Cut, lift, and project methods for $0 - 1$ programming

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Given a polytope

$$K = \{x \in \Re^n \mid Ax \leq b\}$$

Find the linear description of the $0-1$ polytope

$$P := \mathrm{conv}(K \cap \{0,1\}^n)$$

Write P as

$$P = \{x \in \Re^n \mid Ax \leq b,\ A'x \leq b'\}$$

$Ax \leq b$ is a linear programming formulation for P

Example 1: P **:= TSP polytope of** $G = (V, E)$

$$\begin{array}{lll} \min & \sum_{e \in E} c_e x_e & \\ \text{s.t.} & \sum_{e \in \delta(i)} x_e = 2 & (i \in V) \\ & \sum_{e \in \delta(S)} x_e \geq 2 & (\emptyset \subset S \subset V) \\ & x_e \geq 0 & (e \in E) \\ & x \in \{0,1\}^E & \end{array}$$

Example 2: P **:= stable set polytope of** G

$$\begin{array}{lll} \max & \sum_{i \in V} w_i x_i & \\ \text{s.t.} & x_i + x_j \leq 1 & (ij \in E) \\ & x_i \geq 0 & (i \in V) \\ & x \in \{0,1\}^V & \end{array}$$

Example 3: P **:= matching polytope of** G

$$\begin{array}{lll} \max & \sum_{e \in E} w_e x_e & \\ \text{s.t.} & \sum_{e \in \delta(i)} x_e \leq 1 \ (i \in V) & \\ & x_e \geq 0 & (e \in E) \\ & x \in \{0,1\}^E & \end{array}$$

[Edmonds 1965]

$$P = K \cap \{x \mid \sum_{e \subseteq U} x_e \leq \frac{|U|-1}{2} \ \text{ for } U \subseteq V,\ |U| \text{ odd}\}$$

Cutting plane methods

Fact: Given *a integral*
$a^T x \le \beta$ is valid for $K \Longrightarrow a^T x \le \lfloor \beta \rfloor$ is valid for P.

$$\begin{array}{ccc} u_1 & \cdot & a_1^T x \le b_1 \\ & \vdots & \\ u_m & \cdot & a_m^T x \le b_m \end{array}$$

$$a^T x := (\sum_{j=1}^{m} u_j a_j)^T x \le \sum_{j=1}^{m} u_j b_j \le \beta$$

Gomory-Chvátal cuts: [Gomory 1960]

$$K' := \{x \mid u^T A x \le \lfloor u^T b \rfloor \ \forall u \ge 0 \text{ with } u^T A \text{ integral}\}$$

$$K^{t+1} := (K^t)'$$

Theorem: [Chvátal 73, Schrijver 80]
If K is a rational polyhedron, then K' is a rational polyhedron and $P = K^t$ for some integer t.

Example: $K := \{x \in \Re_+^E \mid \sum_{e \in \delta(u)} x_e \le 1 \ \forall v \in V\}$

The blossom inequalities: $\sum_{e \in E(U)} x_e \le \left\lfloor \frac{|U|}{2} \right\rfloor$ ($U \subseteq V$, $|U|$ odd) have Chvátal rank 1
$\leadsto K' \equiv$ matching polytope of G

The Chvátal rank can be large (depends on n and A)

$$K = \{(x,y) \in \Re^2 \mid x \geq 0,\ x - 2ky \leq 0,\ x + 2ky \leq 2k\}$$

has Chvátal rank k

When $K \subseteq [0,1]^n$

Chvátal rank $\quad t \leq O(n^3 \log n)$
[Bockmayr, Eisenbrand, Hartmann, Schulz 1999]

$t \leq O(n^2 \log n)$ [Eisenbrand Schulz 1999]
Exist examples where $t > n$

$t \leq n$ if $K \cap \{0,1\} = \emptyset$ [BEHS99]
If $t = n$, then P has at least 2^n facets [ES99]

Example: [Chvátal, Cook, Hartmann 1989]
Chvátal rank $= n$ and $K \cap \{0,1\}^n = \emptyset$ for

$$K := \{x \in \Re^n \mid \sum_{i \in I} x_i + \sum_{i \notin I} (1 - x_i) \geq \frac{1}{2} \ \ \forall I \subseteq [1,n]\}$$

$$= \{x \in \Re^n \mid \left\| x - \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \right\|_1 \geq \frac{1}{2}\}$$

Extensions of Gomory-Chvátal cuts

Gomory's mixed integer cuts for mixed integer programs:

$$K = \{(x,y) \in \Re^{n+p} \mid Ax + Gy \leq b\}$$

$$P = \mathrm{conv}(\{(x,y) \in K \mid x \in Z^n\})$$

based on the principle: if $x \in Z$, $y \geq 0$, $f := b - \lfloor b \rfloor > 0$ then

$$x + y \geq b \Longrightarrow x + \frac{y}{f} \geq \lceil b \rceil$$

Disjunctive cuts [Balas 1975]
The inequality $a^T(x,y) \leq \beta$ is a *disjunctive cut* if it is valid for

$$\{(x,y) \in K \mid \pi^T x \leq \pi_0 \text{ or } \pi^T x \geq \pi_0 + 1\}$$

for some integral π, π_0

Disjunctive cuts are equivalent to the Gomory mixed integer cuts [Nemhauser Wolsey 1988] and contain GC-cuts

Theorem:
(i) [Cook, Kannan, Schrijver 1990] The disjunctive closure K' is a polyhedron
(ii) [Balas 1979] The disjunctive rank is at most n for mixed $0-1$ programs.

Proof: (ii) $K' \subseteq \mathrm{conv}(K \cap \{(x,y) \mid x_i = 0,1\})$ $\forall i = 1,\ldots,n$

What about optimization ?

