STRENGTHENED SEMIDEFINITE PROGRAMMING BOUNDS FOR CODES

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ABSTRACT. We give a hierarchy of semidefinite upper bounds for the maximum size $A(n,d)$ of a binary code of word length $n$ and minimum distance at least $d$. At any fixed stage in the hierarchy, the bound can be computed (to an arbitrary precision) in time polynomial in $n$; this is based on a result of de Klerk, Pasechnik and Schrijver [2] about the regular $*$-representation for matrix $*$-algebras. The Delsarte bound for $A(n,d)$ is the first bound in the hierarchy, and the new bound of Schrijver [13] is located between the first and second bounds in the hierarchy. While computing the second bound involves a semidefinite program with $O(n^7)$ variables and thus seems out of reach for interesting values of $n$, Schrijver’s bound can be computed via a semidefinite program of size $O(n^5)$, a result which uses the explicit block-diagonalization of the Terwilliger algebra. We propose two strengthenings of Schrijver’s bound with the same computational complexity.

1. INTRODUCTION

We consider the problem of computing the parameter $A(n,d)$, defined as the maximum size of a binary code of word length $n$ and minimum distance at least $d$. With $\mathcal{P}$ denoting the collection of all subsets of $\{1, \ldots, n\}$, we can identify code words in $\{0, 1\}^n$ with their supports; so a code $C$ is a subset of $\mathcal{P}$ and the Hamming distance of $I, J \in \mathcal{P}$ is equal to $|I \Delta J|$. The minimum distance of a code $C$ is the minimum Hamming distance of distinct elements of $C$. If we define the graph $\mathcal{G}(n,d)$ with node set $\mathcal{P}$, two nodes $I, J \in \mathcal{P}$ being adjacent if $|I \Delta J| \in \{1, \ldots, d-1\}$, then a code with minimum distance $d$ corresponds to a stable set in the graph $\mathcal{G}(n,d)$. Therefore, the parameter $A(n,d)$ is equal to the stability number of the graph $\mathcal{G}(n,d)$, i.e., the maximum cardinality of a stable set in $\mathcal{G}(n,d)$.

Schrijver [13] introduced recently an upper bound for $A(n,d)$ which refines the classical bound of Delsarte [3]. While Delsarte bound is based on diagonalizing the (commutative) Bose-Mesner algebra of the Hamming scheme and can be computed via linear programming, Schrijver’s bound is based on block-diagonalizing the (non-commutative) Terwilliger algebra of the Hamming scheme and can be computed via semidefinite programming. In both cases the bounds can be formulated as the optimum of a (linear or semidefinite) program of size polynomial in $n$ (size $O(n)$ for Delsarte bound and size $O(n^5)$ for Schrijver’s bound).

Finding tight upper bounds for the stability number $\alpha(\mathcal{G})$ of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has been the subject of extensive research. Lovász [9] introduced the theta number $\theta(\mathcal{G})$,
which can be computed, e.g., via the semidefinite program:

\[
\vartheta(\mathcal{G}) := \max_{X} \sum_{i \in V} X_{ii} \quad \text{s.t.} \quad X = (X_{ij})_{i,j \in V \cup \{0\}} \succeq 0, \quad X_{00} = 1, \\
X_{0i} = X_{ii} \quad (i \in V), \quad X_{ij} = 0 \quad (ij \in \mathcal{E}).
\]

The theta number can be computed (with arbitrary precision) in time polynomial in the number of nodes of the graph. Moreover, \(\vartheta(\mathcal{G}) = \alpha(\mathcal{G})\) when \(\mathcal{G}\) is a perfect graph (see [5]). Schrijver [12] introduced the strengthening \(\vartheta'(\mathcal{G})\) of \(\vartheta(\mathcal{G})\) obtained by adding the nonnegativity constraint \(X \succeq 0\) to the program (1) and proved that \(\vartheta'(\mathcal{G}(n,d))\) coincides with Delsarte bound.

Various methods have been proposed in the litterature for constructing tighter semidefinite upper bounds for the stability number of a graph, in particular, by Lovász and Schrijver [10] and more recently by Lasserre [6, 7]. In both cases a hierarchy of upper bounds for \(\alpha(\mathcal{G})\) is obtained with the property that the bound reached at the \(\alpha(\mathcal{G})\)-th iteration coincides in fact with \(\alpha(\mathcal{G})\). It turns out that Lasserre’s hierarchy refines the hierarchy of Lovász and Schrijver (see [8]).

For \(k \geq 1\), denote by \(\ell^{(k)}(\mathcal{G})\) the bound in Lasserre’s hierarchy at the \(k\)-th iteration; see Section 3.1 for the precise definition. It is known (and easy to see) that, for fixed \(k\), one can compute (with arbitrary precision) the parameter \(\ell^{(k)}(\mathcal{G})\) in time polynomial in the number of nodes of the graph \(\mathcal{G}\). However, for the coding problem, the graph \(\mathcal{G}(n,d)\) has \(2^n\) nodes and such complexity is prohibitive for large \(n\). A first contribution of this paper (see Section 3.2) is to show that, for fixed \(k\), the bound \(\ell^{(k)}(\mathcal{G}(n,d))\) can be computed (with arbitrary precision) in time polynomial in \(n\). This result is based on a result of de Klerk, Pasechnik and Schrijver [2], recalled in Section 2.1, about reducing the size of invariant semidefinite programs using the regular \(*\)-representation for the algebra of invariant matrices under action of a group.

The first bound \(\ell^{(1)}(\mathcal{G})\) in the hierarchy is equal to the theta number \(\vartheta(\mathcal{G})\); its strengthening obtained by adding nonnegativity is equal to \(\vartheta'(\mathcal{G})\) which, for the graph \(\mathcal{G} = \mathcal{G}(n,d)\), coincides with the bound of Delsarte for the parameter \(A(n,d)\). It turns out that the bound of Schrijver [13] for \(A(n,d)\) lies between \(\ell^{(1)}_+(\mathcal{G})\) and \(\ell^{(2)}_+(\mathcal{G})\), the strengthenings of \(\ell^{(1)}(\mathcal{G})\) and \(\ell^{(2)}(\mathcal{G})\) obtained by adding certain bounds on the variables. While Schrijver’s bound can be computed via a semidefinite program of size \(O(n^3)\) and thus computed in practice for reasonable values of \(n\), a practical computation of \(\ell^{(2)}_+(\mathcal{G}(n,d))\) seems out of reach for interesting values of \(n\) since one would have to solve a semidefinite program with \(O(n^7)\) variables.

In Section 3.3, we introduce two bounds \(\ell_+(\mathcal{G}(n,d))\) and \(\ell_{++}(\mathcal{G}(n,d))\) satisfying

\[
\ell^{(2)}_+(\mathcal{G}(n,d)) \leq \ell_{++}(\mathcal{G}(n,d)) \leq \ell_+(\mathcal{G}(n,d)) \leq \ell^{(1)}_+(\mathcal{G}(n,d));
\]

they are at least as good as Schrijver’s bound, and their computation still relies on solving a semidefinite program of size \(O(n^3)\). This complexity result follows from the fact that the new bounds, analogously to Schrijver’s bound, require the positive semidefiniteness of certain matrices lying in the Terwilliger algebra (or a variation of it) whose dimension is \(O(n^3)\) and for which the explicit block-diagonalization has been given by Schrijver [13].
Some notation. We group here some notation that will be used throughout the paper. We set \( V := \{1, \ldots, n\} \) and \( \mathcal{P} := \mathcal{P}(V) \) denotes the collection of all subsets of the set \( V \). For a finite set \( \mathcal{V} \) and an integer \( k \geq 1 \), we set \( \mathcal{P}_k(\mathcal{V}) := \{ I \subseteq \mathcal{V} \mid |I| \leq k \} \) and \( \mathcal{P}_{\geq k}(\mathcal{V}) := \{ I \subseteq \mathcal{V} \mid |I| = k \} \). We let \( \text{Sym}(\mathcal{V}) \) denote the set of all permutations of the set \( \mathcal{V} \) and we set \( \text{Sym}(n) := \text{Sym}(\mathcal{V}) \) when \( |\mathcal{V}| = n \). The letter \( \mathcal{G} \) will be used to denote a graph, with node set \( \mathcal{V} \) and edge set \( \mathcal{E} \), while the letter \( G \) will be used to denote a group (e.g., of automorphisms of \( \mathcal{G} \)).

2. Algebraic preliminaries

2.1. Preliminaries on invariant matrices. Let \( G \) be a finite group acting on a finite set \( \mathcal{X} \); that is, we have a homomorphism \( h : G \to \text{Sym}(\mathcal{X}) \), where \( \text{Sym}(\mathcal{X}) \) is the group of permutations of \( \mathcal{X} \). For \( \sigma \in G \), \( h(\sigma) \) is a permutation of \( \mathcal{X} \) and \( M_\sigma \) is the associated \( \mathcal{X} \times \mathcal{X} \) permutation matrix with

\[
(M_\sigma)_{x,y} = \begin{cases} 
1 & \text{if } h(\sigma)(x) = y, \\
0 & \text{otherwise}.
\end{cases}
\]

The set:

\[
\mathcal{A} := \left\{ \sum_{\sigma \in G} \lambda_\sigma M_\sigma \mid \lambda_\sigma \in \mathbb{R} \ (\sigma \in G) \right\}
\]

is a matrix \( * \)-algebra; that is, \( \mathcal{A} \) is closed under addition, scalar and matrix multiplication, and conjugation.

Any \( \sigma \in G \) acts on matrices indexed by the set \( \mathcal{X} \). Namely, for a \( \mathcal{X} \times \mathcal{X} \) matrix \( N \) and \( \sigma \in G \), set

\[
\sigma(N) := (N_{\sigma(x),\sigma(y)})_{x,y \in \mathcal{X}}.
\]

The matrix \( N \) is said to be invariant under the action of \( G \) if \( \sigma(N) = N \) for all \( \sigma \in G \).

Then the commutant algebra \( \mathcal{A}^G \) of the algebra \( \mathcal{A} \), defined by

\[
\mathcal{A}^G := \{ N \in \mathbb{C}^{\mathcal{X} \times \mathcal{X}} \mid NM = MN \ \forall M \in \mathcal{A} \},
\]

consists precisely of the \( \mathcal{X} \times \mathcal{X} \) matrices \( N \) that are invariant under the action of \( G \); \( \mathcal{A}^G \) again a matrix \( * \)-algebra.

