A Journey From Some Classic Polyhedral Results to the Goemans-Williamson Approximation Algorithm for Max-Cut

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The Chinese postman problem, minimum T-joins, and maximum Eulerian subgraphs

Chinese postman problem (CPP): [Guan 60] Given a graph $G = (V, E)$ and edge lengths $\ell : E \to Q_+$, find a minimum length tour traversing each edge at least once.

Definition: G is *Eulerian* if there exists a tour traversing each edge exactly once ($\Longleftrightarrow$ each node has even degree).

(CPP) Find $F \subseteq E$ minimizing $\ell(F)$ s.t. $E \cup F$ is Eulerian

$\Longleftrightarrow$ Find $F \subseteq E$ minimizing $\ell(F)$ s.t. $E \setminus F$ is Eulerian

$\Longleftrightarrow$ Find a minimum length T_0-join F, where
$T_0 := \{v \in V \mid \deg_G(v) \text{ odd}\}$

Definition: $F \subseteq E$ is a T**-join** if $\deg_F(v) = \text{odd}$ if $v \in T$, $\deg_F(v) = \text{even}$ if $v \notin T$

Then, F is an edge disjoint union of $\frac{|T|}{2}$ paths joining two nodes of T, and of circuits.

For $T = \{s, t\}$, finding a minimum length T-join
$\Longleftrightarrow$ finding a minimum length $s - t$ path

Edmonds-Johnson algorithm (1973) for finding a minimum length T-join in a graph G with edge lengths $\ell \in Q^E$ and $T \subseteq V$.

1. Reduction to the case of nonnegative edge lengths. Define

- $c_e := |\ell_e| \ (e \in E)$
- $E_- := \{e \in E \mid \ell_e < 0\}$
- $T_- := \{v \in V \mid |\delta(v) \cap E_-| \text{ odd}\}$
- $T' := T \Delta T_-$

for $F \subseteq E$

$$\boxed{\begin{array}{ccc} F \text{ is a } T\text{-join} & \Longleftrightarrow & F\Delta E_- \text{ is a } T'\text{-join} \\ \text{of minimum weight} & & \text{of minimum weight} \\ \text{w.r.t. } \ell & & \text{w.r.t. } c \end{array}}$$

Proof:

$$|\delta(v) \cap (F\Delta E_-)| \equiv |\delta(v) \cap F| + |\delta(v) \cap E_-| \text{ is odd}$$
$$\Longleftrightarrow v \in (T \setminus T_-) \cup (T_- \setminus T) = T'$$

$$\begin{aligned} c(F\Delta E_-) &= c(F \setminus E_-) + c(E_- \setminus F) \\ &= w(F \setminus E_-) - w(E_- \setminus F) \\ &= w(F) - w(E_-) \end{aligned}$$

Finding a minimum length T-join in a graph with non-negative edge lengths $\ell \in Q_+^E$

2. For every $u \neq v \in T$, compute the length $d(u,v)$ of a shortest (w.r.t. ℓ) path P_{uv} from u to v in G.

3. Find a perfect matching $M := \{u_1v_1, \ldots, u_tv_t\}$ ($t := |T|/2$) of minimum weight in (K_T, d)

Claim 1. If $e \in P_{u_1v_1} \cap P_{u_2v_2}$, then $\ell_e = 0$.

Pf: $M' := M \setminus \{u_1v_1, u_2v_2\} \cup \{u_1u_2, v_1v_2\}$ is a perfect matching in K_T with $d(M') = d(M) - 2\ell_e$

Claim 2. $F_0 := \Delta_{i=1}^t P_{u_iv_i}$ is a minimum T-join.

Pf: F_0 is a T-join with length $\ell(F_0) = d(M)$.
Let F' be a minimum T-join; thus, $F' = Q_1 \cup \ldots \cup Q_t$, where Q_i are edge disjoint $T-T$ paths. Let M' be the corresponding perfect matching in K_T. Then,

$$d(M') \leq \sum_i \ell(Q_i) = \ell(F') \leq \ell(F_0) = d(M)$$

Hence, equality holds throughout, showing that F_0 is a minimum T-join.

Another reduction of minimum T-join to perfect matching

$$G, \ell, T \qquad \longrightarrow \qquad G', \ell'$$

minimum T-join in G, ℓ, T	$\longleftrightarrow$	minimum perfect matching in G', ℓ'

• for each edge $e = (u, v)$ in G, create an edge (u_e, v_e) in G'
• any two nodes v_e, v_f are adjacent in G'
• for each node $v \in T$ with even degree, add new node adjacent to all v_e in G'
• for each node $v \in V \setminus T$ with odd degree, add a new node adjacent to all v_e in G'

Application: Derive the T-join polytope $\mathrm{JOIN}^{\square}(G, T)$ as projection of the perfect matching polytope of G'
Hence, for $T = \emptyset$, we find the Eulerian subgraph polytope $\mathrm{EUL}^{\square}(G)$ of G

$\longrightarrow$ direct proof using switching

The switching operation

Given $C \subseteq E$, define the **switching by** C mapping:

$$\begin{aligned} r_C : \Re^E &\longrightarrow \Re^E \\ x &\mapsto y = r_C(x) \end{aligned}$$

$$\boxed{y_e := \begin{cases} 1 - x_e & \text{if } e \in C \\ x_e & \text{if } e \in E \setminus C \end{cases}}$$

- $r_C(\chi^F) = \chi^{F \Delta C}$
- When C is a Eulerian subgraph, $r_C(\mathrm{EUL}^\square(G)) = \mathrm{EUL}^\square(G)$
- When C is a T-join, $r_C(\mathrm{EUL}^\square(G)) = \mathrm{JOIN}^\square(G, T)$
- If $y = r_C(x)$, then

$$v^T x \leq v_0 \iff (v^C)^T y \leq v_0 - v(C),$$

where v^C is defined by

$$\boxed{v_e^C := \begin{cases} -v_e & \text{if } e \in C \\ v_e & \text{if } e \in E \setminus C \end{cases}}$$

Hence, all valid inequalities for the polytope $\mathrm{EUL}^\square(G)$ arise from the valid inequalities for the cone $\mathrm{EUL}(G)$ via switching by Eulerian subgraphs.