Theorem: [Eisenbrand 1999], [Caprara Letchford 2001]
The membership problem in the Chvátal closure, or in the disjunctive closure of K is a co-NP-complete problem.

$\leadsto$ Optimizing over the Chvátal closure or over the disjunctive closure cannot be done in polynomial time if P $\neq$ NP

Goal:

Construct a relaxation K' satisfying the following properties:

- $P = \text{conv}(K \cap \{0,1\}^n) \subseteq K' \subseteq K$
- one can optimize over K' in polynomial time
- one can find P in a finite number of steps (say, n)

$\leadsto$ **lift and project methods**

Lift and Project Methods

- Balas-Ceria-Cornuéjols [BCC] (1993)
- Lovász-Schrijver [LS] (1991)
- Sherali-Adams [SA] (1990)
- Lasserre (2000)

Common features:
Construct a hierarchy of *linear* (BCC, SA, LS) or *positive semidefinite* (LS_+, Lasserre) relaxations of P:

$$P \subseteq K^n \subseteq K^{n-1} \subseteq \ldots \subseteq K^2 \subseteq K^1 \subseteq K$$

- $P = K^n$
- One can optimize a linear objective function in polynomial time over K^t for *fixed* t assuming

- existence of a separation oracle for K
(BCC, LS, LS_+, SA)
- the number of rows of A is polynomial in n (Lasserre)

It is known that

$$\text{SA} \subseteq \text{LS} \subseteq \text{BCC}.$$

We show that

$$\text{Lasserre} \subseteq \text{SA} \cap \text{LS}_+$$

Some notation

$$K = \{x \in \Re^n \mid Ax \leq b\}$$
$$= \{x \in \Re^n \mid g_\ell^T(1\ x) \geq 0\ \forall \ell = 1, \ldots, m\}$$

$$\widetilde{K} = \{(x_0, x) \in \Re^{n+1} \mid Ax \leq x_0 b\}$$
$$= \{y \in \Re^{n+1} \mid g_\ell^T y \geq 0\ \forall \ell = 1, \ldots, m\}$$

the *homogenization* of K

$$P = \mathrm{conv}(K \cap \{0,1\}^n)$$

We assume that $K \subseteq [0,1]^n$

The Balas-Ceria-Cornuéjols construction

- Choose an index in $[1, n]$, say 1
- Multiply $Ax \leq b$ by x_1 and $1 - x_1$:

$$x_1(b - Ax) \geq 0, \quad (1 - x_1)(b - Ax) \geq 0$$

- Linearize by setting $y_i := x_1 x_i$, $y_1 = x_1$ (i.e., $x_1^2 = x_1$)

$\rightsquigarrow$ polytope $M_1(K)$ defined by

$$bx_1 - Ay \geq 0, \quad b - Ax - bx_1 + Ay \geq 0$$

- Project $M_1(K)$ on the x-space $\Re^n \rightsquigarrow P_1(K)$

Fact: $P \subseteq P_1(K) = \text{conv}(K \cap \{x \mid x_1 = 0, 1\})$

$$P \subseteq N_0(K) := \bigcap_{j=1}^{n} P_j(K)$$

Define iteratively:

$$P_{i_1 \ldots i_k}(K) := P_{i_k}(P_{i_{k-1} \ldots i_1}(K))$$
$$= \text{conv}(K \cap \{x \mid x_{i_1}, \ldots, x_{i_k} = 0, 1\})$$

$$P = P_{i_1 \ldots i_n}(K)$$

$$P_1(K) = \mathrm{conv}(K \cap \{x \mid x_1 = 0, 1\})$$

Proof:

$$x \in \mathrm{conv}(K \cap \{x \mid x_1 = 0, 1\})$$

$$\Updownarrow$$

$$x = \alpha u + (1-\alpha)v \text{ for some } 0 \leq \alpha \leq 1,$$
$$u, v \in K,\ u_1 = 1 \text{ and } v_1 = 0$$

$$\Updownarrow$$

setting $y := \alpha u$

$$\begin{cases} Ay \leq bx_1 \\ A(x-y) \leq (1-x_1)b \\ y_1 = x_1 \end{cases}$$

$$\Updownarrow$$

$$(x, y) \in M_1(K) \text{ for some } y$$

$$\Updownarrow$$

$$x \in P_1(K)$$

The Lovász-Schrijver construction

• Multiply $Ax \leq b$ by *all* x_i and $1 - x_i$:

$$x_i(b - Ax) \geq 0, \quad (1 - x_i)(b - Ax) \geq 0 \quad \forall i = 1, \ldots, n$$

$$e_i^T(1\,x)\cdot g_\ell^T(1\,x) \geq 0, \ (e_0 - e_i)^T(1\,x)\cdot g_\ell^T(1\,x) \geq 0 \ \forall i, \ \forall j \leq m$$

$$g_\ell^T(1\,x)(1\,x)^T e_i \geq 0, \ g_\ell^T(1\,x)(1\,x)^T(e_0 - e_i) \geq 0 \ \ \forall i, \ \forall j$$

• Linearize by setting $Y := (1\,x)(1\,x)^T$

$\rightsquigarrow$ set $M(K)$ of symmetric matrices $Y = (y_{ij})_{i,j=0}^n$ satisfying

(i) $y_{ii} = y_{0i} \ \forall i = 1, \ldots, n$

(ii) $Ye_i, \ Y(e_0 - e_i) \in \tilde{K} \ \ \forall i = 1, \ldots, n$

• Project $M(K)$ on the x-space $\rightsquigarrow N(K)$

Adding Positive Semidefiniteness:

• $M_+(K) := M(K) \cap \{Y \mid Y \succeq 0\} \rightsquigarrow N_+(K)$

$$P \subseteq N_+(K) \subseteq N(K) \subseteq N_0(K) \subseteq K$$

The Lovász-Schrijver t-th iterate:

$N^t(K) := N(N^{t-1}(K)), \ N_+^t(K) := N_+(N_+^{t-1}(K))$

$$N^t(K) \subseteq P_{i_1 \ldots i_k}(K)$$

$$P = N^n(K)$$

The Lovász-Schrijver construction: Multiplying by stronger inequalities

• Let $U = \{x \in \Re^n \mid u_r^T(1\ x) \geq 0 \ \ \forall r = 1, \ldots, R\}$ be another linear relaxation of P

• $M(K, U) :=$ linearization of the system

$$u_r^T(1\ x) \cdot g_\ell^T(1\ x) \geq 0 \ \ \forall r \ \ \forall \ell$$

That is, $M(K, U) \equiv$ all matrices Y satisfying

(i) $y_{ii} = y_{0i} \ \ \forall i = 1, \ldots, n$

(ii) $Y\widetilde{U}^* \subseteq \widetilde{K}$

• Project $M(K, U)$ on x-space $\rightsquigarrow N(K, U) \supseteq P$

$$N(K, K) \subseteq N(K, U) \subseteq N(K, [0, 1]^n) \equiv N(K)$$

$$\text{if} \ \ K \subseteq U \subseteq [0, 1]^n$$

$$N'(K) := N(K, K)$$

$$(N')^t(K) := N((N')^{t-1}(K), K)$$

studied for max-cut [La00]

Adding Positive Semidefiniteness: $\rightsquigarrow N_+(K, U)$

The Sherali-Adams construction

The Sherali-Adams t-th iterate:

• Multiply $Ax \leq b$ by each product $\prod_{i \in I} x_i \prod_{j \in J} (1 - x_j)$ for disjoint $I, J \subseteq V$ with $|I \cup J| = t$.

• Linearize by setting $y_I := \prod_{i \in I} x_i$ and $x_i^2 = x_i \rightsquigarrow R_t(K)$

• Project $R_t(K)$ on the x-space $\rightsquigarrow S_t(K)$.

$$S_1(K) = N(K)$$

$$S_t(K) \subseteq N^t(K) \text{ [LS91]}$$

$$S_t(K) \subseteq N(S_{t-1}(K)) \text{ [La01]}$$

$$\prod_{i \in I} x_i \prod_{j \in J} (1 - x_j) = \sum_{I \subseteq H \subseteq I \cup J} (-1)^{|H \setminus I|} y_H$$

$$y = (y_I)_{|I| \leq t+1} \longleftrightarrow Y := (y_{I \cup J})_{|I| \leq 1, |J| \leq t}$$

$$y \in R_t(K) \iff \begin{array}{l} Y \left(\sum_{I \subseteq H \subseteq U} (-1)^{|H \setminus I|} e_H \right) \in \tilde{K} \\ \forall I \subseteq U \ \forall U \subseteq V,\ |U| \leq t \end{array}$$

Example (continued):

$$K = \{x \in \Re^n \mid \sum_{i \in I} x_i + \sum_{i \notin I} (1 - x_i) \geq \frac{1}{2} \; \forall I \subseteq [1,n]\}$$

$$K \cap \{0,1\}^n = \emptyset \;\rightsquigarrow\; P = \emptyset$$

Fact: n iterations are needed for finding P for any of the following procedures:

- using Gomory-Chvátal cuts [CCH 1989]
- using the N_+ operator [Goemans-Tunçel 2000]
- using the N_+ operator combined with Gomory-Chvátal cuts [Cook Dash 2000]
- using the N_+ operator combined with disjunctive cuts [Cornuéjols Li 2001]
- using the Sherali-Adams construction [La 2001]

Show: $(\frac{1}{2}, \ldots, \frac{1}{2})$ belongs to the $(n-1)$-th iterate

Example: TSP problem

$$K = \{x \in \Re^E \mid \sum_{e \in \delta(v)} x_e = 2 \ (v \in V) \\ \sum_{e \in \delta(S)} x_e \geq 2 \ (\emptyset \subset S \subset V) \\ 0 \leq x \leq 1\}$$

Fact: At least $\frac{n}{8}$ iterations are needed for finding the TSP polytope starting from K

- [CCH 1989] using Gomory-Chvátal cuts

- [CD 2000] using the N_+ operator combined with Gomory-Chvátal cuts (and at most $n+1$ iterations)

Definitions for the Lasserre construction

$$\mathcal{P}(V) := \text{set of all subsets of } V = [1,n]$$

$$y \in \Re^{\mathcal{P}(V)} \rightsquigarrow M_V(y) := (y_{I\cup J})_{I,J\subseteq V},\ M_t(y) := (y_{I\cup J})_{|I|,|J|\le t}$$

its *moment matrices*

$$g, y \in \Re^{\mathcal{P}(V)} \rightsquigarrow g * y \in \Re^{\mathcal{P}(V)}$$

$$g * y(H) := \sum_{J\subseteq V} g(J)y(J\cup H)$$

Fact: $x \in K \cap \{0,1\}^n \Longrightarrow y^x := \left(\prod_{i\in I} x_i\right)_{I\subseteq V}$ satisfies:

$$M_V(y^x) \succeq 0,\ M_V(g_\ell * y^x) \succeq 0\ \forall \ell = 1,\ldots,m$$

Proof: $M_V(y^x) = y^x(y^x)^T$
$M_V(g_\ell * y^x) = g_\ell^T(1\ x) \cdot y^x(y^x)^T$

The Lasserre construction

For $t \ge 0$, $P_t(K)$ is the set of $y \in \Re^{\mathcal{P}_{2t+2}}$ satisfying:

$$M_{t+1}(y) \succeq 0,\ M_t(g_\ell * y) \succeq 0\ \forall \ell = 1,\ldots,m$$

and $Q_t(K)$ is the projection on $\Re^n$ of $P_t(K) \cap \{y \mid y_\emptyset = 1\}$. Then,

$$P \subseteq Q_n(K) \subseteq \ldots \subseteq Q_1(K) \subseteq Q_0(K) \subseteq K$$

$$P = Q_n(K)$$

Proof of Lasserre [2001]: using results about moment theory and representation of polynomials as sums of squares