The orbit of \((x,y) \in \mathcal{X} \times \mathcal{X} \) under action of \( G \) is the set \( \{(\sigma(x),\sigma(y)) \mid \sigma \in G \} \). Let \( \mathcal{O}_1, \ldots, \mathcal{O}_N \) denote the orbits of the set \( \mathcal{X} \times \mathcal{X} \) under the action of the group \( G \) and, for \( i = 1, \ldots, N \), let \( D_i \) be the \( \mathcal{X} \times \mathcal{X} \) matrix:

\[
(D_i)_{x,y} = \begin{cases} 
1 & \text{if } (x,y) \in \mathcal{O}_i \\
0 & \text{otherwise}.
\end{cases}
\]

Then, \( D_1, \ldots, D_N \) form a basis of the commutant \( \mathcal{A}^G \) (as vector space) and \( D_1 + \cdots + D_N = J \) (the all-ones matrix). We normalize the \( D_i \) to

\[
D_i := \frac{D_i}{\sqrt{\langle D_i, D_i \rangle}}
\]
for \( i = 1, \ldots, N \). (For two \( N \times N \) matrices \( A, B \), \( \langle A, B \rangle := \text{Tr}(A^T B) = \sum_{i,j=1}^{N} A_{ij} B_{ij} \).) Then, \( \langle D_i, D_j \rangle = 1 \) if \( i = j \) and 0 otherwise. The multiplication parameters \( \gamma_{i,j}^k \) are defined by

\[
D_i D_j = \sum_{k=1}^{N} \gamma_{i,j}^k D_k
\]

for all \( i, j = 1, \ldots, N \). Define the \( N \times N \) matrices \( L_1, \ldots, L_N \) by

\[
(L_k)_{i,j} := \gamma_{i,j}^k \quad \text{for } k, i, j = 1, \ldots, N.
\]

De Klerk, Pasechnik and Schrijver [2] show:

**Theorem 1.** The mapping \( D_k \mapsto L_k \) is a \(*\)-isomorphism, known as the regular \(*\)-representation of \( \mathcal{A}^G \). In particular, given real scalars \( x_1, \ldots, x_N \),

\[
\sum_{i=1}^{N} x_i D_i \succeq 0 \iff \sum_{i=1}^{N} x_i L_i \succeq 0.
\]

This result has important algorithmic applications, as it permits to give more compact formulations for invariant semidefinite programs. Consider a semidefinite program:

\[
\min \langle C, Y \rangle \quad \text{s.t. } \langle A_\ell, Y \rangle \leq b_\ell \ (\ell = 1, \ldots, m), \ Y \succeq 0
\]

in the \( X \times X \) matrix variable \( Y \). Assume that the program (7) is invariant under action of the group \( G \); that is, \( C \) is invariant under action of \( G \) and, for every matrix \( Y \) feasible for (7) and \( \sigma \in G \), the matrix \( \sigma(Y) \) is again feasible for \( Y \). (This holds, e.g., if the class of constraints is invariant under action of \( G \), i.e., if for each \( \ell \in \{1, \ldots, m\} \) and \( \sigma \in G \), there exists \( \ell' \in \{1, \ldots, m\} \) such that \( \sigma(A_\ell) = A_{\ell'} \) and \( b_\ell = b_{\ell'} \). Then, if \( Y \) is feasible for (7) then the matrix \( Y_\sigma := \frac{1}{|G|} \sum_{\sigma \in G} \sigma(Y) \) too is feasible for (7), with the same objective value as \( Y \). Therefore, in (7), one can assume without loss of generality that \( Y \) is invariant under action of \( G \); that is, \( Y \) is of the form \( Y = \sum_{i=1}^{N} x_i D_i \) with \( x_1, \ldots, x_N \in \mathbb{R} \). Then the objective function reads \( \langle C, Y \rangle = \sum_{i=1}^{N} c_i x_i \), after setting \( C = \sum_{i=1}^{N} c_i D_i \), and the constraints in (7) become linear constraints in \( x \). As a direct application of Theorem 1, we find:

**Corollary 2.** Consider the program (7) in the \( X \times X \) matrix variable \( Y \). If (7) is invariant under the action of the group \( G \), then it can be equivalently reformulated as

\[
\min \sum_{i=1}^{N} c_i x_i \quad \text{s.t. } a_\ell^T x \leq b_\ell \ (\ell = 1, \ldots, m), \ \sum_{i=1}^{N} x_i L_i \succeq 0.
\]

The program (8) involves \( N \times N \) matrices and \( N \) variables. Here, \( N \) is the dimension of the algebra \( \mathcal{A}^G \) (the set of \( X \times X \) invariant matrices under the action of the group \( G \)), typically much smaller than \( |X| \).

To use computationally this result, one needs to know explicitly the matrices \( L_1, \ldots, L_N \), which involves computing the cardinality of the orbits of \( X \times X \) and the multiplication parameters \( \gamma_{i,j}^k \) in (4). De Klerk, Pasechnik and Schrijver [2] apply this technique for computing tighter bounds for the crossing number of a complete bipartite graph. We
apply it in Section 3.2 for reducing the size of the semidefinite programs permitting to compute the hierarchy of semidefinite bounds for the parameter $A(n,d)$.

Example 3. Let $X := \mathcal{P}$, the collection of all subsets of the set $V = \{1, \ldots, n\}$, and $G := \text{Sym}(V)$, the group of permutations of $V$. Each $\pi \in G$ induces a permutation of $X$, again denoted by $\pi$, by letting $\pi(I) := \{\pi(i) \mid i \in I\}$ for $I \in \mathcal{P}$. Two pairs $(I, J), (I', J') \in \mathcal{P}$ lie in the same orbit [i.e., $I' = \pi(I), J' = \pi(J)$ for some $\pi \in G$] if and only if $|I| = |I'|$, $|J| = |J'|$ and $|I \cap J| = |I' \cap J'|$. Therefore, the commutant algebra $A^G$ is generated by the matrices $M_{i,j}^t$ $(i, j, t \in \mathbb{Z}_+)$, where

$$
(M_{i,j}^t)_{I,J} := \begin{cases} 1 & \text{if } |I| = i, |J| = j, |I \cap J| = t, \\ 0 & \text{otherwise} \end{cases}
$$

for $I, J \in \mathcal{P}$; $A^G =: \mathcal{A}_n$ is known as the Terwilliger algebra of the Hamming scheme (Terwilliger [15]).

Example 4. Consider again the set $X := \mathcal{P}$, but now the group is $G := \text{Aut}(\mathcal{P})$, the automorphism group of $\mathcal{P}$. The group $G$ consists of the permutations $\sigma \in \text{Sym}(V)$ preserving the symmetric difference, i.e., for which $|\sigma(I) \Delta \sigma(J)| = |I \Delta J|$ for all $I, J \in \mathcal{P}$.

Thus,

$$
G = \{s_A \mid A \subseteq V, s_A \in \text{Sym}(V)\}
$$

where, for a set $A \subseteq V$, $s_A$ is the permutation of $\mathcal{P}$ mapping any $I \in \mathcal{P}$ to $s_A(I) := A \Delta I$; we have $|G| = 2^{n1}/2$. Two pairs $(I, J), (I', J') \in \mathcal{P}$ lie in the same orbit [i.e., $I' = \sigma(I), J' = \sigma(J)$ for some $\sigma \in G$] if and only if $|I \Delta J| = |I' \Delta J'|$. Therefore, the algebra $A^G$ is generated by the matrices $M_k$ $(k = 0, 1, \ldots, n)$ where

$$
(M_k)_{I,J} := \begin{cases} 1 & \text{if } |I \Delta J| = k, \\ 0 & \text{otherwise} \end{cases}
$$

for $I, J \in \mathcal{P}$; $A^G =: \mathcal{B}_n$ is known as the Bose Mesner algebra of the Hamming scheme. The Bose-Mesner algebra is a subalgebra of the Terwilliger algebra, as $M_k = \sum_{i,j=0}^{n} M_{i,j}^{(i+j-k)/2}$ for $k = 0, 1, \ldots, n$.

In fact, it is known from invariant theory and $C^*$-algebra theory that the algebra $A^G$ can be block-diagonalized. Therefore, there exists a semidefinite program equivalent to the invariant program (7), where the matrix $Y$ is replaced by a block-diagonal matrix with possibly repeated blocks; see, e.g., Gatermann and Parrilo [4]. Such program is typically more compact than the program (8). However, finding explicitly the block-diagonalization is a nontrivial task in general. An advantage of the above mentioned reduction method, based on the regular $*$-representation, is that it involves the matrices $L_t$ which are explicitly defined in terms of the matrices $D_i$ generating the algebra. Nevertheless, Schrijver [13] was able to determine explicitly the block-diagonalization for the Terwilliger algebra; we recall this result in the next section as we will need it for the computation of our stronger bounds for the coding problem.

2.2. Block-diagonalization of the Terwilliger algebra. While the Bose-Mesner algebra $\mathcal{B}_n$ is a commutative algebra and thus can be diagonalized (see [3]), the Terwilliger algebra $\mathcal{A}_n$ is a non-commutative algebra. Its dimension is $\dim \mathcal{A}_n = \binom{n+3}{3}$, which is the number of triples $(i, j, t)$ for which $M_{i,j}^t \neq 0$. As $\mathcal{A}_n$ is a matrix $*$-algebra containing the
identity, it can be block-diagonalized, which means the following: There exists a unitary \( \mathcal{P} \times \mathcal{P} \) complex matrix \( U \) (i.e., \( U^*U = I \)) and positive integers \( m \) and \( p_0, q_0, \ldots, p_m, q_m \) such that the set \( U^*A_nU := \{ U^*MU \mid M \in \mathcal{A}_n \} \) is equal to the collection of block-diagonal matrices
\[
\begin{pmatrix}
C_0 & 0 & \cdots & 0 \\
0 & C_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & C_m
\end{pmatrix}
\]
where each \( C_k \) \( (k = 0, 1, \ldots, m) \) is a block-diagonal matrix with \( q_k \) identical blocks \( B_k \) of order \( p_k \):
\[
C_k = \begin{pmatrix}
B_k & 0 & \cdots & 0 \\
0 & B_k & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_k
\end{pmatrix}
\]
thus \( 2^n = \sum_{k=0}^{m} p_kq_k \) and \( \sum_{k=0}^{m} p_k^2 = \dim \mathcal{A}_n = \binom{n+3}{3} \). By deleting copies of identical blocks, it follows that \( \mathcal{A}_n \) is isomorphic to the algebra
\[
\bigoplus_{k=0}^{m} \mathbb{C}^{p_k \times p_k} = \left\{ \begin{pmatrix}
B_0 & 0 & \cdots & 0 \\
0 & B_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_m
\end{pmatrix} \mid B_k \in \mathbb{C}^{p_k \times p_k} \text{ for } k = 0, 1, \ldots, m \right\}.
\]

An important fact for our purpose is that this isomorphism preserves positive semidefiniteness. The existence of a unitary matrix \( U \) with the above properties is standard \( C^* \)-algebra theory (see, e.g., [14]). Schrijver [13] has constructed explicitly this matrix \( U \) and the image of a matrix \( M \in \mathcal{A}_n \) in the algebra (12). We recall some facts from [13] needed for our treatment; we refer to [13] for details and proofs.

