Lift-and-Project Methods for $0-1$ Programming

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Given a polytope

$$\boxed{K = \{x \in \Re^n \mid Ax \leq b\}}$$

Find the linear description of the $0-1$ polytope

$$\boxed{P := \mathrm{conv}(K \cap \{0,1\}^n)}$$

That is, find a linear system $A'x \leq b'$ such that

$$\boxed{P = \{x \in \Re^n \mid Ax \leq b,\ A'x \leq b'\}}$$

Example 1: P **:= TSP polytope of** $G = (V, E)$

$$\begin{array}{lll} \min & \sum_{e \in E} c_e x_e & \\ \text{s.t.} & \sum_{e \in \delta(i)} x_e = 2 & (i \in V) \\ & \sum_{e \in \delta(S)} x_e \geq 2 & (\emptyset \subset S \subset V) \\ & x_e \geq 0 & (e \in E) \\ & x \in \{0,1\}^E & \end{array}$$

Example 2: P **:= stable set polytope of** G

$$\begin{array}{lll} \max & \sum_{i \in V} w_i x_i & \\ \text{s.t.} & x_i + x_j \leq 1 & (ij \in E) \\ & x_i \geq 0 & (i \in V) \\ & x \in \{0,1\}^V & \end{array}$$

Example 3: P **:= matching polytope of** G

$$\begin{array}{lll} \max & \sum_{e \in E} w_e x_e & \\ \text{s.t.} & \sum_{e \in \delta(i)} x_e \leq 1 \ (i \in V) & \\ & x_e \geq 0 & (e \in E) \\ & x \in \{0,1\}^E & \end{array}$$

Theorem [Edmonds 1965]:

$$P = K \cap \{x \mid \sum_{e \subseteq U} x_e \leq \frac{|U| - 1}{2} \ \text{ for } U \subseteq V, \ |U| \text{ odd}\}$$

Contents of the talk

• Brief reminder on cutting planes

• lift-and-project methods:

(i) Balas-Ceria-Cornuéjols

(ii) Lovász-Schrijver

(iii) Sherali-Adams

(iv) Lasserre

• Application to the stable set polytope

• Application to Max-Cut

More about Lasserre's approach:

• Application to polynomial programming

• Algebraic background: Representation of nonnegative polynomials as sums of squares and the dual theory of moment sequences

Cutting plane methods

Fact: Given a *integral*
$a^T x \le \beta$ is valid for $K \Longrightarrow a^T x \le \lfloor \beta \rfloor$ is valid for P.

$$\begin{array}{ccc} u_1 & \cdot & a_1^T x \le b_1 \\ & \vdots & \\ u_m & \cdot & a_m^T x \le b_m \end{array}$$

$$a^T x := (\sum_{j=1}^{m} u_j a_j)^T x \le \sum_{j=1}^{m} u_j b_j \le \beta$$

Gomory-Chvátal cuts: [Gomory 1960]

$$K' := \{x \mid u^T A x \le \lfloor u^T b \rfloor \; \forall u \ge 0 \text{ with } u^T A \text{ integral}\}$$

$$K^{t+1} := (K^t)'$$

Theorem: [Chvátal 73, Schrijver 80]
If K is a rational polyhedron, then K' is a rational polyhedron and $P = K^t$ for some integer t.

Example: $K := \{x \in \Re_+^E \mid \sum_{e \in \delta(u)} x_e \le 1 \; \forall v \in V\}$

The blossom ineq: $\sum_{e \in E(U)} x_e \le \left\lfloor \frac{|U|}{2} \right\rfloor$ ($U \subseteq V$, $|U|$ odd)
have Chvátal rank 1
$\rightsquigarrow K' \equiv$ matching polytope of G

The Chvátal rank can be large (depends on n and A)

$$K = \{(x, y) \in \Re^2 \mid x \geq 0,\ x - 2ky \leq 0,\ x + 2ky \leq 2k\}$$

has Chvátal rank k

$\boxed{\text{When } K \subseteq [0,1]^n}$

Bound on the Chvátal rank:

$\boxed{t \leq O(n^2 \log n)}$

[Eisenbrand Schulz 1999]

Exist examples where $t > n$

$\boxed{t \leq n \text{ if } K \cap \{0,1\}^n = \emptyset}$

[Bockmayr, Eisenbrand, Hartmann, Schulz 1999]

Example: [Chvátal, Cook, Hartmann 1989]
Chvátal rank $= n$ and $K \cap \{0,1\}^n = \emptyset$ for

$$K := \{x \in \Re^n \mid \sum_{i \in I} x_i + \sum_{i \notin I} (1 - x_i) \geq \frac{1}{2}\ \ \forall I \subseteq [1, n]\}$$

$$= \{x \in \Re^n \mid \left\| x - \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \right\|_1 \leq \frac{n-1}{2}\}$$

What about optimization ?