Elementary proof for the Lasserre construction

Let Z be the $0-1$ matrix indexed by $\mathcal{P}(V)$ with columns the vectors y^x for $x \in \{0,1\}^n$

$$Z(I,J) = 1,\ Z^{-1}(I,J) = (-1)^{|J \setminus I|} \text{ if } I \subseteq J$$
$$Z(I,J) = Z^{-1}(I,J) = 0 \text{ otherwise}$$

Fact 1: Every 0-1 polytope is projection of a $(2^n - 1)$-simplex

Fact 2: [LS 91] $y \in \text{cone}(y^x \mid x \in \{0,1\}^n)$
$\iff Z^{-1}y \geq 0 \iff M_V(y) \succeq 0$

Proof: $M_V(y) = Z^T \text{diag}(Z^{-1}y) Z$

Fact 3: $y \in \text{cone}(y^x \mid x \in K \cap \{0,1\}^n)$
$\iff Z^{-1}y \geq 0$ and $(Z^{-1}y)_J = 0$ if $x := \chi^J \notin K$
$\iff Z^{-1}y \geq 0$ and $(Z^{-1}y)_J = 0$ if $g_\ell^T(1\ \chi^J) < 0$
$\iff (Z^{-1}y)_J \cdot g_\ell^T(1\ \chi^J) \geq 0\ \ \forall J \subseteq V$

$\iff M_V(y) \succeq 0$ and $M_V(g_\ell * y) \succeq 0\ \ \forall \ell = 1, \ldots, m$

Proof: $M_V(g_\ell * y) = Z^T \text{diag}(u) Z$
setting $u_J := (Z^{-1}y)_J \cdot g_\ell^T(1\ \chi^J)$

Corollary 1: $\quad Q_n(K) = P$

The Lasserre and Sherali-Adams relaxations are two variations of the common idea of relaxing the PSD conditions:

$$M_V(y) \succeq 0,\ M_V(g_\ell * y) \succeq 0 \text{ for } \ell = 1, \ldots, m$$

• by $M_{t+1}(y) \succeq 0$, $M_t(g_\ell * y) \succeq 0$
$\rightsquigarrow$ Lasserre relaxation $Q_t(K)$

• by $M_W(y) \succeq 0,\ M_U(g_\ell * y) \succeq 0$
for all $W \in \mathcal{P}_{t+1}(V),\ U \in \mathcal{P}_t(V)$
$\rightsquigarrow$ Sherali-Adams relaxation $S_t(K)$

Proof:

The t-th SA relaxation is obtained by linearizing the following inequalities for all $I \subseteq U \subseteq V$ with $|U| = t$:

$$\prod_{i \in I} x_i \prod_{j \in U \setminus I} (1 - x_i) \cdot g_\ell^T (1\ x) \geq 0$$

$$\left(\sum_{I \subseteq H \subseteq U} (-1)^{|H \setminus I|} y_H \right) \cdot \left(\sum_J (g_\ell)_J\ y_J \right) \geq 0$$

$$\sum_{I \subseteq H \subseteq U} (-1)^{|H \setminus I|} g_\ell * y(H) \geq 0$$

That is, $Z^{-1}(g_\ell * y) \geq 0$

where Z is the zeta matrix of $\mathcal{P}(U)$

Equivalently, $M_U(g_\ell * y) \succeq 0$

The Lasserrre hierarchy refines both the Sherali-Adams and the Lovász-Schrijver hierarchies

Theorem: [La01] For any $t \geq 1$

$$Q_t(K) \subseteq S_t(K), \quad Q_t(K) \subseteq N_+^t(K)$$

$$Q_1(K) \subseteq N_+(K) \subseteq Q_0(K)$$

setting $Y := M_1(y)$, we have:

$$g_\ell * y(\emptyset) = g_\ell^T Y e_0, \quad g_\ell * y(i) = g_\ell^T Y e_i$$

$$Q_t(K) \subseteq N_+(Q_{t-1}(K))$$

Tool: $M_{t+1}(y) \succeq 0 \Longrightarrow M_t(e_k * y),\ M_t((e_\emptyset - e_k) * y) \succeq 0$

Proof: Set $\mathcal{P}_1 := \{I \in \mathcal{P}_t(V) \mid k \notin I\}$, $\mathcal{P}_1' := \{I \in \mathcal{P}_t(V) \mid k \in I\}$, $\mathcal{P}_2 := \{I \cup \{k\} \mid I \in \mathcal{P}_1\}$.

$$M_{t+1}(y) = \begin{array}{c} \\ \mathcal{P}_1 \\ \mathcal{P}_1' \\ \mathcal{P}_2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{P}_1 & \mathcal{P}_1' & \mathcal{P}_2 & \dots \end{array} \\ \begin{pmatrix} A & D & B & \dots \\ D & C & D & \dots \\ B & D & B & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}, \quad M_t(e_k * y) = \begin{array}{c} \\ \mathcal{P}_1 \\ \mathcal{P}_1' \end{array} \begin{array}{c} \begin{array}{cc} \mathcal{P}_1 & \mathcal{P}_1' \end{array} \\ \begin{pmatrix} B & D \\ D & C \end{pmatrix} \end{array}$$

$$M_t((e_\emptyset - e_k) * y) = \begin{array}{c} \\ \mathcal{P}_1 \\ \mathcal{P}_1' \end{array} \begin{array}{c} \begin{array}{cc} \mathcal{P}_1 & \mathcal{P}_1' \end{array} \\ \begin{pmatrix} A - B & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

Prove: $Q_t(K) \subseteq N_+(Q_{t-1}(K))$ **by induction on** $t \geq 1$**.**
Assume $M_{t+1}(y),\ M_t(g_\ell * y) \succeq 0$. Set $Y := M_1(y)$. We show that $Y \in M_+(Q_{t-1}(K))$, that is,

$$Ye_k,\ Y(e_\emptyset - e_k) \in Q_{t-1}(K).$$

As Ye_k is the projection on $\Re^{\mathcal{P}_1(V)}$ of $e_k * y$, it suffices to verify that

$$M_t(e_k * y),\ M_{t-1}(g_\ell * (e_k * y)) \succeq 0$$

Use identity: $g_\ell * (e_k * y) = e_k * (g_\ell * y)$ and **Tool**.
Idem for $Y(e_\emptyset - e_k)$.

