + |F| - 1).$$

We have: $L_1 \leq L_2$ (since x satisfies (5.2)) and $L_1 \leq 1$, $L_2 \geq 0$ (since $x \in [0,1]^E$); thus, $[L_1, L_2] \cap [0,1] \neq \emptyset$. With the notation of the proof of Proposition 5.3, one finds that

$$L_1 = 1 - \min(z(P) \mid P \text{ is an } ab' - \text{path in } \tilde{G}),$$

$$L_2 = \min(z(P) \mid P \text{ is an } ab - \text{path in } \tilde{G}).$$

Hence one can compute α in polytime. One can then determine the extension y of x to H by iteratively applying this procedure.

The distance matrix completion problem for graphs in $\mathcal{H}$ can be treated in a similar manner. Indeed, given $G = (V,E)$ in $\mathcal{H}$ and $a \in \mathbf{R}^E_+$, set $x_e := \sqrt{a}_e$ for $e \in E$. Then, a is completable to a distance matrix if and only if x satisfies the linear inequalities:

$$x_e - \sum_{f\in C\setminus e} x_f \leq 0 \ \text{ for all circuits } C \text{ in } G \text{ and all } e \in C. \tag{5.3}$$

(Cf. [26].) Again one can test in polynomial time whether $x \geq 0$ satisfies (5.3) (simply, test for each edge $e = ab \in E$ whether $x_e \leq \min(x(P) \mid P \text{ is an } ab - \text{path in } G)$). An algorithm analogous to the one exposed in the psd case permits to construct a distance matrix completion. Therefore, we have shown:

THEOREM 5.4. *One can construct a real psd (distance matrix) completion or decide that none exists in polynomial time in the real number model for the graphs containing no homeomorph of K_4.* $\square$

It is an open question whether the above result extends to the bit model of computation, even for the simplest case of circuits.

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