Theorem [Eisenbrand 1999]:
Testing membership in the first Chvátal closure is a co-NP-complete problem.

$\rightsquigarrow$ Optimizing over the first Chvátal closure is a hard problem

Goal:

Construct a relaxation K' satisfying the following properties:

- $P = \mathrm{conv}(K \cap \{0,1\}^n) \subseteq K' \subseteq K$
- one can optimize over K' in polynomial time
- one can find P in a finite number of steps (say, n)

$\rightsquigarrow$ **lift-and-project methods**

Lift and Project Methods

- Balas-Ceria-Cornuéjols [BCC] (1993)
- Lovász-Schrijver [LS] (1991)
- Sherali-Adams [SA] (1990)
- Lasserre (2000)

Common features:

Construct a hierarchy of *linear* (BCC, SA, LS) or *positive semidefinite* (LS_+, Lasserre) relaxations of P:

$$\boxed{P \subseteq K^n \subseteq K^{n-1} \subseteq \ldots \subseteq K^2 \subseteq K^1 \subseteq K}$$

satisying:

- $P = K^n$

- One can optimize a linear objective function in polynomial time over K^t for *fixed* t assuming

- existence of a separation oracle for K (BCC, LS, SA)
- the number of rows of A is polynomial in n (Lasserre)

The following inclusions hold:

$$\boxed{\begin{array}{l} \text{SA} \subseteq \text{LS} \subseteq \text{BCC} \\ \text{Lasserre} \subseteq \text{SA} \cap \text{LS}_+ \end{array}}$$

Some notation:

$$K = \{x \in \Re^n \mid Ax \leq b\}$$

$$P = \mathrm{conv}(K \cap \{0,1\}^n)$$

[Assume that the constraints $0 \leq x_i \leq 1$ $(i = 1, \ldots, n)$ are present in the system $Ax \leq b$]

It will be convenient to embed the polytope K in the space $\Re^{n+1}$

$$\tilde{K} = \left\{\lambda\binom{1}{x} \mid x \in K\right\} = \left\{\binom{x_0}{x} \in \Re^{n+1} \mid Ax \leq bx_0\right\}$$

The cone $\tilde{K}$ is the **homogenization** of K

Rewrite the system $Ax \leq b$ as

$$a_\ell^T x \leq b_\ell, \ \ \text{equivalently,} \ \ g_\ell^T \binom{1}{x} \geq 0 \quad (\ell = 1, \ldots, m)$$

$$\text{setting } g_\ell := \begin{pmatrix} b_\ell \\ -a_\ell \end{pmatrix}$$

$$\text{Then, } \tilde{K} = \{y \in \Re^{n+1} \mid g_\ell^T y \geq 0 \ \ (\ell \leq m)\}$$

A common recipe for the lift-and-project methods

(1) Multiply the system $Ax \leq b$ by an inequality $c^T x \leq \gamma$, which is valid for P

$\rightsquigarrow$ non-linear system: $(\gamma - c^T x)(b_\ell - a_\ell^T x) \geq 0$ $(\ell \leq m)$ valid for P

(2) Linearize by setting $y_{ij} := x_i x_j$ $(i \neq j)$ and $x_i^2 = 1$

$\rightsquigarrow$ linear system in variable (x, y)

(3) Project back on the space $\Re^n$ of the x-variable

$\rightsquigarrow$ linear relaxation K' of P satisfying:

$$\boxed{P \subseteq K' \subseteq K}$$

The methods vary:

- in the choice of multipliers [some, or all of the inequalities $x_i \geq 0$, $1 - x_i \geq 0$,...]
- in the way of iterating the basic step

The Balas-Ceria-Cornuéjols construction

(0) Choose an index in $[1, n]$, say 1

(1) Multiply $Ax \leq b$ by x_1 and $1 - x_1$:

$$\boxed{x_1(b - Ax) \geq 0, \quad (1 - x_1)(b - Ax) \geq 0}$$

(2) Linearize by setting $y_i := x_1 x_i$, $y_1 = x_1$ $(x_1^2 = x_1)$
$\rightsquigarrow$ polytope $M_1(K)$ in the (x, y)-space defined by:

$$\boxed{y_1 = x_1, \quad bx_1 - Ay \geq 0, \quad b - Ax - bx_1 + Ay \geq 0}$$

(3) Project $M_1(K)$ on the x-subspace $\Re^n \rightsquigarrow P_1(K)$

Fact: $\boxed{P \subseteq P_1(K) = \text{conv}(K \cap \{x \mid x_1 = 0, 1\})}$

$$\boxed{P \subseteq N_0(K) := \bigcap_{j=1}^{n} P_j(K)}$$

(4) Define iteratively:

$$P_{i_1 \ldots i_k}(K) := P_{i_k}(P_{i_{k-1} \ldots i_1}(K))$$
$$= \text{conv}(K \cap \{x \mid x_{i_1}, \ldots, x_{i_k} = 0, 1\})$$

$$\boxed{P = P_{i_1 \ldots i_n}(K)}$$

$$\boxed{P_1(K) \stackrel{?}{=} \operatorname{conv}(K \cap \{x \mid x_1 = 0, 1\})}$$