Algorithmic properties

Assume that one can solve in polynomial time the (weak) separation problem for K.

• [LS91] The same holds for $M(K)$, $M_+(K)$ and thus for the projections $N(K)$, $N_+(K)$; also for $N^t(K)$, $N_+^t(K)$ for any *fixed* t.

$\rightsquigarrow$ One can optimize in polynomial time any linear objective function (with an arbitrary precision) over $N^t(K)$, $N_+^t(K)$ for fixed t

• The same holds for the SA relaxation $S_t(K)$

• The result holds for the Lasserre relaxation $Q_t(K)$ and the LS relaxations $(N')^t(K)$, $(N')_+^t(K)$ under the additional assumption that the number m of rows in A is part of the input data

Application to the stable set polytope of $G = (V, E)$

$$P := \mathrm{ST}(G) = \mathrm{conv}(\chi^S \mid S \subseteq V \text{ stable set})$$

$$K := \mathrm{FR}(G) = \{x \in \Re^V_+ \mid x_i + x_j \leq 1 \ \forall ij \in E\}$$

$$\mathrm{ODD}(G) = \mathrm{FR}(G) \cap \{x \mid \sum_{i \in V(C)} x_i \leq \left\lfloor \frac{|C|}{2} \right\rfloor \ \forall C \text{ odd circuit in } G\}$$

$$\mathrm{ST}(G) \subseteq N_+(\mathrm{FR}(G)) \subseteq N(\mathrm{FR}(G)) \subseteq N_0(\mathrm{FR}(G)) \subseteq \mathrm{ODD}(G)$$

Theorem: [LS 1991] $N(\mathrm{FR}(G)) = \mathrm{ODD}(G)$

$\rightsquigarrow$ compact representation of $\mathrm{ST}(G)$ for G t-perfect

The N_+ operator does help

• clique, odd antihole, wheel inequalities ... are valid for $N_+(\mathrm{FR}(G))$

• $N_+(\mathrm{FR}(G)) \subseteq \mathrm{TH}(G)$

$$\mathrm{TH}(G) := \{x \in \Re^V \mid (1\ x) = Ye_0 \text{ for } Y \succeq 0 \text{ with } Y_{ij} = 0 \ \forall ij \in E\}$$

$\rightsquigarrow$ basic SDP relaxation for $\mathrm{ST}(G)$ giving the Lovász theta function

Proof: Let $Y \in M_+(\mathrm{FR}(G))$ and $ij \in E$. Then, $(Ye_j)_i + (Ye_j)_j \leq (Ye_j)_0$, i.e., $y_{ij} + y_{jj} \leq y_{0j} \Longrightarrow y_{ij} = 0$

$\rightsquigarrow$ $N_+(\mathrm{FR}(G)) = \mathrm{ST}(G)$ for G perfect

Bounds on the N-rank of $\mathrm{FR}(G)$ [LS 1991]

$$\frac{n}{\alpha(G)} - 2 \leq N\text{-rank } \leq n - \alpha(G) - 1$$

$$N_+\text{-rank } \leq \alpha(G)$$

The SA relaxations:

No clear improvement: $\frac{n}{\alpha(G)} - 2$ is still a lower bound for the Sherali-Adams rank

The Lasserre relaxations

Proposition: [La 2001] For $t \geq 1$, $Q_t(\mathrm{FR}(G))$ is the projection of

$$M_{t+1}(y) \succeq 0,\ y_{ij} = 0\ \forall ij \in E$$

$$Q_1(\mathrm{FR}(G)) \subseteq N_+(\mathrm{FR}(G)) \subseteq \mathrm{TH}(\mathrm{FR}(G)) \subseteq Q_0(\mathrm{FR}(G))$$

$$\text{Lasserre rank } \leq \alpha(G) - 1 \quad \text{if } \alpha(G) \geq 2$$

A ‘bad’ example

$G :=$ line graph of K_{2n+1}
$\mathrm{ST}(G) =$ matching polytope of K_{2n+1}

[Stephen and Tunçel 1999] N_+-rank $= \alpha(G) = n$, even starting with the stronger relaxation:

$$\sum_{e \in \delta(i)} x_e \leq 1 \quad (i \in V)$$

General setting of the methods

• The Balas-Ceria-Cornuéjols, Sherali-Adams and Lovász-Schrijver methods apply, more generally, to the case when $F = \{0,1\}^n \cap K$ where K is an arbitrary convex body and to mixed $0-1$ linear programming:

$$K = \{(x, y) \in \Re^{n+p} \mid Ax + Gy \leq b\}$$

$$F = \{(x, y) \in K \mid x \in \{0,1\}^n\}$$

• The Sherali-Adams and Lasserre methods apply, more generally, to the case when $F = \{0,1\}^n \cap K$ and K is a semi-algebraic set

$$K = \{x \in \Re^n \mid g_\ell(x) \geq 0 \;\; \forall \ell = 1, \ldots, m\}$$

where g_ℓ are polynomials which can be assumed to have degree ≤ 1 in every variable x_i