It turns out that \( U \) is real valued, \( m = \lfloor \frac{n}{3} \rfloor \) and, for \( k = 0, 1, \ldots, \lfloor \frac{n}{3} \rfloor \), the block \( B_k \) has order \( p_k = n - 2k + 1 \) and multiplicity \( q_k = \binom{n}{k} - \binom{n}{k-1} \). In particular, the block \( B_0 \) has order \( n + 1 \) and multiplicity 1. We now describe explicitly the matrix \( U \). For this, for \( k = 1, \ldots, \lfloor \frac{n}{3} \rfloor \), define
\[
\mathcal{L}_k := \{ b \in \mathbb{R}^P \mid M_{k-1,k}^{k-1}b = 0 \text{ and } b_I = 0 \text{ if } |I| \neq k \}.
\]
Let \( \mathcal{B}_k \) be a basis of \( \mathcal{L}_k \). Then \( |\mathcal{B}_k| = \binom{n}{k} - \binom{n}{k-1} \) and \( \sum_{I \in \mathcal{P}} b_I = 0 \) for \( b \in \mathcal{L}_k \). Set \( \mathcal{B}_0 := \{ b_0 \} \) where \( b_0 := (1, 0, \ldots, 0)^T \in \mathbb{R}^P \) (the nonzero entry being indexed by \( 0 \in \mathcal{P} \)) and define
\[
\mathcal{Q} := \{(k, b, i) \mid k \in \{0, \ldots, \lfloor \frac{n}{3} \rfloor \}, b \in \mathcal{B}_k, i \in \{k, k+1, \ldots, n-k\} \}.
\]
Then \( |\mathcal{Q}| = 2^n = |\mathcal{P}| \). For \( (k, i, b) \in \mathcal{Q} \), define the vector
\[
u_{k,i,b} := \left( \begin{pmatrix} n - 2k \\ i - k \end{pmatrix} \right)^{-1/2} M_{k,k}^b b \in \mathbb{R}^P.\]
Finally define $U$ as the $P \times Q$ matrix whose columns are the vectors $u_{k,i,b}$ for $(k, i, b) \in Q$. The following is shown in [13].

**Proposition 5.** [13] The matrix $U$ is orthogonal, i.e., $U^T U = I$. Moreover, for a matrix $M = \sum_{i,j,t=0}^n x_{i,j}^t M_{i,j} \in \mathbb{A}_n$ (with $x_{i,j}^t \in \mathbb{R}$), the matrix $U^T MU$ is a block-diagonal matrix determined by the partition of $Q$ into the classes $Q_{k,b} := \{(k, i, b) \mid k \leq i \leq n - k\}$ (for $k = 0, \ldots, \lfloor n/2 \rfloor$, $b \in B_k$). For a given integer $k = 0, \ldots, \lfloor n/2 \rfloor$, the blocks corresponding to the classes $Q_{k,b}$ (for $b \in B_k$) are all identical to the following matrix:

\[
B_k(x) := \left( \sum_{t} \frac{(n-2k)}{(i-k)} \left( \frac{(n-2k)}{(j-k)} \right)^{\frac{1}{2}} \beta_{i,j,k}^t x_{i,j}^t \right)_{i,j=k}^{n-k},
\]

after setting

\[
\beta_{i,j,k}^t := \sum_{u=0}^{n} (-1)^{t-u} \binom{n}{u} \binom{n-2k}{n-k-u} \binom{n-k-u}{i-u} \binom{n-k-u}{j-u}
\]

for $i, j, k, t \in \{0, \ldots, n\}$. As $\mathbb{A}_n$ is isomorphic to the algebra (12), we have:

\[
\sum_{i,j,t=0}^n x_{i,j}^t M_{i,j} \succeq 0 \iff B_k(x) \succeq 0 \quad \text{for } k = 0, 1, \ldots, \lfloor n/2 \rfloor.
\]

The above property (15) is the key tool used in [13] and in the present paper, which allows reducing semidefinite programs involving matrices in the Terwilliger algebra to semidefinite programs of size $O(n^3)$.

We will deal in this paper with matrices of the form

\[
\tilde{M} = \begin{pmatrix}
  d & c^T \\
  c & M
\end{pmatrix}, \quad \text{where } M = \sum_{i,j,t=0}^n x_{i,j}^t M_{i,j} \in \mathbb{R}, \quad d, c = \sum_{i=0}^n c_i \chi_{\mathbb{P}\mathbb{V}^i(V)}.
\]

Recall that $\mathbb{P}_{=i}(V) = \{I \subseteq V \mid |I| = i\}$ and $\chi_{\mathbb{P}\mathbb{V}^i(V)} \in \{0, 1\}^P$ whose $I$-th entry is 1 if and only if $I \in \mathbb{P}_{=i}(V)$.

**Lemma 6.** The matrix $\tilde{M}$ from (16) is positive semidefinite if and only if $B_k(x) \succeq 0$ for $k = 1, \ldots, \lfloor n/2 \rfloor$, and

\[
\tilde{B}_0(x) := \begin{pmatrix}
  d & c^T \\
  c & B_0(x)
\end{pmatrix} \succeq 0, \quad \text{where } c := (c_i \binom{n}{i})_{i=0}^n.
\]

**Proof.** Setting

\[
\tilde{U} := \begin{pmatrix}
  1 & 0 \\
  0 & U^T
\end{pmatrix},
\]

we have:

\[
\tilde{U}^T \tilde{M} \tilde{U} = \begin{pmatrix}
  d & c^T U \\
  U^T c & U^T M U
\end{pmatrix}.
\]

It suffices now to verify that $(c^T U)_{k,i,b} = c^T u_{k,i,b} = 0$ for $(k, i, b) \in Q$ with $k \geq 1$, and that $(c^T U)_{0,i,b} = c_i \binom{n}{i}^{\frac{1}{2}}$ for $i = 0, \ldots, n$. This is direct verification using the above definitions;
details are omitted. Hence, $\tilde{U}^T \tilde{M} \tilde{U}$ is block-diagonal, with blocks $\tilde{B}_0(x)$ (with multiplicity 1) and $B_k(x)$ (with multiplicity $q_k$) for $k = 1, \ldots, \left[ \frac{n}{4} \right]$. The lemma now follows.

3. Semidefinite programming bounds for the stability number of a graph

3.1. Lasserre’s construction. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph. A stable set in $\mathcal{G}$ is a set $S \subseteq \mathcal{V}$ containing no edge and the stability number $\alpha(\mathcal{G})$ of $\mathcal{G}$ is the maximum cardinality of a stable set in $\mathcal{G}$. Recall $\mathcal{P}_k(\mathcal{V}) = \{ I \subseteq \mathcal{V} \mid |I| \leq k \}$ for an integer $k$. Given a stable set $S$ in $\mathcal{G}$, define $x = (x_I)_{I \in \mathcal{P}_k(\mathcal{V})} \in \{0,1\}^{\mathcal{P}_k(\mathcal{V})}$ and $y = (y_{IJ})_{I \in \mathcal{P}_k(\mathcal{V})} \in \{0,1\}^{\mathcal{P}_k(\mathcal{V})}$ with $x_I = 1$ (resp., $y_{IJ} = 1$) if and only if $I \subseteq S$, for $I \in \mathcal{P}_k(\mathcal{V})$ (resp., for $I \in \mathcal{P}_k(\mathcal{V})$). Then $y$ and the matrix $Y := xx^T$ satisfy:

\begin{align}
Y & \succeq 0 \\
Y_{I,J} &= y_{I \cup J} & \text{(for } I, J \in \mathcal{P}_k(\mathcal{V})) \\
Y_{I,J} &= y_{I \cup J} = 0 & \text{if } I \cup J \text{ contains an edge} & \text{(for } I, J \in \mathcal{P}_k(\mathcal{V})) \\
Y_{0,0} &= y_0 = 1 \\
0 & \leq y_I \leq y_J & \text{if } J \subseteq I & \text{(for } I, J \in \mathcal{P}_k(\mathcal{V})).}
\end{align}

We refer to (19) as the edge condition and to (18) as the moment condition. A matrix $Y$ satisfying (18) is known as a moment matrix and is denoted as $Y = M_k(y)$ (see [6, 7, 8]). Under the assumption (17), the edge condition (19) is, in fact, equivalent to $y_{ij} = 0$ (for $ij \in \mathcal{E}$). (Here and below, we set $y_{ij} := y_{\{i\},\{j\}}, y_i := y_{\{i\}},$ etc.) Under (17), relation (21) holds for $I \in \mathcal{P}_k(\mathcal{V})$; indeed, the principal submatrix of $M_k(y)$ indexed by $\{I, J\}$ has the form \( \begin{pmatrix} y_I & y_J \\ y_J & y_J \end{pmatrix} \), whose positive semidefiniteness implies $0 \leq y_I \leq y_J$. On the other hand, $M_0(y) \succeq 0$ implies $|y_0| \leq \max(y_i, y_J)$; indeed the principal submatrix of $M_0(y)$ indexed by $\{i, j\}$ has the form \( \begin{pmatrix} y_i & y_j \\ y_j & y_j \end{pmatrix} \), whose positive semidefiniteness implies $y_{ij} \leq y_i y_J \leq \max(y_i^2, y_J^2)$. Similarly, $M_k(y) \succeq 0$ implies that $|y_{ij,k}|$ is at most the largest two values among $y_{ij}, y_{ik}, y_{jk}$; indeed the principal submatrix of $M_k(y)$ indexed by $\{i, j\}, \{i, k\}, \{j, k\}$ has the form \( \begin{pmatrix} y_{ij} & y_{ik} & y_{jk} \\ y_{ik} & y_{ik} & y_{ik} \\ y_{jk} & y_{ik} & y_{jk} \end{pmatrix} \), whose positive semidefiniteness implies $y_{ij,k}^2 \leq \min(y_{ij}, y_{ik}, y_{jk}) y_{ik}$ assuming, say, that $y_{ij} \leq y_{ik} \leq y_{jk}$.