Proof:

$$x \in \operatorname{conv}(K \cap \{x \mid x_1 = 0, 1\})$$

$$\Updownarrow$$

$$x = \alpha u + (1-\alpha)v \text{ for some } 0 \le \alpha \le 1,$$
$$u, v \in K,\ u_1 = 1 \text{ and } v_1 = 0$$

$$\Updownarrow$$

[setting $y := \alpha u$ and using the fact that $\alpha = x_1$]

$$\begin{cases} Ay \le bx_1 \\ A(x-y) \le (1-x_1)b \\ y_1 = x_1 \end{cases}$$

$$\Updownarrow$$

$$(x, y) \in M_1(K) \text{ for some } y$$

$$\Updownarrow$$

$$x \in P_1(K)$$

The Lovász-Schrijver construction: The N operator

(1) Multiply $Ax \le b$ by x_i and $1 - x_i$ for *all* $i = 1, \ldots, n$

$$\boxed{(b - Ax)x_i \ge 0, \quad (b - Ax)(1 - x_i) \ge 0 \quad \forall i = 1, \ldots, n}$$

equivalently:

$$\boxed{g_\ell^T \binom{1}{x}\binom{1}{x}^T e_i \ge 0, \; g_\ell^T \binom{1}{x}\binom{1}{x}^T (e_0 - e_i) \ge 0 \;\; \forall i \le n, \; \forall \ell \le m}$$

(2) Linearize by setting $Y := \binom{1}{x}\binom{1}{x}^T$

$$\boxed{\begin{array}{ll} \leadsto \; M(K) := \{Y = (y_{ij})_{i,j=0}^n \mid & Y \text{ symmetric} \\ & Ye_0 = \mathrm{diag}(Y) \quad [y_{ii} = y_{0i} \;\; \forall i] \\ & Ye_i, \; Y(e_0 - e_i) \in \tilde{K} \;\; \forall i\} \end{array}}$$

(3) Project $M(K)$ on the x-subspace

$$\boxed{\leadsto N(K) := \{x \in \Re^n \mid \exists Y \in M(K) \text{ s.t. } \binom{1}{x} = Ye_0\}}$$

(4) Define iteratively: $N^t(K) := N(N^{t-1}(K))$

Adding positive semidefiniteness: The N_+ operator

$$\boxed{M_+(K) := M(K) \cap \{Y \mid Y \succeq 0\}}$$

$$\boxed{N_+(K) := \{x \in \Re^n \mid \exists Y \in M_+(K) \text{ s.t. } \binom{1}{x} = Ye_0\}}$$

Iterate: $N_+^t(K) := N_+(N_+^{t-1}(K))$

Facts:

(1) $P \subseteq N_+(K) \subseteq N(K) \subseteq K$

(2) $N(K) \subseteq P_i(K)$ [BCC relaxation] $\forall i$

(3) $N^t(K) \subseteq P_{i_1 \dots i_k}(K)$

(4) $P = N^n(K)$

Proof of (2): Let $x \in N(K)$ and $Y \in M(K)$ s.t. $\binom{1}{x} = Ye_0$.

$$\text{Then, } \binom{1}{x} = Ye_i + Y(e_0 - e_i), \quad \text{with}$$

$$Ye_i \in \tilde{K},\ Y(e_0 - e_i) \in \tilde{K}$$

$$(Ye_i)_i = Y_{ii} = Y_{0i} = (Ye_i)_0, \quad Y(e_0 - e_i)_i = Y_{0i} - Y_{ii} = 0$$

$$\text{Thus, } \binom{1}{x} \in \Re_+(\tilde{K} \cap \{y \mid y_i = y_0, 0\}),$$

$$\text{that is, } \quad x \in \mathrm{conv}(K \cap \{x \mid x_i = 0, 1\}) = P_i(K)$$

The Sherali-Adams construction

The Sherali-Adams t-th iterate ($t \geq 1$ integer):

(1) Multiply $Ax \leq b$ by all products $\prod_{i \in I} x_i \prod_{j \in J} (1 - x_j)$ for disjoint $I, J \subseteq V$ with $|I \cup J| = t$.