$\longrightarrow$ Enough to find the linear description of the cone $\mathrm{EUL}(G)$

The linear description of the Eulerian subgraph cone $\mathrm{EUL}(G)$

$$Q(G) := \{x \in \Re_+^E \mid x_e - x(\delta(S)\backslash e) \leq 0 \, (S \subseteq V, \, e \in \delta(S))\}$$

$$\mathrm{EUL}(G) \subseteq Q(G)$$

Theorem: $\mathrm{EUL}(G) = Q(G)$

Suppose not. Let G be a counterexample with $|V| + |E|$ minimum. There exists x lying on an extreme ray of $Q(G)$ and $x \notin \mathrm{EUL}(G)$.

1. $x_e > 0$ **for all** $e \in E$**.**
If $x_e = 0$, then the projection y of x on $\Re^{E\backslash e}$ belongs to $Q(G\backslash e) = \mathrm{EUL}(G\backslash e)$. Hence,

$$y = \sum_C \lambda_C \chi^C$$

(summing over cycles of $G\backslash e$), implying that

$$x = \sum_C \lambda_C \chi^C \in \mathrm{EUL}(G)$$

2. $|\delta(v)| \geq 3$ **for all** $v \in V$**.**
If $\delta(S) = \{e\}$, then $x_e = 0$.
If $\delta(S) = \{e, f\}$, then $x_e = x_f$ and the projection y of x on $\Re^{E\backslash e}$ belongs to $Q(G/e) = \mathrm{EUL}(G/e)$. Hence,

$$y = \sum_{C|f\notin C} \lambda_C \chi^C + \sum_{C|f\in C} \lambda_C \chi^C$$

(summing over cycles of G/e), implying that

$$x = \sum_{C|f\notin C} \lambda_C \chi^C + \sum_{C|f\in C} \lambda_C \chi^{C\cup\{e\}} \in \mathrm{EUL}(G)$$

3. When $S = \{v\}$, there is at most one tight equality: $x_e - x(\delta(v) \setminus e) = 0$.
Suppose $x_f - x(\delta(v) \setminus f) = 0$ is another tight equality. Then, $\delta(v) = \{e, f\}$, since $x_g = 0$ for all $g \in \delta(v) \setminus \{e, f\}$.

4. As x is an extreme ray of $Q(G)$, there are $m - 1$ tight equalities for x. Now, $m \geq \frac{3}{2}n$, implying $m \geq n + 2$. [Except if $n = 2$, $m = 3$, which can be settled directly.] Hence, **there is a tight equality:**

$$(*) \qquad x_f - x(\delta(S) \setminus f) = 0$$

where $|S|, |V \setminus S| \geq 2$.

G_1	G_2
contract S to node a_1	contract $V \setminus S$ to node a_2
x_1 projection of x on G_1	x_2 projection of x on G_2

For $i = 1, 2$, $x_i \in Q(G_i) = \mathrm{EUL}(G_i)$, implying that

$$x_i = \frac{1}{k} \sum_{C \in \mathcal{C}_i} \chi^C,$$

where $\mathcal{C}_i$ is a multiset of cycles of G_i, k is an integer

Each $C \in \mathcal{C}_i$ satisfies $(*)$, implying that

$$C \cap \delta(a_i) = \emptyset \text{ or } \{f_i, e_i\}$$

for some $e \in \delta(S) \setminus \{f\}$. Write:

$$kx_i = \sum_{C \in \mathcal{C}_i | f_i \notin C} \chi^C + \sum_{e \in \delta(S) \setminus \{f\}} \sum_{C \in \mathcal{C}_i | f_i, e_i \in C} \chi^C$$

Then,

$$|\{C \in \mathcal{C}_i \mid f_i, e_i \in C\}| = kx_i(e) = kx(e)$$

is a constant not depending on $i = 1, 2$.
Now, any $C_1 \in \mathcal{C}_1$ containing e_1, f_1 and any $C_2 \in \mathcal{C}_2$ containing e_2, f_2 can be combined to a cycle of G:

$$C := C_1 \setminus \{e_1, f_1\} \cup C_2 \setminus \{e_2, f_2\} \cup \{e, f\}$$

Therefore, the two decompositions of x_1 and x_2 can be combined to give a decomposition of x, showing that $x \in \mathrm{EUL}(G)$, a contradiction!

The Max-Cut problem

Given a graph G with edge weights $w \in Q^E$, find a cut $\delta(S)$ of maximum total weight.

Max-Cut is NP-hard, already when $w \equiv 1$ and G is cubic, or planar plus one node [Yannakakis 78, Barahona 82]

Reduction to the partition problem:
$G = K_n$ with weight $b_i b_j$ on edge ij, where $b_1, \ldots, b_n$ are given integers.
$\exists$ cut of weight $\geq \frac{1}{4}(\sum_{i=1}^{n} b_i)^2 \iff$ the sequence $b_1, \ldots, b_n$ can be partitioned.

Max-Cut can be solved in **polynomial time in planar graphs** [Orlova and Dorfman 1972, Hadlock 1975]

$\longrightarrow$ reduction to the maximum Eulerian subgraph problem in the dual graph G^*

A Combinatorial algorithm for solving max-cut in planar graphs

Fact: Let G be a plane graph and let G^* be its planar dual graph. Then,
cuts in $G \longleftrightarrow$ cycles (Eulerian subgraphs) in G^*
max-cut in $G \longleftrightarrow$ max cycle in G^*

$\leadsto$ the problem can be solved using Edmonds-Johnson algorithm (reduction to the maximum weight perfect matching problem)

LP approach to max-cut in planar graphs

$$\mathrm{CUT}^{\square}(G) := \mathrm{conv}(\chi^{\delta(S)} \mid S \subseteq V)$$

$$\begin{aligned}\mathrm{MET}^{\square}(G) := \{x \in [0,1]^E \mid\ & x(F) - x(C \setminus F) \leq |F| - 1 \\ & (C \text{ cycle},\ F \subseteq C,\ |F| \text{ odd})\}\end{aligned}$$

$$\boxed{\mathrm{CUT}^{\square}(G) \subseteq \mathrm{MET}^{\square}(G)}$$

Question 1. When does equality holds ?