Consider the semidefinite program:

\begin{equation}
\ell^{(k)}(\mathcal{G}) := \max \sum_{i \in \mathcal{V}} y_i \text{ s.t. } M_k(y) \succeq 0, \ y_0 = 1, \ y_{ij} = 0 \text{ (if } ij \in \mathcal{E}).
\end{equation}

Then, $\alpha(\mathcal{G}) \leq \ell^{(k)}(\mathcal{G})$, with equality if $k \geq \alpha(\mathcal{G})$ ([7, 8]). Define $\ell^{(k)}(\mathcal{G})$ as the parameter obtained by adding to (22) the constraints (21); thus,

\[ \alpha(\mathcal{G}) \leq \ell^{(k)}_+(\mathcal{G}) \leq \ell^{(k)}(\mathcal{G}). \]
For $k = 1$, $\ell^{(1)}(\mathcal{G}) = \vartheta(\mathcal{G})$, the Lovász’ theta number, and the stronger bound obtained by adding nonnegativity to (22) is $\vartheta'(\mathcal{G})$, the strengthening of $\vartheta(\mathcal{G})$ introduced by McEliece, Rodemich and Rumsey [11] and Schrijver [12]. The bound $\ell^{(2)}(\mathcal{G})$ is at least as good as the parameter obtained by optimizing over $N_+(\text{TH}(\mathcal{G}))$, the convex relaxation of the stable set polytope of $\mathcal{G}$ obtained by applying the Lovász-Schrijver $N_+$-operator to the theta body $\text{TH}(\mathcal{G})$ ([8]; or see (26)). For $k = 2$, the program (22) has size $O(|\mathcal{V}|^4)$. We now formulate a bound $\ell(\mathcal{G})$, which is weaker than $\ell^{(2)}(\mathcal{G})$, but still at least as good as the bound obtained from $N_+(\text{TH}(\mathcal{G}))$, although its computation is more economical since it can be expressed via a semidefinite program of size $O(|\mathcal{V}|^3)$.

Namely, for each $r \in \mathcal{V}$, consider the principal submatrix $Y_r(y)$ of $M_2(y)$ indexed by the set $\mathcal{P}_2(\mathcal{V};r) := \mathcal{P}_1(\mathcal{V}) \cup \{\{r, i\} \mid i \in \mathcal{V}\}$; thus the matrices $Y_r(y)$ involve only variables $y_I$ for $I \in \mathcal{P}_2(\mathcal{V})$. Define

$$\ell(\mathcal{G}) := \max \sum_{i \in \mathcal{V}} y_i \quad \text{s.t.} \quad y_0 = 1, \quad y_{ij} = 0 \ (ij \in \mathcal{E}), \quad Y_r(y) \succeq 0 \ (r \in \mathcal{V}),$$

and $\ell_+(\mathcal{G})$ as the parameter obtained by adding to (23) the constraints: $0 \leq y_{ijk} \leq y_{jk}$ for distinct $i, j, k \in \mathcal{V}$ (coming from (21)). Obviously,

$$\ell^{(2)}(\mathcal{G}) \leq \ell(\mathcal{G}) \leq \ell^{(1)}(\mathcal{G});$$

analogously for the $\ell_+$ parameters. We will see in Section 3.3 that, for the graph $\mathcal{G} = \mathcal{G}(n, d)$, the matrices involved in (23) lie in (a variation of) the Tverrilliger algebra, which allows reformulating the parameters $\ell(\mathcal{G}(n, d)), \ell_+(\mathcal{G}(n, d))$ via semidefinite programs of size $O(n^3)$.

From the moment condition (18), the matrix $Y_r(y)$ has the block structure:

$$Y_r(y) = \begin{pmatrix} 1 & a^T & b_r^T \\ a & A & B_r \\ b_r & B_r & B_r \end{pmatrix},$$

where $A := (y_{ij})_{i,j \in \mathcal{V}}, B_r := (y_{i,j,r})_{i,j \in \mathcal{V}}$ are symmetric $\mathcal{V} \times \mathcal{V}$ matrices, and $a := (y_k)_{k \in \mathcal{V}}, b_r := (y_{kr})_{k \in \mathcal{V}}$. As $b_r$ coincides with the $r$-th column of $A$ and of $B_r$, by applying some column/row manipulation to $Y_r(y)$, one deduces that

$$Y_r(y) \succeq 0 \iff B_r \succeq 0 \quad \text{and} \quad \tilde{C}_r := \begin{pmatrix} 1 - y_r & a^T - b_r^T \\ a & A - B_r \end{pmatrix} \succeq 0,$$

which permits to reduce the size of the matrices involved in program (23). Setting

$$\text{TH}(\mathcal{G}) = \{x \in \mathbb{R}^{\mathcal{P}_1(\mathcal{V})} \mid \exists y \in \mathbb{R}^{\mathcal{P}_2(\mathcal{V})} \text{ s.t. } M_1(y) \succeq 0, \quad y_{ij} = 0 \ (ij \in \mathcal{E}), \quad x_I = y_I \ (I \in \mathcal{P}_1(\mathcal{V}))\},$$

$$N_+(\text{TH}(\mathcal{G})) = \{x \in \mathbb{R}^{\mathcal{V}} \mid \exists y \in \mathbb{R}^{\mathcal{P}_2(\mathcal{V})} \text{ s.t. } M_1(y) \succeq 0, \quad y_0 = 1, \quad x_i = y_i \ (i \in \mathcal{V}), \quad (y_{I \cup \{r\}})_{I \in \mathcal{P}_1(\mathcal{V})}, \quad (y_I - y_{I \cup \{r\}})_{I \in \mathcal{P}_1(\mathcal{V})} \in \text{TH}(\mathcal{G})\}$$

one can verify that

$$\ell(\mathcal{G}) \leq \max_{x \in N_+(\text{TH}(\mathcal{G}))} \sum_{i \in \mathcal{V}} x_i.$$
To see it, let $y$ be feasible for (23); then $x := (y_i)_{i \in \mathcal{V}} \in N_+(\text{TH}(G))$. Indeed, the vector $(y_{U \cup \{r\}})_{i \in \mathcal{P}_1(\mathcal{V})}$ is equal to the first column of the principal submatrix of $Y_r(y)$ indexed by $\{r\} \cup \{\{r, i\} \mid i \in \mathcal{V}\}$, and $(y_{U \cup \{r\}})_{i \in \mathcal{P}_1(\mathcal{V})}$ is the first column of the matrix $\tilde{C}_r$ in (25).

3.2. The semidefinite programming bounds $\ell^k(G)$ for the coding problem. Let $G$ be a group of automorphisms of the graph $G = (\mathcal{V}, \mathcal{E})$; that is, $G \subseteq \text{Sym}(\mathcal{V})$ and each $\sigma \in G$ preserves edges, i.e., $ij \in \mathcal{E} \implies \sigma(i)\sigma(j) \in \mathcal{E}$. Then $G$ acts on the set $\mathcal{P}_k(\mathcal{V})$ indexing matrices in the program (22), by letting $\sigma(I) = \{\sigma(i) \mid i \in I\}$ for $\sigma \in G$, $I \in \mathcal{P}_k(\mathcal{V})$.

LEMMA 7. Let $G$ be a group of automorphisms of $G$. Then the program (22) is invariant under the action of $G$.

Proof. Set $Y = M_k(y)$. The objective function is of the form $\sum_{i \in \mathcal{V}} y_i = \sum_{i \in \mathcal{V}} Y_{i,i} = \langle C, Y \rangle$, where $C$ is invariant under action of $G$, since the set $\{\{i\}, \{i\} \mid i \in \mathcal{V}\}$ is a union of orbits of $\mathcal{P}_k(\mathcal{V}) \times \mathcal{P}_k(\mathcal{V})$ (in fact, a single orbit if $G$ is vertex-transitive). The constraint $y_0 = Y_{0,0} = 1$ is of the form $\langle A, Y \rangle = 1$ where $A$ is invariant, since the set $\{(0,0)\}$ is an orbit. The class of edge constraints (19) is invariant under action of $G$: If $I \cup J$ contains an edge $ij$ and $\sigma \in G$, then $\sigma(I) \cup \sigma(J)$ contains the edge $\sigma(i)\sigma(j)$ and thus the equation: $y_{\sigma(I) \cup \sigma(J)} = Y_{\sigma(I) \cup \sigma(J)} = 0$ is again an edge constraint. Similarly, the class of moment constraints (18) is also invariant under action of $G$.

By Corollary 2, the parameter $\ell^k(G)$ can therefore be formulated as the optimum of a semidefinite program in $N$ variables involving $N \times N$ matrices, where $N$ is the number of orbits of the set $\mathcal{P}_k(\mathcal{V}) \times \mathcal{P}_k(\mathcal{V})$ under the action of the group $G$. We now apply this technique to the graph $G = G(n, d)$ and to the group $G = \text{Aut}(\mathcal{P})$, the group of automorphisms of $\mathcal{P}$ (introduced in (10)). Recall that $G(n, d)$ has node set $\mathcal{P}$, the collection of subsets of $\{1, \ldots, n\}$, with an edge $(I, J)$ if $|I \Delta J| \in \{1, \ldots, d-1\}$ for $I, J \in \mathcal{P}$. Thus $G$ also acts on the set $\mathcal{P}_k(\mathcal{P}) = \{A \subseteq \mathcal{P} \mid |A| \leq k\}$, indexing the matrix variable in program (22). We show:

THEOREM 8. For any fixed $k$, one can compute (to an arbitrary precision) the parameter $\ell^k(G(n, d))$ from (22) in time polynomial in $n$. The same holds for the parameter $\ell^k(G)$ obtained by adding the constraints (21) to (22).

Proof. Let $k$ be fixed and let $N_k$ denote the number of orbits of the set $\mathcal{P}_k(\mathcal{P}) \times \mathcal{P}_k(\mathcal{P})$ under the action of the group $G$. As mentioned above, the parameter $\ell^k(G(n, d))$ can be expressed via a semidefinite program of the form (8), involving $N_k \times N_k$ matrices and $N_k$ variables. Hence, to show Theorem 8, it suffices to verify that $N_k$ is bounded by a polynomial in $n$ and that the new program equivalent to (22) can be constructed in time polynomial in $n$.

To begin with, it is useful to have a way to identify the orbits of the set $\mathcal{P}_k(\mathcal{P}) \times \mathcal{P}_k(\mathcal{P})$.