(2) Linearize by setting $y_I := \prod_{i \in I} x_i$ and $x_i^2 = x_i$
$\rightsquigarrow$ polytope $R_t(K)$

(3) Project $R_t(K)$ on the x-subspace $\rightsquigarrow S_t(K)$

Facts:

(1) $S_1(K) = N(K)$

(2) $S_t(K) \subseteq N(S_{t-1}(K))$

(3) $S_t(K) \subseteq N^t(K)$

Example (continued):

$$K = \{x \in \Re^n \mid \sum_{i \in I} x_i + \sum_{i \notin I}(1 - x_i) \geq \frac{1}{2} \; \forall I \subseteq [1, n]\}$$

$$K \cap \{0, 1\}^n = \emptyset \; \rightsquigarrow \; P = \emptyset$$

Fact: n iterations are needed for finding P for any of the following procedures:

- using Gomory-Chvátal cuts [CCH 1989]
- using the N_+ operator [Goemans-Tunçel 2000]
- using the N_+ operator combined with Gomory-Chvátal cuts [Cook Dash 2000]
- using the N_+ operator combined with disjunctive cuts [Cornuéjols Li 2001]
- using the Sherali-Adams construction [La 2001]

Show: $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ belongs to the $(n-1)$-th iterate

This is an example where positive semidefiniteness does not help!

Proposition [Goemans-Tunçel 2000]:
If $K \cap \{x \mid x_i = 1\} = K \cap \{x \mid x_i = 0\} + e_i$ for all i, then

$$N_+(K) = N(K) = \cap_{i=1}^n \text{conv}(K \cap \{x \mid x_i = 0, 1\})$$

Intermezzo: A canonical lifting for $0/1$ polytopes

$$x \in \{0,1\}^n \rightsquigarrow \zeta^x := \Big(\prod_{i\in I} x_i\Big)_{I\subseteq V} \in \{0,1\}^{\mathcal{P}(V)}$$

$$\zeta^x = \Big(1\ x_1 \ldots x_n\ x_1x_2\ \ldots\ldots\ \prod_{i=1}^{n} x_i\Big)$$

$Z :=$ matrix with columns ζ^x ($x \in \{0,1\}^n$)

Equivalently, Z is the square $0/1$ matrix indexed by $\mathcal{P}(V)$ with entries

$$Z(I,J) = 1 \Longleftrightarrow I \subseteq J$$

Z is known the *Zeta matrix* of the lattice $\mathcal{P}(V)$

$$Z = \begin{array}{c} \emptyset \\ 1 \\ \vdots \\ n \\ 12 \\ \vdots \\ \vdots \\ V \end{array} \left(\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right)$$

Fact 1: Z is non-singular; its inverse is Z^{-1}, the *Möbius matrix* of $\mathcal{P}(V)$, with entries

$$Z^{-1}(I,J) = \begin{cases} (-1)^{|J\setminus I|} & \text{if } I \subseteq J \\ 0 & \text{otherwise} \end{cases}$$

Note: (1) $\zeta^x(\zeta^x)^T \succeq 0$; **(2)** $\zeta^x(\zeta^x)^T(I,J) = \zeta^x(I \cup J)$
Thus, the (I,J)th entry of the matrix $\zeta^x(\zeta^x)^T$ depends *only on the union* $I \cup J$.

Definition: Given $y \in \Re^{\mathcal{P}(V)}$, let $M_V(y)$ denote the matrix indexed by $\mathcal{P}(V)$ with entries

$$\boxed{M_V(y)(I,J) := y_{I\cup J} \quad (I, J \subseteq V),}$$

the **(combinatorial) moment matrix** of y.
Example: $M_V(\zeta^x) = \zeta^x(\zeta^x)^T$

Fact 2: $\boxed{M_V(y) = Z\mathrm{diag}(Z^{-1}y)Z^T}$

Proof: The (I,J)th entry of $Z\mathrm{diag}(Z^{-1}y)Z^T$ is equal to

$$\sum_H Z(I,H)Z(J,H)(Z^{-1}y)_H = \sum_{H|I\cup J\subseteq H} \left(\sum_{K|K\subseteq H} (-1)^{|K\setminus H|} y_K\right)$$
$$= \sum_K y_K \left(\sum_{H|I\cup J\subseteq H\subseteq K} (-1)^{|K\setminus H|}\right) = y(I \cup J)$$

as the inner sum is equal to 1 if $K = I \cup J$, and to 0 otherwise.