Answer. Yes, for planar graphs [precisely, for graphs with no K_5-minor]

Question 2. Can one optimize over $\mathrm{MET}^{\square}(G)$ in polynomial time ?

Answer. Yes [Barahona and Mahjoub 1986]

Solution 1: $\mathrm{MET}^{\square}(G)$ is equal to the projection on $\Re^E$ of $\mathrm{MET}^{\square}(K_n)$, the metric polytope of the complete graph K_n, which is defined by the *triangle inequalities*:

$$x_{ij} - x_{ik} - x_{jk} \leq 0, \quad x_{ij} + x_{ik} + x_{jk} \leq 2$$

for all $i, j, k \in V$.

$\longrightarrow$ One can solve planar max-cut via LP

Solution 2: One can solve the separation problem over $\mathrm{MET}^{\Box}(G)$ in polynomial time.

Given $x \in [0,1]^E$, decide whether x satisfies the system:

$$\sum_{e \in C \setminus F} x_e + \sum_{e \in F} (1 - x_e) \geq 1$$

for all cycles C, $F \subseteq C$ with $|F|$ odd.

• Consider two disjoint copies $G' = (V', E')$ and $G'' = (V'', E'')$ of G. Let H be the graph on $V' \cup V''$ whose edge set contains $E' \cup E''$ and all pairs (u', v''), (u'', v') for $(u, v) \in E_G$.

• For each edge $e = (u, v) \in E_G$, assign weight x_e to $(u', v'), (u'', v'')$ and $1 - x_e$ to $(u', v''), (u'', v')$.

• For $v \in V$, compute the minimum length μ_v of a $v' - v''$ path in H. Then, $\min_{v \in V} \mu_v$ is equal to the minimum value of $\sum_{e \in C \setminus F} x_e + \sum_{e \in F} (1 - x_e)$ (over all feasible pairs (C, F))

The cut polytope for graphs with no K_5-minor

Theorem [Seymour 1981, Barahona-Mahjoub 1986]:

$$\boxed{\mathrm{CUT}^{\square}(G) = \mathrm{MET}^{\square}(G) \Longleftrightarrow G \text{ has no } K_5\text{-minor}}$$

Sketch of proof:

1. The class $\mathcal{G} := \{G \mid \mathrm{CUT}^{\square}(G) = \mathrm{MET}^{\square}(G)\}$ is **closed under taking minors**

2. $K_5 \notin \mathcal{G}$. Indeed, $2/3(1, \ldots, 1)$ belongs to $\mathrm{MET}^{\square}(K_5)$, but not to $\mathrm{CUT}^{\square}(K_5)$, as it violates the inequality:

$$\sum_{1 \le i < j \le 5} x_{ij} \le 6,$$

valid for $\mathrm{CUT}^{\square}(K_5)$

3. Wagner's decomposition theorem: Let G be a maximal graph with no K_5-minor. Then, G can be obtained by means of clique k-sums ($k \le 3$) from planar graphs and copies of V_8.

4. V_8 and planar graphs belong to $\mathcal{G}$

5. Barahona's 'gluing' theorem: Let G be the clique k-sum of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with $k \le 3$. Let

$$\mathrm{CUT}^{\square}(G_i) = \{x \in \Re^{E_i} \mid A_i x \le b_i\}$$

be the linear description of $\mathrm{CUT}^{\square}(G_i)$. Then,

$$\mathrm{CUT}^{\square}(G) = \{x \in \Re^{E} \mid A_1 x \le b_1,\ A_2 x \le b_2\}$$

is the linear description of $\mathrm{CUT}^{\square}(G)$

The graph V_8:

Clique k-sums of graphs:

Proof of the 'gluing' theorem

Case of $0,1$-sum

Assume $x \in \Re^E$ satisfies $A_1 x \leq b_1,\ A_2 x \leq b_2$.
We show that $x \in \mathrm{CUT}^{\Box}(G)$,

For $i = 1, 2$, let x_i be the projection of x on $\Re^{E(G_i)}$. Then,

$$A_1 x_1 \leq b_1,\ A_2 x_2 \leq b_2 \Longrightarrow x_1 \in \mathrm{CUT}^{\Box}(G_1),\ x_2 \in \mathrm{CUT}^{\Box}(G_2)$$

$$\Longrightarrow x_1 = \sum_i \alpha_i \delta_{G_1}(S_i),\ x_2 = \sum_j \beta_j \delta_{G_2}(T_j)$$

with $\alpha_i, \beta_j \geq 0$, $\sum_i \alpha_i = \sum_j \beta_j = 1$, and, w.l.o.g., $v_0 \notin S_i, T_j$.

Therefore,

$$x = \sum_{i,j} \alpha_i \beta_j \delta_G(S_i \cup T_j) \in \mathrm{CUT}^{\Box}(G)$$

Case of 3-sum

$$K := \{v_1, v_2, v_3\} \text{ is the common clique}$$

By assumption,

$$x_1 = \sum_{i \in I} \alpha_i \delta_{G_1}(S_i), \quad x_2 = \sum_{j \in J} \beta_j \delta_{G_2}(T_j)$$

W.l.o.g., $|S_i \cap K| = 0, 1$ for all $i \in I$. Set

$I_0 := \{i \in I \mid S_i \cap K = \emptyset\}$, $I_1 := \{i \in I \mid S_i \cap K = \{v_1\}\}$

$I_2 := \{i \in I \mid S_i \cap K = \{v_2\}\}$, $I_3 := \{i \in I \mid S_i \cap K = \{v_3\}\}$

Define analogously the sets J_0, J_1, J_2, J_3 for x_2.