Intermezzo: The moment problem

$(S, +)$ commutative semigroup
$y = (y_\alpha)_{\alpha \in S} \in \Re^S \rightsquigarrow$ *moment matrix* $M(y) := (y_{\alpha+\beta})_{\alpha,\beta \in S}$
Given $F \subseteq S$, y is a F-*moment sequence* if

$$y_\alpha = \int_{S^*} f(\alpha) d\mu(f) \quad (\alpha \in S)$$

for some non negative measure μ supported by F, where S^* is the set of non zero multiplicative functions $f : S \longrightarrow \Re$

Fact: y is a F-moment sequence $\Longrightarrow M(y) \succeq 0$

Classic results for the semigroup $(Z_+^n, +)$:

• $n = 1$, $F = \Re$, y is a moment sequence $\Longleftrightarrow M(y) \succeq 0$ [Hamburger 1920]

• $n = 1$, $F = \Re_+$, y is a moment sequence $\Longleftrightarrow M(y) \succeq 0$ and $M(e_1 * y) \succeq 0$ [Stieltjes 1894]

• [Schmüdgen 1991] F compact semi-algebraic
$F = \{x \in \Re^n \mid g_\ell(x) \geq 0 \ \forall \ell = 1, \ldots, m\}$ where g_ℓ are polynomials
y is a moment sequence $\Longleftrightarrow M(y) \succeq 0$, $M(\prod_{\ell \in L} g_\ell * y) \succeq 0$ for all $L \subseteq [1, m]$

Result for the semigroup $(\mathcal{P}(V), \cup)$:

For $F = \{x \in \{0,1\}^n \mid g_\ell(x) \geq 0 \ \forall \ell = 1, \ldots, m\}$,
y is a moment sequence $\Longleftrightarrow M_V(y) \succeq 0$, $M_V(g_\ell * y) \succeq 0$ $\forall \ell = 1, \ldots, m$

Intermezzo continued: Representation of positive polynomials as sums of squares - dual of the theory of moments

Theorem: [Haviland 1935]
y is a F-moment sequence $\Longleftrightarrow y^T p \geq 0$ for all polynomials p nonnegative on F

$M(y) \succeq 0 \Longleftrightarrow y^T p \geq 0$ for all polynomials p that are sums of squares

Theorem: [Schmüdgen 1991]
For F compact semi-algebraic set in $\Re^n$,
p polynomial *positive* on $F \Longrightarrow p = \sum_{L \subseteq [1,m]} (p_L)^2 \left(\prod_{\ell \in L} g_\ell \right)$

Theorem: [Putinar 1993]
For F compact semi-algebraic set in $\Re^n$ such that

(*) $\exists$ polynomial $u \in \Sigma(g)$ for which the set $\{x \mid u(x) \geq 0\}$ is compact,

p is positive on $F \Longrightarrow p \in \Sigma(g)$

setting

$$\Sigma(g) := \{(p_0)^2 + (p_1)^2 g_1 + \ldots + (p_m)^2 g_m \mid p_0, \ldots, p_m \text{ polynomials}\}$$

The Lasserre approach for polynomial programming

Problem: Given a semi-algebraic set

$$F = \{x \in \Re^n \mid g_\ell(x) \geq 0 \;\; \forall \ell = 1, \ldots, m\}$$

approximate the polytope $P = \mathrm{conv}(F)$ and the program

$$p^* := \min g_0(x) \quad \text{subject to} \quad x \in F$$

Fact: $x \in \Re^n \rightsquigarrow y^x := (x_1^{\alpha_1} \ldots x_n^{\alpha_n})_{\alpha \in Z_+^n}$
$x \in F \Longrightarrow y^x$ is a F-moment sequence

Primal SDP relaxation:

$$\begin{array}{rll} p_t^* := & \min & g_0^T y \\ & \text{s.t.} & M_{t+1}(y) \succeq 0 \\ & & M_{t-v_\ell+1}(g_\ell * y) \succeq 0 \; \forall \ell \end{array}$$

Dual SDP relaxation:

$$\begin{array}{rll} \rho_t^* := & \max & -X(0,0) - \sum\limits_\ell g_\ell(0) Z_\ell(0,0) \\ & \text{s.t.} & \langle X, B_\alpha \rangle + \sum\limits_\ell \langle Z_\ell, C_\alpha^\ell \rangle = (g_0)_\alpha \;\; (\alpha \neq 0) \\ & & X, Z_\ell \succeq 0 \end{array}$$

setting

$$M_{t+1}(y) = \sum_{\alpha \neq 0} y_\alpha B_\alpha + B_0,$$

$$M_{t-v_\ell+1}(g_\ell * y) = \sum_{\alpha \neq 0} y_\alpha C_\alpha^\ell + g_\ell(0) C_0^\ell$$

$$v_\ell = \lfloor \frac{\text{degree of } g_\ell}{2} \rfloor$$

$$\mathcal{Q}_t(F) := \text{projection on } \Re^n \text{ of } \begin{array}{l} M_{t+1}(y) \succeq 0 \\ M_{t-v_\ell+1}(g_\ell * y) \succeq 0 \end{array}$$

Theorem: [Lasserre 2000]
If F satisfies Putinar's assumption **(*)** then

$$\lim_{t\to\infty} p_t^* = p^*, \qquad \bigcap_{t\geq v-1} \mathcal{Q}_t(F) = \text{conv}(F)$$

Sketch of proof: Given $\epsilon > 0$, $g_0(x) - p^* + \epsilon$ is positif on $F \Longrightarrow$ [by Putinar's theorem] it belongs to $\Sigma(g)$.
$\rightsquigarrow$ dual feasible solution (X, Z_ℓ) with value $p^* - \epsilon$
This shows: $\forall \epsilon > 0\ \exists t$ such that $p^* - \epsilon \leq p_t^* \leq p^*$.