Corollary [LS91]: Given $y \in \Re^{\mathcal{P}(V)}$

$$\boxed{y \in \Re_+(\zeta^x \mid x \in \{0,1\}^n) \Longleftrightarrow Z^{-1}y \geq 0 \Longleftrightarrow M_V(y) \succeq 0}$$

Taking into account additional constraints:

$$K = \{x \in \Re^n \mid Ax \le b\} = \{x \in \Re^n \mid g_\ell^T \binom{1}{x} \ge 0\ \forall \ell\}$$

Definition: Given $y \in \Re^{\mathcal{P}(V)}$, define the vector $g_\ell * y \in \Re^{\mathcal{P}(V)}$ with entries

$$(g_\ell * y)_H := \sum_K (g_\ell)_K y_{H \cup K} \quad \forall H \subseteq V$$

[Extend g_ℓ to $\Re^{\mathcal{P}(V)}$ by zeros: $(g_\ell)_K := 0$ for $|K| \ge 2$]

Note: $g_\ell * \zeta^x = g_\ell^T \binom{1}{x}\ \zeta^x \Longrightarrow M_V(g_\ell * \zeta^x) \succeq 0$ if $x \in K$

Fact 3: For $y \in \Re^{\mathcal{P}(V)}$

$$\boxed{M_V(g_\ell * y) = Z\ \mathrm{diag}(u) Z^T, \text{ where } u_H := (Z^{-1}y)_H \cdot g_\ell^T \binom{1}{\chi^H}\ \forall H}$$

Corollary: For $y \in \Re^{\mathcal{P}(V)}$

$$\begin{aligned} y \in \Re_+(\zeta^x \mid x \in K) &\Longleftrightarrow Z^{-1}y \ge 0,\ (Z^{-1}y)_H = 0 \ \text{ for } \chi^H \notin K \\ &\Longleftrightarrow M_V(y) \succeq 0,\ M_V(g_\ell * y) \succeq 0\ \forall \ell \end{aligned}$$

Theorem:

P is equal to the projection on $\Re^n$ of the set

$$\{y \in \Re^{\mathcal{P}(V)} \mid y_\emptyset = 1,\ M_V(y) \succeq 0,\ M_V(g_\ell * y) \succeq 0\ \forall \ell\}$$

The Lasserre hierarchy of relaxations

Idea: Relax the conditions:

$$M_V(y) \succeq 0,\ M_V(g_\ell * y) \succeq 0$$

by requiring positive semidefiniteness of certain principal submatrices only.

Definition: Set $\mathcal{P}_t(V) := \{I \subseteq V \mid |I| \leq t\}$, for $t \geq 1$

$M_t(y) :=$ principal submatrix of $M_V(y)$ indexed by $\mathcal{P}_t(V)$

Thus, $M_n(y) = M_V(y)$

$Q_t(K) :=$ projection on $\Re^n$ of the set

$\{y \in \Re^{\mathcal{P}_{2t+2}(V)} \mid y_0 = 1,\ M_{t+1}(y) \succeq 0,\ M_t(g_\ell * y) \succeq 0\ \forall \ell\}$

$$\boxed{P = Q_n(K) \subseteq \ldots \subseteq Q_t(K) \subseteq \ldots \subseteq Q_0(K) \subseteq K}$$

Note:

- Lasserre (2001) proves the finite convergence using Curto-Fialkow's result on 'flat extensions' of moment sequences.
- His construction is motivated by results on the representation of positive polynomials as sums of squares of polynomials.

The Lasserre and Sherali-Adams relaxations are two variations of the common idea of relaxing the psd conditions:

$$M_V(y) \succeq 0, \ M_V(g_\ell * y) \succeq 0 \ \text{ for } \ell = 1, \ldots, m$$

• by $M_{t+1}(y) \succeq 0$, $M_t(g_\ell * y) \succeq 0$
$\rightsquigarrow$ Lasserre relaxation $Q_t(K)$

• by $M_W(y) \succeq 0, \ M_U(g_\ell * y) \succeq 0$
for all $W \in \mathcal{P}_{t+1}(V), \ U \in \mathcal{P}_t(V)$
$\rightsquigarrow$ Sherali-Adams relaxation $S_t(K)$

where $M_U(y)$ is the principal submatrix of $M_V(y)$ indexed by $\mathcal{P}(U)$

To prove: The SA relaxation can be alternatively described by the psd conditions:

$$M_W(y) \succeq 0,\ M_U(g_\ell * y) \succeq 0,\ \ \forall W \in \mathcal{P}_{t+1}(V),\ U \in \mathcal{P}_t(V),\ \ell$$

Work out the linearization of the constraints:

$$(b_\ell - a_\ell^T x) \cdot \left(\prod_{i \in I} x_i \prod_{j \in J} (1 - x_j) \right)$$

$$\boxed{\begin{aligned} \prod_{i \in I} x_i \prod_{j \in J} (1 - x_j) &= \sum_{I \subseteq H \subseteq I \cup J} (-1)^{|H \setminus I|} \prod_{h \in H} x_h \\ &\overset{linearization}{=} \sum_{I \subseteq H \subseteq I \cup J} (-1)^{|H \setminus I|} y_H \end{aligned}}$$