Therefore,

$$x_1 = \sum_{i \in I_0} \alpha_i \delta_{G_1}(S_i) + \sum_{i \in I_1} \alpha_i \delta_{G_1}(S_i) + \sum_{i \in I_2} \alpha_i \delta_{G_1}(S_i) + \sum_{i \in I_3} \alpha_i \delta_{G_1}(S_i)$$

$$x_2 = \sum_{j \in J_0} \beta_j \delta_{G_2}(T_j) + \sum_{j \in J_1} \beta_j \delta_{G_2}(T_j) + \sum_{j \in J_2} \beta_j \delta_{G_2}(T_j) + \sum_{j \in J_3} \beta_j \delta_{G_2}(T_j)$$

Set $\alpha^k := \sum_{i \in I_k} \alpha_i, \quad \beta^k := \sum_{j \in J_k} \beta_j \quad$ for $k = 0, 1, 2, 3$.

Claim: $\alpha^k = \beta^k$ for $k = 0, 1, 2, 3$.

Proof: Using the decomposition of x_1, we find that

$$x_{v_1v_2} = \alpha^1 + \alpha^2,\ x_{v_1v_3} = \alpha^1 + \alpha^3,\ x_{v_2v_3} = \alpha^2 + \alpha^3$$

This implies:

$$\alpha^1 = \frac{x_{v_1v_2} + x_{v_1v_3} - x_{v_2v_3}}{2}$$

Using the decomposition of x_2, we find also that

$$\beta^1 = \frac{x_{v_1v_2} + x_{v_1v_3} - x_{v_2v_3}}{2}$$

Therefore, $\alpha^1 = \beta^1$ and, analogously, $\alpha^2 = \beta^2$, $\alpha^3 = \beta^3$. This implies that $\alpha^0 = \beta^0$, since $\sum_{k=0}^{3} \alpha^k = \sum_{k=0}^{3} \beta^k = 1$.

Therefore, one can combine the decompositions of x_1 and x_2 into the following decomposition of x:

$$x = \sum_{k=0,1,2,3} \frac{1}{\alpha^k} \left(\sum_{\substack{i \in I_k \\ j \in J_k}} \alpha_i \beta_j \delta_G(S_i \cup T_j) \right)$$

Max-Cut and weakly bipartite graphs

For non-negative edge weights, max-cut is equivalent to the problem of finding a maximum weight bipartite subgraph in G

$$\mathrm{BIP}^{\square} := \mathrm{conv}(\chi^F \mid F \subseteq E,\ (V, F) \text{ bipartite})$$

$$\mathrm{ODD}^{\square} := \{x \in [0,1]^E \mid x(C) \leq |C| - 1\,\forall C \text{ odd cycle in } G\}$$

$$\mathrm{BIP}^{\square}(G) \subseteq \mathrm{ODD}^{\square}(G)$$

Question 1. When does equality holds?
G is said to be *weakly bipartite* if equality holds

Question 2. Can one optimize in polynomial time over $\mathrm{ODD}^{\square}(G)$**?**

Yes, as the separation problem over $\mathrm{ODD}^{\square}(G)$ **can be solved in polynomial time.** [Barahona Grötschel 1981]

Solution: Let $x \in [0,1]^E$ and $w_e := 1 - x_e$ $(e \in E)$.

$$\boxed{x \in \mathrm{BIP}^{\square}(G) \iff w(C) \geq 1 \quad \text{for all odd cycles } C}$$

Check whether $w(C) \geq 1$ for every odd cycle C.
Equivalently, check whether $w(P) \geq 1 - w_e$ for every edge $e = (u, v) \in E$ and every even $u - v$ path P

Computing a shortest even $u - v$ path in G

$$\begin{array}{ccc} \text{even } u - v \text{ path } P & \longleftrightarrow & \text{perfect matching } M \\ \text{in } G & & \text{in } H \end{array}$$

$$w(P) = w'(M)$$

Characterization of weakly bipartite graphs

$$\mathrm{MET}^{\square}(G) \subseteq \mathrm{ODD}^{\square}(G)$$

Theorem. [Poljak 92] When $w \geq 0$,

$$\max_{x \in \mathrm{MET}^{\square}(G)} w^T x = \max_{x \in \mathrm{ODD}^{\square}(G)} w^T x$$

Corollary. Graphs with no K_5-minor are weakly bipartite

Call H an *odd minor* of G if H is obtained from G by deleting nodes and edges, and by contracting all edges in a cut.

Theorem [Guenin 98]:
G is weakly bipartite $\Longleftrightarrow$ K_5 is not an odd minor of G

How good is the relaxation of $\mathrm{CUT}^\square(K_n)$ by the triangle inequalities?

$$lp(G) := \max w^T x \text{ s.t. } x \in \mathrm{MET}^\square(G)$$
$$mc(G) := \max w^T x \text{ s.t. } x \in \mathrm{CUT}^\square(G)$$

$$\boxed{mc(G) \leq lp(G) \leq w(E) \leq 2 \cdot mc(G)}$$

Theorem: [Poljak Tuza 92]
Let $G_{n,p}$ be a random graph on n nodes where edges are chosen independently with probability p.
• ('sparse' case) If $p(n)n \longrightarrow \infty$ and $p(n)n^{1-a} \longrightarrow 0$ for all $a > 0$, then $\frac{lp(G_{n,p})}{mc(G_{n,p})} \longrightarrow 2$ almost surely
• ('dense case') If $p(n) = \Omega(\sqrt{\frac{\log n}{n}})$, then $\frac{lp(G_{n,p}}{mc(G_{n,p})} \longrightarrow \frac{4}{3}$ almost surely.

Stronger valid inequalities: Hypermetric inequalities

Given integers $b_1, \ldots, b_n$, $\sigma := \Sigma_{i=1}^n b_i$, the inequality:

$$\boxed{\sum_{1 \le i < j \le n} b_i b_j x_{ij} \le \left\lfloor \frac{\sigma^2}{4} \right\rfloor}$$

is valid for the cut polytope $\mathrm{CUT}^{\square}(K_n)$.