Polynomial $0-1$ programming

$$F = \{x \in \Re^n \mid \begin{array}{l} g_\ell(x) \geq 0 \ \forall \ell = 1, \ldots, m \\ h_i(x) = x_i - x_i^2 = 0 \ \forall i = 1, \ldots, n\} \end{array}$$

Lasserre [2001] shows finite convergence:

$$\mathcal{Q}_{n+v-1}(F) = \text{conv}(F)$$

using results of Curto-Fialkow about moment sequences

Elementary proof: Reduction to moment matrices indexed by $\mathcal{P}(V)$

$$\alpha \in Z_+^n \rightsquigarrow \overline{\alpha} \in \{0,1\}^n \text{ with } \overline{\alpha}_i := \min(\alpha_i, 1)$$

$$M(h_i * y) = 0 \rightsquigarrow y_{\overline{\alpha}} = y_\alpha \quad \forall \alpha \in Z_+^n$$

For $t \geq n$,

$$M_t(y) = (y_{\alpha+\beta})_{\substack{\alpha,\beta\in Z_+^n \\ |\alpha|,|\beta|\leq t}} \succeq 0 \Longleftrightarrow M_n(y) = (y_{\alpha+\beta})_{\substack{\alpha,\beta\in Z_+^n \\ |\alpha|,|\beta|\leq n}} \succeq 0$$

$$\Longleftrightarrow M_V(y) = (y_{\alpha+\beta})_{\alpha,\beta\in\{0,1\}^n} \succeq 0$$

Application to Max-Cut

$$\max x^T A x \text{ subject to } x \in \{\pm 1\}^n$$

Cut polytope: $\mathrm{CUT}(K_n) := \mathrm{conv}(xx^T \mid x \in \{\pm 1\}^n)$

Metric polytope: $\mathrm{MET}(K_n) :=$ all symmetric matrices X with $\mathrm{diag}X = e$ and

$$X_{ij} + X_{ik} + X_{jk} \geq -1,\ X_{ij} - X_{ik} - X_{jk} \geq -1 \ \ \forall i,j,k$$

For a graph $G = (V, E)$, project on $\Re^E$

$$\rightsquigarrow \ \ \mathrm{CUT}(G),\ \mathrm{MET}(G)$$

Theorem: [Barahona Mahjoub 1986]
$\mathrm{CUT}(G) \subseteq \mathrm{MET}(G)$ with equality iff G has no K_5 minor
$\rightsquigarrow$ max-cut can be solved in polynomial time for graphs with no K_5 minor

• max-cut is NP-hard for graphs with no K_6 minor [Barahona 1982] and hard to approximate with a ratio $> 16/17$ [Hastad 1997]

• [Goemans Williamson 1995] $\exists$ approximation algorithm for max-cut with ratio 0.878
$\rightsquigarrow$ using the basic SDP relaxation

$$\max \langle A, X \rangle \text{ subject to } X \succeq 0,\ \mathrm{diag}X = e$$

How to construct stronger relaxations for the cut polytope ?

Strategy 1: Apply the lift and project construction to $K :=$ the metric polytope (in the edge space)

Strategy 2: Apply the lift and project construction to $K := [-1,1]^n$ (in the node space) and project on the edge space

Strategy 1: Apply the Lovász-Schrijver construction to $K := \mathrm{MET}(G)$, $P := \mathrm{CUT}(G)$ [La 2000]

$$\rightsquigarrow \ N(\mathrm{MET}(G))$$
$$\rightsquigarrow \ N(G) := \pi_E(N(\mathrm{MET}(K_n)))$$

$\mathrm{CUT}(G) \subseteq N(G) \subseteq N(\mathrm{MET}(G)) \subseteq \mathrm{conv}(\mathrm{MET}(G) \cap \{x \mid x_e$

Basic property: $x \in \mathrm{MET}(G)$
$x_{ij} = \pm 1 \Longrightarrow x_{ik} = \pm x_{jk}$ for any triangle ijk

Theorem:

- $\mathrm{CUT}(G) = \mathrm{conv}(\mathrm{MET}(G) \cap \{x \mid x_{e_1}, \ldots, x_{e_t} = \pm 1\})$ $\Longleftrightarrow G/\{e_1, \ldots, e_t\}$ has no K_5 minor
- $\exists\ t$ edges whose contraction produces a graph with no K_5-minor $\Longleftrightarrow \mathrm{CUT}(G) = N^t(\mathrm{MET}(G))$

Corollary:

- N-rank of $\mathrm{MET}(G) \leq n-4$ with equality if $G = K_n$, $n = 4, 5, 6, 7$
- projected N-rank $\leq n - \alpha(G) - 3$

$\mathrm{CUT}(K_6) = N'(\mathrm{MET}(K_6)) \subset N_+(\mathrm{MET}(K_6)) \subset N(\mathrm{MET}(K_6))$

Theorem: N'-rank of $\mathrm{MET}(G) \leq n-5$ for $n \geq 6$

Theorem: The projected N-rank is minor monotone

What are the minimal forbidden minors for the class of graphs with projected N-rank ≤ 1 ?