Therefore:

$$(b_\ell - a_\ell^T x) \cdot \left(\prod_{i \in I} x_i \prod_{j \in J} (1 - x_j) \right) = g_\ell^T \binom{1}{x} \cdot \left(\prod_{i \in I} x_i \prod_{j \in J} (1 - x_j) \right)$$

$$= \left(\sum_{K \subseteq V} (g_\ell)_K \prod_{k \in K} x_k \right) \left(\sum_{I \subseteq H \subseteq I \cup J} (-1)^{|H \setminus I|} \prod_{h \in H} x_h \right)$$

$$= \sum_{I \subseteq H \subseteq I \cup J} (-1)^{|H \setminus I|} \left(\sum_K (g_\ell)_K \prod_{h \in H \cup K} x_h \right)$$

$$\overset{linearization}{=} \sum_{I \subseteq H \subseteq I \cup J} (-1)^{|H \setminus I|} \underbrace{\left(\sum_K (g_\ell)_K y_{H \cup K} \right)}_{(g_\ell * y)_H}$$

Therefore, the SA relaxation is defined by the linear inequalities:

$$(\mathbf{1}) \qquad \sum_{I \subseteq H \subseteq W} (-1)^{|H \setminus I|} y_H \geq 0 \quad (I \subseteq W)$$

for all $W \in \mathcal{P}_{t+1}(V)$

$$(\mathbf{2}) \qquad \sum_{I \subseteq H \subseteq U} (-1)^{|H \setminus I|} (g_\ell * y)_H \geq 0 \quad (I \subseteq U)$$

for all $U \in \mathcal{P}_t(V)$ and $\ell \leq m$

Now:

$$(\mathbf{1}) \iff Z^{-1} y \geq 0 \quad (\text{where } Z := \text{ Zeta matrix of } \mathcal{P}(W))$$

$$\iff M_W(y) \succeq 0$$

$$(\mathbf{2}) \iff Z^{-1}(g_\ell * y) \geq 0 \; (\text{where } Z := \text{ Zeta matrix of } \mathcal{P}(U))$$

$$\iff M_U(g_\ell * y) \succeq 0$$

The Lasserrre hierarchy refines both the Sherali-Adams and the Lovász-Schrijver hierarchies

Theorem: [La01] For any $t \geq 1$

$$\boxed{\begin{array}{l} (1)\ Q_t(K) \subseteq S_t(K) \\[1em] (2)\ Q_t(K) \subseteq N_+(Q_{t-1}(K)) \\[1em] (3)\ Q_t(K) \subseteq N_+^t(K) \end{array}}$$

Proof of (2): Let $x \in Q_t(K)$. Then, $x = (y_1, \ldots, y_n)$ for some $y \in \Re^{\mathcal{P}_{2t+2}(V)}$ s.t.

$$y_0 = 1,\ M_{t+1}(y) \succeq 0,\ M_t(g_\ell * y) \succeq 0.$$

We show that

$$Y := M_1(y) \in M_+(Q_{t-1}(K));$$

equivalently, for $k = 1, \ldots, n$,

$$Ye_k,\ Y(e_0 - e_k) \in \widetilde{Q_{t-1}}(K)$$

Consider Ye_k. We have to find a vector $u \in \Re^{\mathcal{P}_{2t}(V)}$ s.t.

$$M_t(u) \succeq 0,\ M_{t-1}(g_\ell * u) \succeq 0,\ u_I = (Ye_k)_I \text{ for } |I| \leq 1.$$

Answer: Take $u := e_k * y$.

Lemma 1: $M_{t+1}(y) \succeq 0 \Longrightarrow M_t(e_k * y) \succeq 0$

Proof: Define

$$\mathcal{I}_1 := \{I \mid |I| \le t,\ k \in I\}$$
$$\mathcal{I}_2 := \{I \mid |I| = t+1,\ k \in I\}$$
$$\mathcal{I}_1 - k := \{I \setminus \{k\} \mid I \in \mathcal{I}_1\}$$
$$\mathcal{I}_2 - k := \{I \setminus \{k\} \mid I \in \mathcal{I}_2\}$$