Proof:

$$\sum_{ij} b_i b_j \delta(S)_{ij} = \sum_{i \in S,\ j \notin S} b_i b_j = (\sum_{i \in S} b_i)(\sigma - \sum_{i \in S} b_i) \le \frac{\sigma^2}{4}$$

- for $\sigma = 0$, get the *negative type inequalities* $\longrightarrow \mathrm{NEG}_n$
- for $\sigma = 1$, get the *hypermetric inequalities* $\longrightarrow \mathrm{HYP}_n$

Properties:
(i) $\mathrm{CUT}^{\square}(K_n) \subseteq \mathrm{HYP}_n \subseteq \mathrm{NEG}_n$

(ii) $\mathrm{NEG}_n \simeq \mathrm{PSD}_{n-1}$
$\longrightarrow$ the cone NEG_n is not polyhedral.
$\longrightarrow$ One can solve the separation problem over NEG_n in polynomial time

Theorem. [Deza Grishukhin La 93]
The cone HYP_n is polyhedral

An affine trasformation:

$$\begin{array}{rcl} \varphi : \Re^{\binom{n}{2}} & \longrightarrow & Sym_{n-1} \\ x = (x_{ij}) & \mapsto & P = (p_{ij})_{i,j=1}^{n-1} \end{array}$$

$$\boxed{\begin{array}{l} p_{ii} = x_{in} \\ p_{ij} = \frac{1}{2}(x_{in} + x_{jn} - x_{ij}) \end{array}}$$

$$\boxed{\begin{array}{ccc} \begin{array}{c} x = \delta(S) \\ [n \notin S] \end{array} & \longleftrightarrow & P = \chi^S (\chi^S)^T \\ \\ x \in \mathrm{CUT}^{\square}(K_n) & \longleftrightarrow & P \in \mathrm{conv}(yy^T \mid y \in \{0,1\}^{n-1}) \\ \\ x \in \mathrm{HYP}_n & \longleftrightarrow & \begin{array}{c} a^T P a - a^T diag(P) \geq 0 \\ \forall a \in Z^{n-1} \end{array} \\ \\ x \in \mathrm{NEG}_n & \longleftrightarrow & \begin{array}{c} a^T P a \geq 0 \ \forall a \in Z^{n-1} \\ \text{i.e., } P \succeq 0 \end{array} \end{array}}$$

Proof: $P = \varphi(x)$, $b \in Z^n$, $a := (b_1, \ldots, b_{n-1}) \in Z^{n-1}$

$$\begin{aligned} \sum_{1 \leq i < j \leq n} b_i b_j x_{ij} &= \sum_{i=1}^{n-1} b_i b_n p_{ii} + \sum_{1 \leq i < j \leq n-1} b_i b_j (p_{ii} + p_{jj} - 2p_{ij}) \\ &= \sum_{i=1}^{n-1} p_{ii} b_i (\sigma - b_i) - 2 \sum_{1 \leq i < j \leq n-1} b_i b_j p_{ij} \\ &= \sigma \cdot a^T diag(P) - a^T P a \end{aligned}$$

More about hypermetrics

Let $x \in \mathrm{HYP}_n$, $P = \varphi(x)$. As $P \succeq 0$, $\exists\, v_1, \ldots, v_{n-1} \in \Re^k$ ($k = \text{rank } P$) such that

$$p_{ij} = v_i^T v_j \text{ for all } i, j \leq n-1$$

$$x_{ij} = \|v_i - v_j\|^2 \text{ for all } i, j \leq n, \text{ setting } v_n := 0$$

Properties of the vectors $v_1, \ldots, v_n$**.** [Assouad 82]

1. The vectors $v_1, \ldots, v_n$ lie on a sphere S; i.e., there exists $c \in \Re^k$ s.t.

$$\|v_i\|^2 = 2 v_i^T c \quad \forall i \leq n$$

2. $L := Z(v_1, \ldots, v_{n-1})$ is a lattice

3. The sphere S is *empty* in the lattice L, i.e.,

$$\|\sum_i a_i v_i - c\| \geq \|c\| \quad \forall a \in Z^{n-1}$$

$x \in \mathrm{HYP}_n \longleftrightarrow v_1, \ldots, v_n$ is a generating subset of the set of vertices of a Delaunay polytope

Theorem. [DGL93] There are finitely many distinct (up to affine equivalence) Delaunay polytopes in $\Re^k$

Corollary. HYP_n is polyhedral

Complexity results about hypermetric inequalities

Problem: Given x, does x belong to HYP_n?
Complexity: co-NP [Avis-Grishukhin 93]
Tool: $\max_i |b_i| \leq n!$ for hypermetric facets: $\sum_{i<j} b_i b_j x_{ij} \leq 0$

Problem: Given x and an integer k, does x satisfy all hypermetric inequalities with $\sum_i |b_i| \leq 2k+1$?
Complexity: co-NP-complete [AG93]

Problem: Given x, does x belong to HYP_n? If not, find a violated hypermetric inequality with $\sum_i |b_i|$ as small as possible.
Complexity. NP-hard [AG93]

The analogous results hold for negative type inequalities ! [Avis 2002]

Open problem: What is the complexity of the separation problem for hypermetric inequalities?

Equivalently: Given an ellipsoid

$$E := \{x \in \Re^n \mid x^T P x - x^T diag P \geq 0\}$$

where $P \succeq 0$ (rational valued), what is the complexity of deciding whether there is an integral point in the interior of E?

Back to the inequality:

$$\boxed{(1)\ \sum_{1\le i<j\le n} b_i b_j x_{ij} \le \left\lfloor \frac{(\Sigma_i\, b_i)^2}{4} \right\rfloor, \quad \text{for } b \in Z^n}$$

Theorem. The separation problem for the class of inequalities:

$$\boxed{(2)\ \sum_{1\le i<j\le n} b_i b_j x_{ij} \le \frac{(\Sigma_i\, b_i)^2}{4}, \quad \text{for } b \in Z^n}$$

can be solved in polynomial time.