Consider $(A, B) \in \mathcal{P}_k(\mathcal{P}) \times \mathcal{P}_k(\mathcal{P})$ with $r := |A|$ and $s := |B|$. If $r = s = 0$ then $A = B = \emptyset$, the empty subset of $\mathcal{P}$, and the orbit of $(\emptyset, \emptyset)$ just consists of the pair $(\emptyset, \emptyset)$. We can now assume that $r + s \geq 1$. Let $\mathcal{A} = (A_1, \ldots, A_r)$ be an ordering of the elements
of $\mathcal{A}$; similarly, $\mathcal{B} = (B_1, \ldots, B_s)$ is an ordering of the elements of $\mathcal{B}$. Then one can define the $(r + s) \times n$ incidence tableau of $(\mathcal{A}, \mathcal{B})$, whose rows are the incidence vectors $\chi^{A_1}, \ldots, \chi^{A_r}, \chi^{B_1}, \ldots, \chi^{B_s}$ (in that order) of the sets $A_1, \ldots, A_r, B_1, \ldots, B_s$. Define the function $\varphi_{\mathcal{A}, \mathcal{B}} : \{0,1\}^r \times \{0,1\}^s \rightarrow \mathbb{Z}_+$ where, for $(u, v) \in \{0,1\}^r \times \{0,1\}^s$, $\varphi_{\mathcal{A}, \mathcal{B}}(u, v)$ is the multiplicity of $(u, v)$ as a column of the incidence tableau of $(\mathcal{A}, \mathcal{B})$. Thus $\varphi_{\mathcal{A}, \mathcal{B}}$ belongs to the set $\Phi_{r,s}$ consisting of the functions $\phi : \{0,1\}^r \times \{0,1\}^s \rightarrow \{0,1, \ldots, n\}$ satisfying: 
\[ \sum_{a \in \{0,1\}^r, b \in \{0,1\}^s} \phi(a, b) = n \quad \text{and, for all } i \neq j \in \{1, \ldots, r\} \text{ (resp., } i \neq j \in \{1, \ldots, s\}, \text{ there exists } (u, v) \in \{0,1\}^r \times \{0,1\}^s \text{ for which } \phi(a, b) \geq 1 \text{ and } u_i \neq u_j \text{ (resp., } v_i \neq v_j). \]

Let $\mathcal{A}'$ (resp., $\mathcal{B}'$) be another ordered sequence of $r$ (resp., of $s$) distinct elements of $\mathcal{P}$ and let $\mathcal{A}' = (\mathcal{A}', \mathcal{B}')$. Then, $\mathcal{A}' = (\sigma(A_1), \ldots, \sigma(A_r))$ and $\mathcal{B}' = (\sigma(B_1), \ldots, \sigma(B_s))$ for some $\sigma \in G$ if and only if $\phi(a, b) + \phi(1 - a, 1 - b) = \phi'(a, b) + \phi'(1 - a, 1 - b)$ for all $(a, b) \in \{0,1\}^r \times \{0,1\}^s$. (Here, $1 := (1, \ldots, 1)$ denotes the all-ones vector of the suitable size.) Moreover, $\mathcal{A}' = (A_{\alpha(1)}, \ldots, A_{\alpha(r)})$ and $\mathcal{B}' = (B_{\alpha(1)}, \ldots, B_{\alpha(s)})$ for some permutations $\alpha \in \text{Sym}(r)$, $\beta \in \text{Sym}(s)$ if and only if $\phi'(a, b) = \phi(\alpha(a), \beta(b))$ for all $(a, b) \in \{0,1\}^r \times \{0,1\}^s$, setting $\alpha(a) := (a_{\alpha(1)}, \ldots, a_{\alpha(r)})$, $\beta(b) := (b_{\beta(1)}, \ldots, b_{\beta(s)})$. For two elements $\phi, \phi' \in \Phi_{r,s}$, write $\phi \sim \phi'$ if 
\[ \phi'(a, b) + \phi'(1 - a, 1 - b) = \phi(\alpha(a), \beta(b)) + \phi(1 - \alpha(a), 1 - \beta(b)) \quad \forall (a, b) \in \{0,1\}^r \times \{0,1\}^s \]
for some $\alpha \in \text{Sym}(r)$, $\beta \in \text{Sym}(s)$. This defines an equivalence relation on $\Phi_{r,s}$. We can now characterize orbits in the following way: Two pairs $(\mathcal{A}, \mathcal{B}), (\mathcal{A}', \mathcal{B}')$ belong to the same orbit of $\mathcal{P}_k(\mathcal{P}) \times \mathcal{P}_k(\mathcal{P})$ under action of $G$ if and only if $|\mathcal{A}| = |\mathcal{A}'| := r$, $|\mathcal{B}| = |\mathcal{B}'| := s$ and $\varphi_{\mathcal{A}, \mathcal{B}} \sim \varphi_{\mathcal{A}', \mathcal{B}'}$ for some respective orderings $\mathcal{A}, \mathcal{B}, \mathcal{A}', \mathcal{B}'$. Thus each orbit of $\mathcal{P}_k(\mathcal{P}) \times \mathcal{P}_k(\mathcal{P})$ corresponds to an equivalence class of $\cup_{0 \leq r, s \leq k} \Phi_{r,s}$. Hence the number $N_k$ of orbits of $\mathcal{P}_k(\mathcal{P}) \times \mathcal{P}_k(\mathcal{P})$ is at most 
\[ 1 + \sum_{r \geq s \geq 1} (n + 1)(r + s - 1), \]
giving:

$$N_k \leq O(n^{2k-1}).$$

We now verify that the matrices $L_i$ ($i = 1, \ldots, N_k$) (as defined in (5)) can be constructed in time polynomial in $n$.

For this one first needs to be able to compute in time polynomial in $n$ the cardinality of the orbits of $\mathcal{P}_k(\mathcal{P}) \times \mathcal{P}_k(\mathcal{P})$. Given $\phi_0 \in \Phi_{r,s}$ ($0 \leq r, s \leq k, r + s \geq 1$), one has to count the number $L_{\phi_0}$ of pairs $(\mathcal{A}, \mathcal{B}) \in \mathcal{P}_c(\mathcal{P}) \times \mathcal{P}_c(\mathcal{P})$ for which $\varphi_{\mathcal{A}, \mathcal{B}} \sim \phi_0$ for some orderings $\mathcal{A}, \mathcal{B}$ of $\mathcal{A}, \mathcal{B}$. Given $\phi \sim \phi_0$, there are $\ell_\phi := n! / \prod_{v \in \{0,1\}^s} \phi(v)$ pairs $(\mathcal{A}, \mathcal{B})$ for which $\varphi_{\mathcal{A}, \mathcal{B}} \sim \phi_0$. Therefore, $L_{\phi_0} = \frac{1}{r+s} \sum_{\phi \sim \phi_0} \ell_\phi$, which can be computed in time polynomial in $n$ since one can enumerate the equivalence class of $\phi_0$ in time polynomial in $n$.

Next we verify that one can compute in time polynomial in $n$ the multiplicity parameters $\gamma_{i,j,k}$ from (4), used for defining the matrices $L_i$ in (5). For this, given $(\mathcal{A}, \mathcal{B}) \in \mathcal{P}_c(\mathcal{P}) \times \mathcal{P}_c(\mathcal{P})$ with respective orderings $\mathcal{A}, \mathcal{B}$, given an integer $0 \leq t \leq k$, and given $\phi_0 \in \Phi_{r,t}$, $\psi_0 \in \Phi_{s,t}$, one has to count the number $L_{\phi_0,\psi_0}$ of elements $\mathcal{C} \in \mathcal{P}_t(\mathcal{P})$ for which $\varphi_{\mathcal{A}, \mathcal{B}_c} \sim \phi_0$ and $\varphi_{\mathcal{B}, \mathcal{C}_t} \sim \psi_0$ for some ordering $\mathcal{C}_t$ of $\mathcal{C}$. Set $\xi := \varphi_{\mathcal{A}, \mathcal{B}_c}$. Given $\phi \sim \phi_0$ and $\psi \sim \psi_0$, we first count the number $\ell_{\phi, \psi}$ of ordered sequences $\mathcal{C}_t$ of $t$ elements of
\( \mathcal{P} \) for which \( \varphi_{\tilde{A}, \tilde{C}} = \phi \) and \( \varphi_{\tilde{B}, \tilde{C}} = \psi \). For this let \( x(u, v, w) \in \{0, 1\}^r \times \{0, 1\}^s \times 2^t \) as column of the incidence tableau of \((\tilde{A}, \tilde{B}, \tilde{C})\). The first \( r + s \) rows of the tableau are given and one needs to determine its last \( t \) rows. Then, \( x(u, v, w) \in \{0, 1, \ldots, n\} \) satisfy the system

\[
\sum_{w \in \{0, 1\}^t} x(u, v, w) = \phi(u, w) \quad \forall u \in \{0, 1\}^r, w \in \{0, 1\}^s
\]

\[
\sum_{w \in \{0, 1\}^t} x(u, v, w) = \psi(v, w) \quad \forall v \in \{0, 1\}^s, w \in \{0, 1\}^t
\]

\[
\sum_{w \in \{0, 1\}^t} x(u, v, w) = \xi(u, v) \quad \forall u \in \{0, 1\}^r, v \in \{0, 1\}^s.
\]

As the system (28) has polynomially many variables and equations, its set \( S \) of solutions can be found by complete enumeration and \( |S| \leq (n + 1)^{2^t + s + t} \). Therefore, \( \ell_{\phi, \psi} = \sum_{x \in S} \sum_{u \in \{0, 1\}^r, v \in \{0, 1\}^s} \sum_{w \in \{0, 1\}^t} \xi(u, v, w) \), the number of possible ways to assign the vectors \( w \in 2^t \) as columns of the lower \( t \times n \) part of the tableau. Now, \( L_{\phi, \psi} = \frac{1}{!} \sum_{\psi = 0} \ell_{\phi, \psi} \) can be computed in time polynomial in \( n \) since one can enumerate the equivalence classes of \( \phi_0 \) and \( \psi_0 \).

Remains only to construct the linear constraints corresponding to the moment constraints (18) and the edge constraints (19). Label the orbits of \( \mathcal{P}_k(\mathcal{P}) \times \mathcal{P}_k(\mathcal{P}) \) as \( O_1, \ldots, O_{N_k} \) and determine a pair \((A_i, B_i)\) belonging to each orbit \( O_i \). Then the moment constraints read: \( x_i = x_j \) if \( A_i \cup B_i = \sigma(A_j \cup B_j) \) for some \( \sigma \in G \) (which can be tested in time polynomial in \( n \)), and the edge constraints read: \( x_i = 0 \) if \( A_i \cup B_i \) contains a pair \((I, J)\) with \( |I \Delta J| \in \{1, \ldots, d - 1\} \).