The sets $\mathcal{I}_1$ and $\mathcal{I}_2$ are contained in $\mathcal{P}_{t+1}(V)$ and the sets $\mathcal{I}_1 - k$, $\mathcal{I}_2 - k$, and $\mathcal{I}_1$ partition $\mathcal{P}_t(V)$

$$M_{t+1}(y) = \begin{array}{c} \\ \mathcal{I}_1 \\ \mathcal{I}_2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} \mathcal{I}_1 & \mathcal{I}_2 & \dots \end{array} \\ \begin{pmatrix} A & B & \dots \\ B^T & C & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

$$M_t(e_k * y) = \begin{array}{c} \\ \mathcal{I}_1 \\ \mathcal{I}_1 - k \\ \mathcal{I}_2 - k \end{array} \begin{array}{c} \begin{array}{ccc} \mathcal{I}_1 & \mathcal{I}_1 - k & \mathcal{I}_2 - k \end{array} \\ \begin{pmatrix} A & A & B \\ A & A & B \\ B^T & B^T & C \end{pmatrix} \end{array}$$

Lemma 2: $g_\ell * (e_k * y) = e_k * (g_\ell * y)$

Corollary: $M_t(e_k * y) \succeq 0,\ M_{t-1}(g_\ell * (e_k * y)) \succeq 0$

Algorithmic properties

Assume that one can solve in polynomial time the (weak) separation problem for K.

• [LS91] The same holds for $M(K)$, $M_+(K)$ and thus for the projections $N(K)$, $N_+(K)$; also for $N^t(K)$, $N_+^t(K)$ for any *fixed* t.

$\rightsquigarrow$ One can optimize in polynomial time any linear objective function (with an arbitrary precision) over $N^t(K)$, $N_+^t(K)$ for fixed t

• The same holds for the SA relaxation $S_t(K)$

• The result holds for the Lasserre relaxation $Q_t(K)$ and the LS relaxations $(N')^t(K)$, $(N')_+^t(K)$ under the additional assumption that the number m of rows in A is part of the input data

Back to the Lovász-Schrijver N and N_+ operators:

Two useful lemmas for proving validity of an inequality: $a^T x \le a_0$ for $N(K)$, $N_+(K)$

Lemma 1:
If $a^T x \le a_0$ is valid for $K \cap \{x \mid x_i = 0, 1\}$ for some index $i \in \{1, \ldots, n\}$, then $a^T x \le a_0$ is valid for $N(K)$.

Proof: $N(K) \subseteq \text{conv}(K \cap \{x \mid x_i = 0, 1\}) \;\; \forall i \le n$.

Lemma 2: Assume $a_0, \ldots, a_n \ge 0$.
If $a^T x \le a_0$ is valid for $K \cap \{x \mid x_i = 1\}$ for all indices $i \in \{1, \ldots, n\}$ for which $a_i > 0$, then $a^T x \le a_0$ is valid for $N_+(K)$.

Application to the stable set polytope

$$P := \mathrm{STAB}(G) = \mathrm{conv}(\chi^S \mid S \subseteq V \text{ stable set})$$
$$K := \mathrm{FRAC}(G) = \{x \in \Re_+^V \mid x_i + x_j \leq 1 \ \forall ij \in E\}$$

Note: In both polytopes STAB(G) and FRAC(G):

Setting $x_i = 0$ amounts to *deleting* node i from G

$$\begin{array}{ccc} \mathrm{STAB}(G) \cap \{x \mid x_i = 0\} & \longleftrightarrow & \mathrm{STAB}(G \backslash i) \\ \mathrm{FRAC}(G) \cap \{x \mid x_i = 0\} & \longleftrightarrow & \mathrm{FRAC}(G \backslash i) \\ a^T x \leq a_0 & \longleftrightarrow & \sum_{j \in V \backslash \{i\}} a_j x_j \leq a_0 \quad \text{(del)} \end{array}$$

Setting $x_i = 1$ amounts to *destroying* node i, i.e., deleting its extended neighborhood:

$$\tilde{N}(i) := \{i\} \cup \{j \mid j \text{ adjacent to } i\}$$

$$\begin{array}{ccc} \mathrm{STAB}(G) \cap \{x \mid x_i = 1\} & \longleftrightarrow & \mathrm{STAB}(G \backslash \tilde{N}(i)) \\ \mathrm{FRAC}(G) \cap \{x \mid x_i = 1\} & \longleftrightarrow & \mathrm{FRAC}(G \backslash \tilde{N}(i)) \\ a^T x \leq a_0 & \longleftrightarrow & \sum_{j \in V \backslash \tilde{N}(i)} a_j x_j \leq a_0 - a_i \quad \text{(des)} \end{array}$$

Lemma 1: Consider an inequality $a^T x \leq a_0$.
If there is a node i for which the inequalities (del) and (des), obtained by deleting and destroying node i, are valid for FRAC(G), then $a^T x \leq a_0$ is valid for N(FRAC(G))

Example: Each odd circuit inequality: $\sum_{i \in C} x_i \leq \left\lfloor \frac{|C|}{2} \right\rfloor$ is valid for $N(\mathrm{FRAC}(G))$.