Proof: Let X be the $n \times n$ matrix with zero diagonal and off-diagonal entries x_{ij}. Then,

$$\sum_{1\le i<j\le n} b_i b_j x_{ij} \le \frac{(\Sigma_i\, b_i)^2}{4} \Longleftrightarrow 2 \sum_{i,j=1}^{n} b_i b_j x_{ij} \le (\sum_i b_i)^2 = b^T J b$$

$$\Longleftrightarrow b^T(J - 2X)b \ge 0$$

Therefore, x satisfies all inequalities (2) $\Longleftrightarrow J - 2X \succeq 0$

Via the bijective correspondances:

$$\begin{array}{ccccc} x = (x_{ij}) \in \Re^{\binom{n}{2}} & \mapsto & X \in \mathrm{Sym}_n & \mapsto & Y = J - 2X \in \mathrm{Sym_n} \\ & & \mathrm{diag}(X) = 0 & & \mathrm{diag}(Y) = e \end{array}$$

$$\boxed{\begin{array}{ll} \max & \sum_{ij \in E} w_{ij} x_{ij} \\ \text{s.t.} & \sum_{1 \le i < j \le n} b_i b_j x_{ij} \le \frac{(\sum_i b_i)^2}{4} \ \forall b \in Z^n \end{array}}$$

$$\boxed{\begin{array}{rll} = & \max & \sum_{ij \in E} w_{ij} X_{ij} \\ & \text{s.t.} & J - 2X \succeq 0, \ \mathrm{diag}(X) = 0 \end{array}}$$

$$\boxed{\begin{array}{rll} = \ sdp := & \max & \frac{1}{2} \sum_{ij \in E} w_{ij}(1 - Y_{ij}) \\ & \text{s.t.} & Y \succeq 0, \ \mathrm{diag}(Y) = e \end{array}}$$

This is the **basic SDP relaxation** for Max-Cut

It can also be derived directly from the following formulation of the Max-Cut problem:

$$\boxed{\begin{array}{rll} mc(G, w) = & \max & \frac{1}{2} \sum_{ij \in E} w_{ij}(1 - y_i y_j) \\ & \text{s.t.} & y \in \{\pm 1\}^n \end{array}}$$

Encode a cut $\delta(S)$ by a ± 1 vector y ($y_i = 1 \longleftrightarrow i \in S$) and note that $Y := yy^T$ is feasible for the SDP program

How good is the SDP relaxation?

Obviously: $\frac{mc(G,w)}{sdp} \leq 1$

Theorem [Goemans-Williamson 94]: For non-negative edge weights $w \in \Re^E_+$,

$$\boxed{\frac{mc(G,w)}{sdp} \geq 0.878}$$

(i) Solving the SDP relaxation and
(ii) using the '*random hyperplane rounding technique*',
G-W construct in polynomial time a random cut $\delta(S)$ whose expected weight $E(S)$ satisfies:

$$\boxed{\frac{E(S)}{mc(G,w)} \geq 0.878}$$

The construction can be derandomized [Mahajan-Ramesh 95]

$\rightsquigarrow$ 0.878-approximation algorithm for Max-Cut

Best previous result:
0.5-approximation algorithm [Sahni-Gonzalez 76]
Idea: Assign the nodes to either side of the cut randomly with probability 1/2

Non-approximability result [Håstad 97]
$\not\exists$ 0.917-approximation algorithm for Max-Cut unless P=NP

The G-W approximation algorithm for Max-Cut

(1) SDP optimization phase: Solve the SDP relaxation:

$$sdp = \max \frac{1}{2} \sum_{ij \in E} w_{ij}(1 - Y_{ij}) \ \text{ s.t. } Y \succeq 0, \ \mathrm{diag}(Y) = e$$

Let Y be an optimum solution and let $v_1, \ldots, v_n \in \Re^d$ $(d \leq n)$ be unit vectors s.t. $Y_{ij} = v_i^T v_j$ for all $i, j = 1, .., n$

(2) Random hyperplane rounding phase: Select a random unit vector $u \in \Re^d$.
The hyperplane H with normal r partitions the space into two halfspaces, which induces the cut $\delta(S)$ where

$$S := \{i \in \{1, \ldots, n\} \mid r^T v_i \geq 0\}$$

The expected weight $E(S)$ of the cut $\delta(S)$ satisfies:

$$\boxed{\alpha_{GW} \cdot sdp \overset{?}{\leq} E(S) \leq mc(G, w)}$$

setting:

$$\boxed{\alpha_{GW} := \min_{0 < \vartheta \leq \pi} \frac{2}{\pi} \frac{\vartheta}{1 - \cos \vartheta}; \quad 0.87856 < \alpha_{GW} < 0.87857}$$

Proof:

$\vartheta_{ij} :=$ angle formed by the two vectors v_i and v_j

Thus: $\cos\vartheta_{ij} = v_i^T v_j$

We have:

$$
\begin{aligned}
E(S) &= \text{ expected weight of the random cut } \delta(S) \\
&= \sum_{ij\in E} w_{ij} \text{ prob}\{\text{edge } ij \text{ belongs to the random cut } \delta(S)\} \\
&= \sum_{ij\in E} w_{ij} \text{ prob}\{\text{random hyperplane } H \text{ separates } v_i,\ v_j\} \\
&= \sum_{ij\in E} w_{ij} \frac{\vartheta_{ij}}{\pi} \\
&= \sum_{ij\in E} w_{ij} \left(\frac{2}{\pi}\frac{\vartheta_{ij}}{1-\cos\vartheta_{ij}}\right)\left(\frac{1-\cos\vartheta_{ij}}{2}\right) \\
&\geq \alpha_{GW} \cdot \left(\sum_{ij\in E} w_{ij}\frac{1-\cos\vartheta_{ij}}{2}\right) \\
&= \alpha_{GW} \cdot sdp.
\end{aligned}
$$

Comments about the G-W approximation algorithm

Recall: $0.878 \cdot sdp \leq E(S) \leq mc \leq sdp$

Question 1: Can one improve the bound: $\frac{mc}{sdp} \geq 0.878$?