- K_6 is minimal forbidden minor
- possible candidate:

Strategy 2: Apply the lift and project construction to $K := C_n = [-1,1]^n$ and project on $\Re^E$ (instead of $\Re^n$)

$\leadsto$ relax the SDP condition:

$$M_V(y) := (y_{I\Delta J})_{I,J\subseteq V} \succeq 0$$

• **Sherali-Adams relaxation:** $S_t(C_n) \equiv$ projection on $\Re^{E_n}$ of

$$M_U(y) \succeq 0 \quad \forall U \subseteq V \text{ with } |U| = t+1$$

$\equiv$ polytope defined by all valid inequalities for $\mathrm{CUT}(K_n)$ induced by $\leq t+1$ points

• **Lasserre relaxation:** $Q_t(C_n) \equiv$ projection on $\Re^{E_n}$ of

$$M_{t+1}(y) := (y_{I\Delta J})_{I,J} \succeq 0$$

$$t = 0$$

$\leadsto$ basic SDP relaxation

$$t = 1$$

$\rightsquigarrow$ new SDP relaxation of $\mathrm{CUT}(K_n) \equiv$ projection on $\Re^{E_n}$ of the set of matrices $Y \succeq 0$ indexed by $\{\emptyset\} \cup E_n$ s.t.
(1) $Y_{ij,ik} = Y_{\emptyset,jk}\ \forall i,j,k$
(2) $Y_{ij,rs} = Y_{ir,js} = Y_{is,jr}\ \forall i,j,r,s$

Fact:
- $N'(\mathrm{MET}(K_n)) \subseteq Q_1(C_n)$
- Without condition (2), this is the relaxation of Anjos-Wolkowicz [2000]

Other techniques for generating SDP relaxations

- **Lovász-Schrijver and Lasserre methods**

- **'Direct' approach for quadratic programming:**

$F = \{x \in \Re^n \mid g_\ell(x) := x^T Q_\ell x + q_\ell^T x + \gamma_\ell \geq 0 \, \forall \ell = 1, \ldots, m\}$

$$\begin{aligned} \widehat{F} := \{x \in \Re^n \mid & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \text{ for some } X \text{ satisfying} \\ & \langle Q_\ell, X \rangle + q_\ell^T x + \gamma_\ell \geq 0 \ \forall \ell = 1, \ldots, m\} \end{aligned}$$

$$\operatorname{conv}(F) \subseteq \widehat{F}$$

$$\widehat{F} = \mathcal{Q}_0(F) \quad \equiv \quad \text{Lasserre 0-th relaxation}$$

Explicit quadratic representation for $\widehat{F}$: [Kojima and Tunçel 2000]

$$\widehat{F} = \{x \in \Re^n \mid \sum_\ell t_\ell g_\ell(x) \geq 0 \ \text{ for } t_\ell \geq 0 \text{ with } \sum_\ell t_\ell Q_\ell \preceq 0\}$$

Example of application: [Fujie Tamura 2000]

$$\mathrm{TH}(G) = \{x \in \Re^n \mid \ x^T M x - \sum_{i=1}^{n} M_{ii} x_i \leq 0 \text{ for all } M \succeq 0 \\ \text{with } M_{ij} = 0 \ (i \neq j, ij \notin E)\}$$

How to make linear constraints quadratic ?

$$F := \{x \in \{0,1\}^n \mid Ax = b\}$$

$$Ax = b \;\rightsquigarrow\; (Ax)^T Ax = \langle A^T A, xx^T \rangle = b^T b$$

$$\widehat{F} = \{x \in \Re^n \mid \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \text{ for some } X \text{ satisfying} \\ \operatorname{diag}(X) = x,\ \langle A^T A, X \rangle = b^T b\}$$

$$N_+(F) \subseteq \widehat{F}$$

Taking the dual of the Lagrangian dual (the bidual) [Poljak Rendl Wolkowicz 1995] [Shor 1985] .... [Lemaréchal Oustry 2000]

Example of max-cut:

$$\max x^T Q x \ \text{ subject to } \ x_i^2 = 1 \ (i = 1, \ldots, n)$$

Lagragian dual:

$$\min_{u \in \Re^n} \max_{x \in \Re^n} x^T Q x + \sum_i u_i (1 - x_i^2)$$

$$\min_{u \in \Re^n} \max_{x \in \Re^n} x^T (Q - \mathrm{diag} u) x + u^T e$$

$$\max = +\infty \text{ if } Q - \mathrm{diag} u \not\preceq 0$$
$$\max = u^e \ \text{ if } Q - \mathrm{diag} u \preceq 0$$

$$= \min u^T e \ \text{ subject to } \ \mathrm{diag} u - Q \succeq 0$$

Taking the dual:

$$= \max \langle Q, X \rangle \ \mathrm{diag} X = e, \ X \succeq 0$$

$\leadsto$ basic SDP relaxation

Idea: get stronger relaxations by adding redundant constraints to the max-cut formulation [Anjos Wolkowicz 2000]

max-cut:

$$\begin{array}{ll} \max & \langle Q, X \rangle \\ \text{s.t.} & X_{ii} = 1 \ (i = 1, \ldots, n) \\ & X \succeq 0 \\ & \operatorname{rank}(X) = 1 \\ & X_{ij}^2 = 1 \ (i \neq j = 1, \ldots, n) \\ & X^2 - nX = 0 \\ & X_{ij} = X_{ik} X_{jk} \ (i \neq j \neq k = 1, \ldots, n) \end{array}$$

Bidual:

$$\begin{array}{ll} \max & \sum_{ij \in E_n} Q_{ij} Y_{0,ij} \\ \text{s.t.} & Y \succeq 0 \\ & \operatorname{diag} Y = e \\ & Y_{ik,jk} = Y_{0,ij} \ (i \neq j \neq k = 1 \ldots n) \end{array}$$

Without , replace by

$$Y_{0,ij} = \frac{1}{n-2} \sum_{k \neq i,j} Y_{ik,jk}$$

Fact: $N_+(\mathrm{MET}_n)$ is contained in the AW relaxation

Example of the stable set problem:

$$\begin{array}{ll} \max & w^T x \\ \text{s.t.} & x_i x_j = 0 \ (ij \in E) \\ & x_i^2 - x_i = 0 \ (i = 1 \dots n) \end{array}$$

Bidual:

$$\begin{array}{ll} \max & w^T x \\ \text{s.t.} & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \\ & \text{diag} X = x \\ & X_{ij} = 0 \ (ij \in E) \end{array}$$

$\leadsto$ basic SDP relaxation TH(G)