The bounds (21) become: \( x_i \geq 0 \) if \( 1, \ldots, N_k \) and \( x_i \leq x_j \) if \( A_i \cup B_i \geq \sigma(A_j \cup B_j) \) for some \( \sigma \in G \) (which can be tested in time polynomial in \( n \)).

Therefore, the parameter \( \ell^{(k)}(G(n, d)) \) (or \( \ell^{(k)}_+(G(n, d)) \)) can be computed as the optimum value of a semidefinite program of the form (8) involving \( N_k \times N_k \) matrices, with \( N_k \) variables and \( O(N_k^2) \) linear constraints. As \( N_k = O(n^{2d-4d+1}) \), it can be computed in time polynomial in \( n \) (to any precision), which concludes the proof of Theorem 8.

The result from Theorem 8 is mainly of theoretical value for \( k \geq 2 \). Indeed, for \( k = 2 \), \( N_k = O(n^7) \) and thus the semidefinite program defining \( \ell^{(2)}(G(n, d)) \) is already too large to be solved in practice for interesting values of \( n \) by the currently available software for semidefinite programming.

3.3. **Refining Schrijver’s bound.** We begin with observing that, when a graph \( G \) has a vertex-transitive group \( G \) of automorphisms then, in the program (23), it suffices to require the condition \( Y_r(y) \geq 0 \) for one choice of \( r \in \mathcal{V} \).

**Lemma 9.** Let \( G \) be a group of automorphisms of the graph \( G = (\mathcal{V}, \mathcal{E}) \). The program (23) is invariant under action of \( G \). If \( G \) is vertex-transitive then, in (23), it suffices to require the constraint \( Y_r(y) \geq 0 \) for one choice of \( r \in \mathcal{V} \) (instead of for all \( r \in \mathcal{V} \)).

**Proof.** The first part of the proof is analogous to the proof of Lemma 7. Here, we use the fact that, for \( r \in \mathcal{V} \), \( \sigma \in G \), \( Y_r(\sigma(y)) = \sigma(Y_{\sigma(r)}(y)) \). Hence, if \( y \) is invariant under action
of \( G \), then \( Y_r(y) \geq 0 \) for some \( r \in \mathcal{V} \) implies that \( Y_r(y) \geq 0 \) for all \( r \in \mathcal{V} \).

3.3.1. A compact semidefinite formulation for the bound \( \ell(\mathcal{G}(n,d)) \). In this section we consider the graph \( \mathcal{G} = \mathcal{G}(n,d) \) and the group \( \mathbf{G} = \text{Aut}(\mathcal{P}) \), whose action on the graph \( \mathcal{G}(n,d) \) is indeed vertex-transitive. We set:

\[
\mathcal{X} := \mathcal{P}_2(\mathcal{P}; \emptyset) = \{\emptyset\} \cup \{\{I\} \mid I \in \mathcal{P}\} \cup \{\emptyset, \{I\} \mid I \in \mathcal{P}\}.
\]

Applying Lemma 9, one can reformulate the parameter \( \ell(\mathcal{G}(n,d)) \) as

\[
\ell(\mathcal{G}(n,d)) = \max_{Y(y) \geq 0, y_{\emptyset} = 1} \sum_{I \in \mathcal{P}} y(I)
\]

\[
y_{I,J} = 0 \text{ if } |I \Delta J| \in \{1, \ldots, d-1\}.
\]

\[
y_{I,J} = y_{\sigma(\mathcal{A})} \text{ for } \sigma \in \mathcal{G}, \mathcal{A} \in \mathcal{X},
\]

where the matrix variable \( Y(y) \) is indexed by the set \( \mathcal{X} \) and satisfies: \( Y(y)_{\mathcal{A},B} = y_{\mathcal{A} \cup \mathcal{B}} \) for \( \mathcal{A}, \mathcal{B} \in \mathcal{X} \). By (24), \( Y(y) \) has the form

\[
Y(y) = \begin{pmatrix}
a & a^T & b^T \\
A & B & B \\
b & B & B
\end{pmatrix}
\]

with \( A = (y(I,J))_{I,J \in \mathcal{P}}, B = (y(I,J))_{I,J \in \mathcal{P}}, a = (y(I))_{I \in \mathcal{P}}, \) and \( b = (y(\emptyset, I))_{I \in \mathcal{P}} \). As \( y \) is invariant under action of \( \mathcal{G} \), it follows that \( A_{I,J} = A_{I',J'} \) if \( I' = \sigma(I), J' = \sigma(J) \) for some \( \sigma \in \mathcal{G} \), i.e., if \( |I \Delta J| = |I' \Delta J'| \). That is, the matrix \( A \) belongs to the Bose-Mesner algebra \( B_n \); say,

\[
A = \sum_{k=0}^n x_k M_k \text{ for some real scalars } x_0, \ldots, x_n
\]

where the matrices \( M_k \) are as in (11). Moreover, \( B_{I,J} = B_{I',J'} \) if \( I' = \sigma(I), J' = \sigma(J), \emptyset = \sigma(\emptyset) \) for some \( \sigma \in \mathcal{G} \), i.e., if \( |I' = |I|, |J' = |J| \) and \( |I \cap J| = |I' \cap J'| \). That is, the matrix \( B \) belongs to the Terwilliger algebra \( A_n \); say,

\[
B = \sum_{i,j,t \geq 0} x^t_{i,j} M^t_{i,j} \text{ for some real scalars } x^t_{i,j}
\]

where the matrices \( M^t_{i,j} \) are as in (9) and \( x^t_{i,j} = x^t_{j,i} \) for all \( i, j, t \). The variables \( x_k \) and \( x^t_{i,j} \) are related by

\[
x_k = x^0_{0,k} \text{ for } k = 0, 1, \ldots, n.
\]

(since \( x_k = A_{\emptyset,I} = B_{\emptyset,I} = x^0_{0,k} \) for \( |I| = k \)). Moreover,

\[
x^t_{i,j} = x^t_{i',j'} \text{ if } (i', j', i' + j' - 2t) \text{ is a permutation of } (i, j, i + j - 2t).
\]

Equivalently, \( x^t_{i,j} = x^{t-t}_{i+j-2t,i} = x^{t-t}_{i+j-2t,j} \). (Indeed, let \( I, J \in \mathcal{P} \) with \( i = |I|, j = |J|, t = |I \cap J| \). As \( \sigma := s_j \) maps \( \mathcal{A} := \{\emptyset, I, J\} \) to \( \{\emptyset, J, I \Delta J\} \) and \( y_{\sigma(\mathcal{A})} = y_{\mathcal{A}} \), then \( x^t_{i,j} = y(\emptyset, I, J) = y(\emptyset, J, I \Delta J) = x^{t-t}_{j,i+j-2t} \). The edge inequalities become:

\[
x^t_{i,j} = 0 \text{ if } \{i, j, i + j - 2t\} \cap \{1, \ldots, d-1\} = \emptyset.
\]
and the bounds (21) read:

\(0 \leq x_{i,j}^t \leq x_{i,j}^0\) for \(i, j, t = 0, \ldots, n.\)

From (25), we know that \(Y(y) \geq 0\) if and only if

\[B = \sum_{i,j,t=0}^{n} x_{i,j}^t M_{i,j}^t \geq 0\ \text{and} \ C := \left(1 - \frac{x_{0,0}^0}{c^T} c\right) \geq 0,
\]

where

\[C := A - B = \sum_{i,j,t=0}^{n} (x_{0,i+j-2t}^t - x_{i,j}^t) M_{i,j}^t \ \text{and} \ c := a - b = \sum_{i=0}^{n} (x_{0,0}^0 - x_{0,i}^0) \chi_{P_i(i)}.
\]

(Recall \(\mathcal{P}_i(V) = \{I \subseteq V \mid |I| = i\}.\) Thus \(\tilde{C}\) is of the form (16). For \(k = 0, 1, \ldots, |\frac{n}{2}|\), define the matrices:

\[A_k(x) := \left(\sum_{t=0}^{n} \left(\begin{pmatrix} n - 2k \\ i - k \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} n - 2k \\ j - k \end{pmatrix}^{-\frac{1}{2}} \beta_{i,j,k}^{\frac{1}{2}} x_{0,i+j-2t}^t \right)_{i,j}^{n-k}
\]

and \(B_k(x)\) as in (13), where \(\beta_{i,j,k}^{\frac{1}{2}}\) are as in (14). It follows from Lemma 6 that the positive semidefiniteness of \(Y(y)\) is equivalent to

\[(i) \quad B_k(x) \geq 0 \quad \text{for} \quad k = 0, 1, \ldots, |\frac{n}{2}|
\]

\[(ii) \quad A_k(x) - B_k(x) \geq 0 \quad \text{for} \quad k = 0, 1, \ldots, |\frac{n}{2}|
\]

\[(iii) \quad \left(1 - \frac{x_{0,0}^0}{\tilde{c}} \right) A_k(x) - B_k(x) \geq 0, \ \text{setting} \ \tilde{c} := \left(\frac{1}{\binom{n}{t}} \left(x_{0,0}^0 - x_{0,i}^0\right)\right)_{i=0}^{n}.
\]

(Of course, (39)(iii) implies (ii) for \(k = 0.\) Summarizing, we have shown:

\[\ell(G(n,d)) = \max 2^n x_{0,0}^0 \text{ s.t. } x_{i,j}^t (i, j, t = 0, \ldots, n) \text{ satisfy (35), (36), (39)(i) - (iii)}.
\]

Similarly,

\[\ell_+^-(G(n,d)) = \max 2^n x_{0,0}^0 \text{ s.t. } x_{i,j}^t (i, j, t = 0, \ldots, n) \text{ satisfy (35), (36), (37), (39)(i) - (iii)}.
\]

Hence both parameters can be computed via a semidefinite program of size \(O(n^3).\)

3.3.2. *Comparison with Schrijver’s bound*. Schrijver [13] introduced the following upper bound for the stability number \(A(n,d)\) of the graph \(G(n,d):

\[\ell_{sch}(G(n,d)) := \max \sum_{i=0}^{n} \binom{n}{i} x_{0,i}^0 \text{ s.t. } x_{i,j}^t (i, j, t = 0, \ldots, n) \text{ satisfy (35), (36), (37), (39)(i) - (iii)} \text{ and } x_{0,0}^0 = 1.
\]

As noted in [13], Schrijver’s bound is at least as good as the Delsarte bound, which coincides with \(\vartheta'(G(n,d)) = \ell_+^-(G(n,d)).\) We now show:

**Lemma 10.** The bound \(\ell_+^-(G(n,d))\) is at least as good as Schrijver’s bound \(\ell_{sch}(G(n,d))\) from (42); that is, \(\ell_+(G(n,d)) \leq \ell_{sch}(G(n,d)).\)
Proof. Let \((x^t_{i,j})_{i,j=0}^n\) be feasible for the program (41). Define \(y_{i,j}^t := x^t_{i,j}/x^0_{0,0}\) for all \(i,j,t = 0,\ldots,n\). Then the variables \(y_{i,j}^t\) satisfy (35), (36), (37), (39) (i)-(ii), and \(y_{0,0}^0 = 1\). Remains to verify that \(2^n x^0_{0,0} \leq \sum_{i=0}^n \binom{n}{i} y_{i,0}^0\), i.e., \(2^n (x^0_{0,0})^2 \leq \sum_{i=0}^n \binom{n}{i} x^0_{i,0}\). For this, recall that the conditions (39) (i)-(iii) are equivalent to the positive semidefiniteness of the matrix in (31). In particular, they imply
\[
\begin{pmatrix} 1 & a^T \\ a & A \end{pmatrix} \succeq 0, \quad \text{i.e.,} \quad A - aa^T \succeq 0,
\]
where \(A\) is as in (32), \(a^T = (x^0_{0,0},\ldots,x^0_{0,0})\), \(x_k = x^0_{0,k}\) for \(k = 0,\ldots,n\). Thus, \(aa^T = (x^0_{0,0})^2 J\), where \(J\) is the all-ones matrix. As \(A - (x^0_{0,0})^2 J \succeq 0\), we deduce that \(\langle J, A \rangle \geq (x^0_{0,0})^2 (J, J) = (x^0_{0,0})^2 n^2\). But \(\langle J, A \rangle = \sum_{k=0}^n x_k \langle J, M_k \rangle = \sum_{k=0}^n x_k 2^n \binom{n}{k}\), which gives \(\sum_{k=0}^n x^0_{0,k} \binom{n}{k} \geq 2^n (x^0_{0,0})^2\).

3.3.3. Refining the bound \(\ell_+ (\mathcal{G}(n,d))\). It is possible to define a new bound \(\ell_{++}(\mathcal{G}(n,d))\), at least as good as the bound \(\ell_+ (\mathcal{G}(n,d))\), whose computation still involves a semidefinite program of size \(O(n^3)\). Namely, let us now consider as matrix variable the principal submatrix \(Y (y)\) of \(M_2 (y)\) indexed by the set
\[
\mathcal{X}_2 := \{\emptyset\} \cup \{\{I\} \mid I \in \mathcal{P}\} \cup \{\emptyset, \{I\} \mid I \in \mathcal{P}\} \cup \{\{I, V\} \mid I \in \mathcal{P}\}.
\]
Then, \(Y (y)\) has the block structure:
\[
Y (y) = \begin{pmatrix} 1 & a^T & b^T & c^T \\ a & A & B & C \\ b & B & B & D \\ c & C & D & C \end{pmatrix}
\]
where \(A = (y(I,J))_{I,J \in \mathcal{P}}, B = (y(I,J,I,J))_{I,J \in \mathcal{P}}, C = (y(I,J,V))_{I,J \in \mathcal{P}}, D = (y(I,V,I,V))_{I,J \in \mathcal{P}}, a = (y(I,I))_{I \in \mathcal{P}}, b = (y(I,I,I))_{I \in \mathcal{P}},\) and \(c = (y(I,V))_{I \in \mathcal{P}}\). The matrices \(A, B\) are given by (32), (33). The matrix \(C\) is a permutation of \(B\); namely,
\[
C = \sum_{i,j=0}^n x_{n-i,n-j}^0 M_{i,j}^t.
\]
The matrix \(D\) too belongs to the Terwilliger algebra:
\[
D = \sum_{i,j=0}^n z_{i,j}^t M_{i,j}^t
\]
satisfying \(z_{i,j}^t = z^t_{j,i};\) indeed, \(D_{I,J} = D_{J,I}\) if there exists \(\sigma \in G\) such that \(\sigma(\emptyset) = \emptyset,\) \(\sigma(I) = I', \sigma(J) = J'\) (then \(\sigma(V) = V\)), i.e., if \(|I| = |I'|, |J| = |J'|, |I \cap J| = |I' \cap J'|\). We have the following relations for the variables \(x_{i,j}^t, z_{i,j}^t:\n\]
\[
z_{i,j}^t = x_{n-i,n-j}^0 z_{n-i,n-j}^t \quad \text{for all} \quad i,j,t = 0,\ldots,n
\]
since \(D_{I,J} = y(\emptyset,V,I,J) = y(\emptyset,V,V \Delta I,V \Delta J) = D_{V \Delta I,V \Delta J}\), and
\[
z_{i,j}^t = z_{0,i}^0 = x_{i,0}^0 \quad \text{for} \quad i = 0,\ldots,n
\]
since } y_{0,V,I} = D_{I,I} = D_{0,I} = D_{V,I} = B_{V,I}. {\text{The edge condition for the } z\text{-variables reads:}}
(47) \quad z_{i,j}^t = 0 \quad \text{if } \{i, j, n-i, n-j, i+j-2t\} \cap \{1, \ldots, d-1\} \neq \emptyset \quad \text{for } i, j, t = 0, \ldots, n. 

The bounds (21) imply:
(48) \quad 0 \leq z_{i,j}^t \leq x_{i,j}^t, \quad z_{i,j}^t \leq z_{i,i}^t \quad \text{for } i, j, t = 0, \ldots, n.

As each non-border block of the matrix } Y(y) \text{ in (44) belongs to the Terwilliger algebra, one can block-diagonalize } Y(y). \text{ Indeed, each non-border block in the matrix}
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & U^T & 0 & 0 \\
0 & 0 & U^T & 0 \\
0 & 0 & 0 & U
\end{pmatrix} Y(y) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & U & 0 & 0 \\
0 & 0 & U & 0 \\
0 & 0 & 0 & U
\end{pmatrix} = \begin{pmatrix}
a^T U & b^T U & c^T U \\
U a & U^T A U & U T B U & U T D U \\
U b & U T B U & U T B U & U T D U \\
U c & U T C U & U T D U & U T C U
\end{pmatrix}
\]
is block-diagonal with respect to the same partition, with } \lceil \frac{n}{2} \rceil + 1 \text{ distinct blocks labeled by } k = 0, 1, \ldots, \lceil \frac{n}{2} \rceil. \text{ It follows from Lemma 6 that } a^T U = (\tilde{a}^T, 0, \ldots, 0), \quad b^T U = (\tilde{b}^T, 0, \ldots, 0), \quad c^T U = (\tilde{c}^T, 0, \ldots, 0), \text{ where } \tilde{a} = x_{0,0}^0 \sum_{i=0}^{n-1} \ell_{i}^{\mathcal{P}_{0,0}}(V), \quad \tilde{b} = \sum_{i=0}^{n-1} x_{0,i}^0 \ell_{i}^{\mathcal{P}_{0,i}}(V) \text{ and } \tilde{c} = \sum_{j=0}^{n} x_{0,0}^0 \ell_{j}^{\mathcal{P}_{0,0}}(V) \text{ are indexed by the positions corresponding to the 0-th block.}

Therefore, } Y(y) \succeq 0 \text{ if and only if}
(49) \quad \begin{pmatrix}
1 & \tilde{a}^T & \tilde{b}^T & \tilde{c}^T \\
\tilde{a} & A_0 & B_0 & C_0 \\
\tilde{b} & B_0 & D_0 & D_0 \\
\tilde{c} & C_0 & D_0 & C_0
\end{pmatrix} \succeq 0, \quad \begin{pmatrix}
A_k & B_k & C_k \\
B_k & B_k & D_k \\
C_k & D_k & C_k
\end{pmatrix} \succeq 0 \quad \text{for } k = 1, \ldots, \lceil \frac{n}{2} \rceil 
\]
where } A_k = A_k(x) \text{ is as in (38), } B_k = B_k(x) \text{ is as in (13) and}

\[
C_k = \left( \sum_{i \in \mathcal{I}} \binom{n-2k}{i-k} \int_{j}^{n-2k} \beta_{i,j,k}^{t} z_{i,j}^{n+i+j-2t} \right)^{n-k}_{i,j=k},
\]

\[
D_k = \left( \sum_{i \in \mathcal{I}} \binom{n-2k}{i-k} \int_{j}^{n-2k} \beta_{i,j,k}^{t} z_{i,j}^{n+i+j-2t} \right)^{n-k}_{i,j=k}.
\]

One can now define the bound
(50) \quad \ell_{++}(\mathcal{G}(n,d)) := \max 2^n x_{0,0}^0 \quad \text{s.t. } \quad x_{i,j}^t, \quad z_{i,j}^t \quad (i, j, t = 0, \ldots, n) \text{ satisfy} \\
(35), (36), (37), (45), (46), (47), (48), (49).

Obviously,

\[
A(n,d) \leq \ell_{++}(\mathcal{G}(n,d)) \leq \ell_{+}(\mathcal{G}(n,d)) \leq \ell_{+ch}(\mathcal{G}(n,d)),
\]
and the bound } \ell_{++}(\mathcal{G}(n,d)) \text{ is again expressed via a semidefinite program of size } O(n^3).