Answer=No: Exist graphs achieving equality $\frac{mc}{sdp} = 0.878$ [Feige-Schechtman 2002]

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Question 2: How to design an approximation algorithm with better performance ratio?
I.e., how to improve the bound $\frac{E(S)}{mc} \geq 0.878$?

Ideas:
- Strengthen the basic SDP relaxation by adding valid inequalities, e.g., triangle inequalities
- use an improved rounding procedure

Indeed: Exist graphs s.t. $\frac{E(S)}{sdp} \simeq 0.878$ and $sdp = mc$ [Karloff 99]
Hence, only adding valid inequalities will not help!
One needs a new rounding procedure.

(i) Adding valid inequalities and a local search phase $\rightsquigarrow$ 0.932-approximation algorithm for Max-Cut in graphs with max. degree 3 [Halperin-Livnat-Zwick 2002]

(ii) Mixing the random hyperplane and basic flipping rounding techniques $\rightsquigarrow$ better approximation ratio for graphs with no large cuts [Zwick 1999]

Extensions

• Extension to quadratic problems:

$$\mu^* := \max_{y \in \{\pm 1\}^n} y^T A y$$

When $A \succeq 0$, then $\frac{\mu^*}{sdp} \geq \frac{2}{\pi} \simeq 0.636$ [Nesterov 1997]

Note: Recover Max-Cut when $A = \frac{1}{4}L$, L being the *Laplacian matrix* of the graph, with entries

$$L_{ii} := \sum_j w_{ij}, \ L_{ij} := -w_{ij} \ (i \neq j)$$

Property: $Le = 0$, $L_{ij} \leq 0 \ (i \neq j) \Longrightarrow L \succeq 0$

• Extension to several other combinatorial optimization problems: bisection, Max k-Cut, colouring, MAX SAT, Max Dicut, etc...

Stronger semidefinite relaxations for Max-Cut

Idea: Lift $x \in \{\pm 1\}^n$ to $y \in \{\pm 1\}^N$ $(N > n)$ and look for conditions on the matrix $Y := yy^T$

For an integer $t \geq 1$:

$$x \in \{\pm 1\}^n \leadsto y := \left(\prod_{i \in I} x_i\right)_{\substack{I \subseteq V \\ |I| \leq t}} \leadsto Y := yy^T \text{ satisfies:}$$

$$\boxed{\begin{array}{l} Y \succeq 0 \\ \mathrm{diag}(Y) = 1 \\ Y_{IJ} \text{ depends only on } I \Delta J \end{array}}$$

Definition: Given an integer $t \geq 1$ and a vector $y = (y_I)_{|I| \leq 2t}$, define its moment matrix of order t

$$M_t(y) := (y_{I \Delta J})_{|I|,|J| \leq t}$$

Define the set:

$$\boxed{Q_t(G) := \text{projection on } \Re^E \text{ of } \{y \mid M_t(y) \succeq 0,\ y_\emptyset = 1\}}$$

This gives a hierarchy of SDP relaxations of the cut polytope:

$$\boxed{\mathrm{CUT}^\square(G) \subseteq Q_n(G) \subseteq \ldots \subseteq Q_t(G) \subseteq \ldots \subseteq Q_1(G)}$$

Theorem: $\boxed{\mathrm{CUT}^\square(G) = Q_n(G)}$

Lemma:
The eigenvectors of the matrix $M_n(y)$ are the vectors $y^A := ((-1)^{|I\cap A|})_{I\subseteq V}$ with eigenvalue $y^T y^A$. That is,

$$M_n(y) = \sum_{A\subseteq V} \frac{y^T y^A}{2^n} y^A (y^A)^T$$

Proof:

$$\begin{aligned}
[M_n(y)y^A]_I &= \sum_{J\subseteq V} y_{I\Delta J} y^A_J = \sum_{J\subseteq V} y_{I\Delta J}(-1)^{|J\cap A|} \\
&= \sum_{J\subseteq V} y_{I\Delta J}(-1)^{|(I\Delta J)\cap A|}(-1)^{|I\cap A|} \\
&= \left(\sum_{K\subseteq V} y_K(-1)^{|K\cap A|}\right)(-1)^{|I\cap A|} \\
&= y^T y^A \cdot y^A_I
\end{aligned}$$

This shows that

$$M_n(y)y^A = y^T y^A \cdot y^A$$

A more concise formulation for $Q_t(G)$

$$M_1(y) = \begin{array}{c|c} & \emptyset \quad 1 \ldots n \\ \begin{array}{c}\emptyset\\1\\\vdots\\n\end{array} & \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \end{array}$$

$$M_2(y) = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}, \quad M_3(y) = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

As we project on the subspace $\Re^E$ indexed by **pairs**, we can assume w.l.o.g. that $y_I = 0$ for all $|I|$ **odd**

$\rightsquigarrow$ restrict the index set of $M_t(y)$ to the subsets having the **same parity** as t

$$\rightsquigarrow \widetilde{M_t}(y) := (y(I \Delta J))_{\substack{|I|,|J| \leq t \\ |I|,|J| \equiv t \bmod 2}}$$

$t = 1 \rightsquigarrow$ basic SDP relaxation

$t = 2 \rightsquigarrow$ the **triangle inequalities** are satisfied [AW01]

$$X := \begin{array}{c} \\ \emptyset \\ 12 \\ 13 \\ 23 \end{array} \begin{array}{c} \begin{array}{cccc} \emptyset & 12 & 13 & 23 \end{array} \\ \begin{pmatrix} 1 & y_{12} & y_{13} & y_{23} \\ y_{12} & 1 & y_{23} & y_{13} \\ y_{13} & y_{23} & 1 & y_{12} \\ y_{23} & y_{13} & y_{12} & 1 \end{pmatrix} \end{array} \succeq 0$$

$$\Longrightarrow e^T X e = 4(1 + y_{12} + y_{13} + y_{23}) \geq 0$$

What is the smallest integer $t =: \rho(G)$ for which equality $Q_t(G) = \mathrm{CUT}^{\square}(G)$ holds?