Theorem [Lovász-Schrijver 91]:
$N(\mathrm{FRAC}(G)) = \mathrm{CSTAB}(G)$, the linear relaxation defined by non-negativity, the edge inequalities, and the odd circuit inequalities.

$\rightsquigarrow$ compact representation for $\mathrm{CSTAB}(G)$

$\rightsquigarrow$ compact representation for $\mathrm{STAB}(G)$ for t-perfect graphs

Lemma 2:
Consider an inequality $a^T x \leq a_0$ with all $a_i \geq 0$.
If, for every node i s.t. $a_i > 0$, the inequality (des), obtained by destroying node i, is valid for FRAC(G), then $a^T x \leq a_0$ is valid for N_+(FRAC(G))

Examples: Clique, odd antihole, odd wheel inequalities are valid for N_+(FRAC(G))

Hence, the N_+ operator does help!

Theorem [LS91]: $\boxed{N_+(\mathrm{FRAC}(G)) \subseteq \mathrm{TH}(G)}$

Proof: Let $x \in N_+(\mathrm{FRAC}(G))$, i.e., $\binom{1}{x} = Ye_0$ for some $Y \in M_+(\mathrm{FRAC}(G))$. It suffices to show that

$$Y_{ij} = 0 \quad (ij \in E).$$

Fix an edge $ij \in E$ and consider the ith column Ye_i of Y. Then,

- $Ye_i \geq 0$, implying that $(Ye_i)_j = Y_{ij} \geq 0$
- $(Ye_i)_i + (Ye_i)_j \leq (Ye_i)_0$, implying $Y_{ii} + Y_{ij} \leq Y_{0i} = Y_{ii}$ and, thus, $Y_{ij} \leq 0$.

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How many iterations for finding $\mathrm{STAB}(G)$?

$$t(G) := \text{ smallest } t \text{ s.t. } N^t(\mathrm{FRAC}(G)) = \mathrm{STAB}(G)$$
$$t_+(G) := \text{ smallest } t \text{ s.t. } N^t(\mathrm{FRAC}(G)) = \mathrm{STAB}(G)$$

$$t(G) = 0 \Longleftrightarrow G \text{ bipartite}$$
$$t(G) = 1 \Longleftrightarrow G \text{ is } t\text{-perfect}$$
$$t_+(G) = 1 \text{ for } G \text{ perfect}$$

Facts:

(1) $t(G) \leq \min\limits_{i \in V} \; t(G\backslash i) + 1$

(2) $t_+(G) \leq \max\limits_{i \in V} \; t_+(G\backslash \tilde{N}(i)) + 1$

(3) $t(G) \geq \frac{n}{\alpha(G)} - 2$

Corollary:

$$\boxed{\begin{array}{l} (1)\ t(G) \leq n - \alpha(G) - 1 \\[2em] (2)\ t_+(G) \leq \alpha(G) \end{array}}$$

A worst case example, where $\alpha(G)$ iterations of the N_+ operator are necessary:
when G is the line graph of K_{2n+1}
STAB(G) is the matching polytope of K_{2n+1},
$t_+(G) = \alpha(G) = n$ [Stephen-Tunçel 99]

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The Lasserre relaxations

Proposition: [La 2001]
For $t \geq 1$, $Q_t(\text{FRAC}(G))$ is defined as the projection of

$$M_{t+1}(y) \succeq 0,\ M_t(g_{ij} * y) \succeq 0 \quad (ij \in E)$$

[where $g_{ij} := 1 - x_i - x_j$]
Equivalently, $Q_t(\text{FRAC}(G))$ is the projection of

$$M_{t+1}(y) \succeq 0,\ y_{ij} = 0 \ (ij \in E)$$

Recall: $Q_t(\text{FRAC}(G)) \subseteq N_+^t(\text{FRAC}(G))$.

General setting of the methods

• The Balas-Ceria-Cornuéjols, Sherali-Adams and Lovász-Schrijver methods apply, more generally, to the case when $F = \{0,1\}^n \cap K$ where K is an arbitrary convex body and to mixed $0-1$ linear programming:

$$K = \{(x,y) \in \Re^{n+p} \mid Ax + Gy \leq b\}$$
$$F = \{(x,y) \in K \mid x \in \{0,1\}^n\}$$

• The Sherali-Adams and Lasserre methods apply, more generally, to the case when $F = \{0,1\}^n \cap K$ and K is a semi-algebraic set

$$K = \{x \in \Re^n \mid g_\ell(x) \geq 0 \ \ \forall \ell = 1, \dots, m\}$$

where g_ℓ are polynomials

• The Lasserre method applies to polynomial programming, i.e., to the case when F is an arbitrary semi-algebraic set

$$F = \{x \in \Re^n \mid g_\ell(x) \geq 0 \ \ \forall \ell = 1, \dots, m\}$$

where g_ℓ are polynomials