Summarizing, the parameters } \ell_{ch}, \ell_{+}, \ell_{++} \text{ can all be seen as variations of the Lasserre bound } \ell^{(2)}. \text{ Namely, instead of considering the full matrix variable } M_2(y) \text{ indexed by the set } \mathcal{P}_2(\mathcal{P}), \text{ one considers a principal submatrix of } M_2(y) \text{ indexed by a subset of } \mathcal{P}_2(\mathcal{P}); \text{ namely, by the set } \mathcal{X} \setminus \{0\} \text{ for } \ell_{ch}, \text{ by the set } \mathcal{X} \text{ for } \ell_{+}, \text{ and by the set } \mathcal{X}_+ = \mathcal{X} \cup \{\{I, V\} | I \in \mathcal{P}\} \text{ for } \ell_{++}. \text{ (Recall the set } \mathcal{X} \text{ in (29).)
3.3.4. Reducing the number of variables. The following observation from [13] can be used for reducing the number of variables in the programs (40), (41), (42), (50), and for further refining the corresponding bounds. A well known fact in coding theory is that, if \( d \) is odd then \( A(n,d) = A(n+1,d+1) \), and if \( d \) is even then \( A(n,d) \) is attained by a code with all code words having an even Hamming weight. Therefore, it suffices to compute \( A(n,d) \) for \( d \) even. Moreover, for \( d \) even, \( A(n,d) = \alpha(G_{ev}(n,d)) \), the stability number of the graph \( G_{ev}(n,d) \), defined as the subgraph of \( G(n,d) \) induced by the set

\[
P_{ev} := \{ I \subseteq V \mid |I| \text{ is even} \}.
\]

Therefore, for \( d \) even, one may add the constraints:

\[
y_A = 0 \quad \text{if } A \not\in P_{ev}
\]

for any \( A \in P_{2k}(P) \) to the program (22) defining \( \ell_k(G(n,d)) \), or for any \( A \in P_{3}(P) \) to the program (23) defining \( \ell(G(n,d)) \). Equivalently, one may add the constraints:

\[
x_{i,j}^t = 0 \quad \text{if one of } i \text{ or } j \text{ is odd,}
\]

to the programs (40), (41), (42), (50), as well as as the constraints:

\[
z_{i,j}^t = 0 \quad \text{if one of } i, j, \text{ or } n \text{ is odd}
\]

to (50), and the new programs still define upper bounds for \( A(n,d) \). Namely, define:

\[
\ell_0(G(n,d)) := \max 2^n x_{0,0}^0 \quad \text{s.t. } x_{i,j}^t \ (i,j,t = 0,\ldots,n) \text{ satisfy}
\]

\[
(35), (36), (39)(i) - (iii), (52)
\]

and let \( \ell^0_\pm \), (resp., \( \ell^0_{\text{ech}} \), \( \ell^0_{\text{+++}} \)) be defined analogously by adding (52) (resp., (52), (52)-\( (53) \)) to (41) (resp., (42), (50)).

As \( A(n,d) = \alpha(G_{ev}(n,d)) \), one may also bound \( A(n,d) \) by the parameter \( \ell(G_{ev}(n,d)) \) (and analogously by \( \ell_\pm(G_{ev}(n,d)) \), \( \ell_{\text{ech}}(G_{ev}(n,d)) \)). The subgroup \( G_{ev} := \{ \pi_{SA} \mid A \in P_{ev} \} \) of the group \( G \) (introduced in (10)) acts vertex-transitively on \( P_{ev} \). Hence, applying Lemma 9, \( \ell(G_{ev}(n,d)) \) can be formulated via the analogue of (30), where \( Y(y) \) in (31) is now indexed only by even sets; that is, \( a, b, A \) and \( B \) in (31) are indexed by \( P_{ev} \). Again, \( A \) belongs to the Bose-Mesner algebra and \( B \) belongs to the Terwilliger algebra; that is, for some scalars \( x_k \), \( x_{i,j}^t \), \( A \) (resp., \( B \)) is equal to the principal submatrix of \( \sum_k \text{ even} x_k M_k \) (resp., of \( \sum_{i,j,t \text{ even}} x_{i,j}^t M_{i,j}^t \)) indexed by \( P_{ev} \). Therefore, \( \ell(G_{ev}(n,d)) \) can be computed via the program:

\[
\ell(G_{ev}(n,d)) = \max 2^{n-1} x_{0,0}^0 \quad \text{s.t. } x_{i,j}^t \ (i,j,t = 0,\ldots,n) \text{ satisfy}
\]

\[
(35), (36), (39)(i) - (iii), (52)
\]

where, in (39), we consider only the ‘even half’ of the matrices \( A_k(x) \), \( B_k(x) \), i.e., their principal submatrices indexed by even indices \( i,j \).

**Lemma 11.** \( A(n,d) \leq \ell(G_{ev}(n,d)) \leq \ell_0(G(n,d)) \leq \ell(G(n,d)) \) and analogously for the parameters \( \ell_+, \ell_{\text{ech}}, \ell_{+++} \).

**Proof.** The right and left most inequalities are obvious. To compare the parameters \( \ell(G_{ev}(n,d)) \) and \( \ell_0(G(n,d)) \), it is easiest to use their formulation via (23); for the formulation of \( \ell_0(G(n,d)) \), one should add to (23) the constraint (51) for any \( A \in P_3(P) \). Consider a feasible solution \( y \) for the program (23) defining \( \ell(G_{ev}(n,d)) \). Thus \( y \) is indexed
by $\mathcal{P}_0(\mathcal{P}_{ev})$, $y_{I,J} = 0$ if $|I\Delta J| = 1, \ldots, d - 1$ (for $I, J \in \mathcal{P}_{ev}$) and, for any $I \in \mathcal{P}_{ev}$, the matrix $Y_I(y)$ (indexed by $\mathcal{P}_0(\mathcal{P}_{ev}; I)$) is positive semidefinite. We define a feasible solution $z$ for the program defining $\ell^0(G(n,d))$ in the following way: For $A \in \mathcal{P}_0(\mathcal{P})$, set $z_A := y_A$ if $A \subseteq \mathcal{P}_{ev}$, and $z_A := 0$ otherwise. It is easy to verify that, for each $I \in \mathcal{P}$, the matrix $Y_I(z)$ (indexed by $\mathcal{P}_0(\mathcal{P}; I)$) is positive semidefinite. Thus, $\ell^0(G(n,d)) \geq \sum_{I \in \mathcal{P}} z_I = \sum_{I \in \mathcal{P}_{ev}} y_I$, implying $\ell^0(G(n,d)) \geq \ell(G_{ev}(n,d))$. The reasoning is analogous for the other parameters.

The bound $\ell(G_{ev}(n,d))$ is more economical to compute than $\ell^0(G(n,d))$, since it involves smaller matrices; as a matter of fact, the bound computed by Schrijver [13] is the bound $\ell_{sch}(G_{ev}(n,d))$. For $n$ odd, in view of (53), all variables $z_{i,j}$ can be set to 0 for the computation of $\ell_{+,}(G(n,d))$; from this follows that $\ell_{+}(G_{ev}(n,d)) = \ell_{+,}(G_{ev}(n,d))$ when $n$ is odd.

### 3.3.5. Some computational results.

We have tested the various bounds on several instances $(n,d)$, in particular, on those where Schrijver’s bound gave a improvement on the previously best known upper bound for $A(n,d)$. There are two instances: $(20,8)$ and $(25,6)$, for which we could find an upper bound for $A(n,d)$ (slightly) better than Schrijver’s bound; namely, $[\ell_{+}(G_{ev}(25,6))]$ and $[\ell_{+,}(G_{ev}(20,8))]$ improve the upper bound given by Schrijver by one. See Table 1 below (the values given there are the bounds rounded down to the nearest integer). For other instances $(n,d)$, the bounds $\ell_{+}$ and $\ell_{+,}$ give an improvement over Schrijver’s bound limited to some decimals, thus yielding no improved upper bound on $A(n,d)$. Our computations were made using the NEOS Server for Optimization, which can be accessed at http://www-neos.mcs.anl.gov/, and we used specifically the software DSDP for semidefinite programming.

We indicate in Table 2 the sizes of the semidefinite programs involved in our computations. (In the ‘block sizes’ column in Table 2, $-N$ indicates that the last block is a diagonal matrix of order $N$.)

<table>
<thead>
<tr>
<th>$(n,d)$</th>
<th>Delsarte bound</th>
<th>Schrijver bound $\ell_{sch}(G_{ev}(n,d))$</th>
<th>$\ell_{+}(G_{ev}(n,d))$</th>
<th>$\ell_{+,}(G_{ev}(n,d))$</th>
<th>$\ell^0_{+}(G(n,d))$</th>
<th>$\ell^0_{+,}(G(n,d))$</th>
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</thead>
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<tr>
<td>(20,8)</td>
<td>290</td>
<td>274</td>
<td>274</td>
<td>273</td>
<td>274</td>
<td>273</td>
</tr>
<tr>
<td>(25,6)</td>
<td>48148</td>
<td>47998</td>
<td>47997</td>
<td>47997</td>
<td>47998</td>
<td>47998</td>
</tr>
</tbody>
</table>

**Table 1**

<table>
<thead>
<tr>
<th>bound</th>
<th># var.</th>
<th># blocks</th>
<th>block sizes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell_{+}(G_{ev}(25,6))$</td>
<td>131</td>
<td>127</td>
<td>13 14 12 11 10 9 8 8 7 6 5 4 4 3 3 2 1 1-436</td>
</tr>
<tr>
<td>$\ell_{+}(G_{ev}(20,8))$</td>
<td>43</td>
<td>23</td>
<td>11 12 9 9 9 7 7 7 5 5 5 3 3 3 1 1 1 1-128</td>
</tr>
<tr>
<td>$\ell_{+,}(G_{ev}(20,8))$</td>
<td>68</td>
<td>12</td>
<td>34 27 27 21 21 15 15 9 9 9 3 3 3-221</td>
</tr>
</tbody>
</table>

**Table 2:** # var. means ‘number of variables’, # blocks means ‘number of blocks’
De Klerk and Pasechnik [1] have recently applied the bound of Schrijver [13] and our bound $\ell_+$ for finding tighter upper bounds for the stability number of the orthogonality graph $\Omega(n)$; $\Omega(n)$ is the graph with node set $\mathcal{P}$, with an edge $(I,J)$ if $|I \Delta J| = n/2$ (for $I, J \in \mathcal{P}$). Namely, to obtain an upper bound for the stability number of $\Omega(n)$, they propose to use the program (42) defining Schrijver’s bound, or the program (41) defining the parameter $\ell_+$, replacing the constraint (36) by the constraint:

$$x_{i,j}^t = 0 \text{ if } \{i,j,i+j-2t\} \cap \{n/2\} \neq \emptyset.$$  

The only interesting case is when $n$ is a multiple of 4, since $\Omega(n)$ is the empty graph for $n$ odd and $\Omega(n)$ is a bipartite graph for $n = 2 \mod 4$. The computations made by de Klerk and Pasechnik [1], quoted in Table 3 below, indicate that the bound $\ell_+(\Omega(n))$ may give a much better upper bound for $\alpha(\Omega(n))$ than Schrijver’s method. This contrasts with the situation encountered in the present paper, where the bound $\ell_+$ gave only a moderate improvement upon Schrijver’s bound for the instances of the coding problem we have tested.

<table>
<thead>
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<th>$n$</th>
<th>$\ell_+(\Omega(n))$</th>
<th>Schrijver’s bound</th>
</tr>
</thead>
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<tr>
<td>16</td>
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<td>2304</td>
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<td>21,103,609</td>
<td>21,723,404</td>
</tr>
</tbody>
</table>

Table 3: [1] Comparing the bounds for the orthogonality graph $\Omega(n)$

**References**


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