$$\boxed{\rho(K_3) = \rho(K_4) = 2,\ \rho(K_5) = \rho(K_6) = 3,\ \rho(K_7) = 4}$$

Proposition [La01]:

(i) $Q_t(G/e) = \mathrm{CUT}^{\square}(G/e) \Longrightarrow Q_{t+1}(G) = \mathrm{CUT}^{\square}(G)$

(ii) $Q_t(G) = \mathrm{CUT}^{\square}(G) \Longrightarrow \begin{cases} Q_t(G\backslash e) = \mathrm{CUT}(G\backslash e) \\ Q_t(G/e) = \mathrm{CUT}^{\square}(G/e) \end{cases}$

($\rho(G)$ is 'minor monotone')

$$\rho(G) \leq 1 \Longleftrightarrow G \text{ has no } K_3\text{-minor}$$
$$\rho(G) \leq 2 \Longleftrightarrow G \text{ has no } K_5\text{-minor}$$
$$\rho(G) \leq 3 \Longrightarrow G \text{ has no } K_7\text{-minor}$$

Other minimal forbidden minors?

Theorem: $\rho(K_n) \geq \left\lceil \frac{n}{2} \right\rceil$; i.e., $\mathrm{CUT}^{\square}(K_n) \subset Q_{\lceil n/2 \rceil - 1}(K_n)$

Conjecture: $\rho(K_n) = \left\lceil \frac{n}{2} \right\rceil$; i.e., $\mathrm{CUT}^{\square}(K_n) = Q_{\lceil n/2 \rceil}(K_n)$

$M_{\lceil n/2 \rceil}(y) \succeq 0 \Longrightarrow M_1(y)$ convex combination of cut matrices

Note: Enough to show the theorem for n odd and the conjecture for n even

Sketch of proof of $\rho(K_n) \geq \lceil \frac{n}{2} \rceil$ ($n = 2k+1$ odd)

Goal: show strict inclusion $\mathrm{CUT}^{\square}(K_n) \subset Q_k(K_n)$

Show:

$$\min_{Q_k(K_n)} \sum_{ij} y_{ij} \stackrel{?}{=} -\frac{n}{2} < \min_{\mathrm{CUT}^{\square}(K_n)} \sum_{ij} y_{ij} = \frac{1-n}{2}$$

For this: Construct $M_k(y) \succeq 0$ with $\sum_{ij} y_{ij} = -\frac{n}{2}$

$$\boxed{\begin{array}{l} a_0 := 1,\ a_{2r+2} := -a_{2r}\frac{2r+1}{n-2r-1} \\[2ex] y_I := a_{|I|} \quad \text{for all even sets } I \end{array}}$$

Theorem: $M_k(y) \succeq 0$

Proof:

(1) $Z :=$ principal submatrix of $M_k(y)$ indexed by the k-subsets of $\{1, \ldots, n-1\}$; $D :=$ order of Z.
Show that Z is positive definite.
Hence $M_k(y)$ has at least D positive eigenvalues.

Tools: Z belongs to the Johnson scheme $J(2k, k)$; compute its eigenvalues using hypergeometric series.

(2) Show that $M_k(y)$ has at least $N - D$ zero eigenvalues.

(1) Show that Z is positive definite

$$Z = \sum_{\ell=0}^{k} a_{2\ell} A_\ell$$

where A_ℓ are the 0/1 adjacency matrices of the Johnson scheme $J(2k, k)$, with (I, J) entry 1 iff $|I \Delta J| = 2\ell$

The **distinct eigenvalues** of Z are, for $u = 0, \ldots, k$,

$$\lambda_u := \sum_{\ell=0}^{k} a_{2\ell} \left(\sum_{j=0}^{\ell} (-1)^j \binom{u}{j} \binom{k-u}{\ell-j}^2 \right)$$

$$= \sum_{i=0}^{k} \binom{k-u}{i}^2 \left(\sum_{j=0}^{k} a_{2i+2j} (-1)^j \binom{u}{j} \right)$$

Show:

(i) The inner sum is equal to $a_{2i} \frac{(-k-1/2)_u}{(i-k)_u}$

(ii) $\lambda_u = \frac{(-k-1/2)_u}{(i-k)_u} \frac{(1/2)_u}{(k-u)!} = \frac{1 \cdot 3 \cdot \ldots \cdot (2k+1)}{2^k \cdot k! \cdot (2k-2u+1)} > 0$

Tool: *Gauss identity for hypergeometric series:* when $b \leq 0$ is integer and $x = 1$

$$\boxed{\sum_{i \geq 0} \frac{(a)_i (b)_i}{(c)_i} \frac{x^i}{i!} = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}}$$

$$\Gamma(n+1) = n!, \quad \frac{\Gamma(x+n)}{\Gamma(x)} = (x)_n \text{ for integer } n \geq 0$$

See: $A = B$, Petkovsek-Wilf-Zeilberger 1996; Complexity of semi-algebraic proofs, Grigoriev-Hirsch-Pasechnik 2001

An example where $M_{n/2}(y) \succeq 0$, $M_1(y)$ is a convex combination of cut matrices, and $M_n(y) \not\succeq 0$

$$n = 4,\ a := -1/6$$

$$M_4(y) = \begin{array}{c c} & \begin{array}{cccccccc} \emptyset & 12 & 13 & 14 & 23 & 24 & 34 & 1234 \end{array} \\ \begin{array}{c} \emptyset \\ 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \\ \\ 1234 \end{array} & \begin{pmatrix} 1 & a & a & a & a & a & a & b \\ a & 1 & a & a & a & a & b & a \\ a & a & 1 & a & a & b & a & a \\ a & a & a & 1 & b & a & a & a \\ a & a & a & b & 1 & a & a & a \\ a & a & b & a & a & 1 & a & a \\ a & b & a & a & a & a & 1 & a \\ & & & & & & & \\ b & a & a & a & a & a & a & 1 \end{pmatrix} \end{array}$$

For $b := -1/6,\ M_2(y) \succeq 0,\ M_4(y) \not\succeq 0$

For $b := 1,\ M_4(y) \succeq 0$