On the Sparsity Order of a Graph and its Deficiency in Chordality

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ABSTRACT

Given a graph G on n nodes, let $\mathcal{P}_G$ denote the cone consisting of the positive semidefinite $n \times n$ matrices (with real or complex entries) having a zero entry at every off-diagonal position corresponding to a non edge of G. Then, the sparsity order of G is defined as the maximum rank of a matrix lying on an extreme ray of the cone $\mathcal{P}_G$.

It is known that the graphs with sparsity order 1 are the chordal graphs and a characterization of the graphs with sparsity order 2 is conjectured in [AHMR88] in the real case. We show in this paper the validity of this conjecture. Moreover, we characterize the graphs with sparsity order 2 in the complex case and we give a decomposition result for the graphs with sparsity order ≤ 2 in both real and complex cases. As an application, these graphs can be recognized in polynomial time.

We also indicate how an inequality from [SM94] relating the sparsity order of a graph and its minimum fill-in can be derived from a result concerning the dimension of the faces of the cone $\mathcal{P}_G$.

1991 Mathematics Subject Classification: 05C50, 05C75, 15A48, 15A57.
Keywords and Phrases: positive semidefinite matrix, chordal graph, forbidden induced subgraph, extreme ray of cone, sparsity order.

1 Introduction

In this paper we study the ranks of extremal positive semidefinite matrices with a given sparsity pattern, in continuation of the papers [AHMR88], [HPR89], [McC88, McC93], [GP90], [HLW94], [SM94]. This study is motivated mainly by its application to the completion problem for positive semidefinite matrices (details are given below) and it is also relevant to chordal graphs and Gaussian elimination for sparse positive definite matrices ([Ro70]).

The sparsity order of a graph. Let $G = (V, E)$ be a graph with node set $V = \{1, 2, \ldots, n\}$ and let $\overline{E}$ denote the set of non edges; that is, $\overline{E}$ is the set of pairs ij with $i \neq j \in V$ and $ij \notin E$. Let $\mathcal{P}_G$ denote the set of positive semidefinite $n \times n$ matrices whose ij-th entries are zero for all $ij \in \overline{E}$. All matrices are assumed to have entries in the field $\mathbb{F}$, where $\mathbb{F}$ is equal to $\mathbb{R}$ (the field of real numbers) or

$\mathbb{C}$ (the field of complex numbers). The set $\mathcal{P}_G$ is a closed convex cone. A matrix $X \in \mathcal{P}_G$ is said to be *extremal* if it lies on an extreme ray of the cone $\mathcal{P}_G$. The *sparsity order* of G (also abbreviated as the *order* of G) $\mathrm{ord}_{\mathbb{F}}(G)$ is defined as the maximum rank of an extremal matrix in $\mathcal{P}_G$. We have: $\mathrm{ord}_{\mathbb{F}}(G) \leq n-2$, with equality in the real case when $n \geq 4$ if and only if G is a circuit [HPR89]. When G is the complete graph, $\mathcal{P}_G$ is the cone of all positive semidefinite matrices, every extremal matrix has rank ≤ 1 and, thus, $\mathrm{ord}_{\mathbb{F}}(G) = 1$. The sparsity order of a complete bipartite graph is computed in [GP90] and the graphs with sparsity order 1 are characterized in [AHMR88] (cf. also [PPS89]). Namely,

Theorem 1. *For a graph G, we have: $\mathrm{ord}_{\mathbb{R}}(G) = 1 \iff \mathrm{ord}_{\mathbb{C}}(G) = 1 \iff G$ is chordal, i.e., G does not contain any circuit of length ≥ 4 as an induced subgraph.* ∎

The following two operations are useful for computing the sparsity order. If H is an induced subgraph of G, then

$$\mathrm{ord}_{\mathbb{F}}(H) \leq \mathrm{ord}_{\mathbb{F}}(G)$$

[AHMR88] and if G is the clique sum of two graphs G_1 and G_2, then

$$\mathrm{ord}_{\mathbb{F}}(G) = \max(\mathrm{ord}_{\mathbb{F}}(G_1), \mathrm{ord}_{\mathbb{F}}(G_2)) \tag{1}$$

[HPR89]. A graph G is called a *k-block* if G has order k and every proper induced subgraph of G has order $\leq k-1$. For instance, the circuit C_n is an $(n-2)$-block over the reals if $n \geq 4$ [AHMR88]. For any given k, the number of k-blocks is finite, because a k-block has at most k^2+k-2 (resp. k^2-1) nodes if $\mathbb{F} = \mathbb{R}$ (resp. $\mathbb{F} = \mathbb{C}$) [AHMR88]. In both the real and complex cases, K_1 is the only 1-block and C_4 is the only 2-block; in the real case, there are exactly sixteen 3-blocks: the graphs $A_1 - A_{10}$ and $B_1 - B_6$ shown in Figure 2 [AHMR88]. Helton et al. [HLW94] have classified the 4-blocks over $\mathbb{R}$ having 9 non edges (by relation (5), this is the smallest number of non edges that a 4-block can have); their number is quite large and the classification involves many technical details. This indicates the difficulty of the general problem of classifying k-blocks.

In order to characterize the graphs having order $\leq k$, it suffices to know the minimal (with respect to taking induced subgraphs) graphs among the p-blocks with $p > k$ and this might be more tractable, at least for small values of k. For instance, it follows from Theorem 1 that every k-block ($k \geq 2$) contains a circuit of length ≥ 4 as an induced subgraph. The following is conjectured in [AHMR88] in the case $k = 2$ and $\mathbb{F} = \mathbb{R}$:

Conjecture 2. *A graph G satisfies: $\mathrm{ord}_{\mathbb{R}}(G) \leq 2$, if and only if G does not contain as an induced subgraph a circuit on $n \geq 5$ nodes nor a 3-block. Equivalently, over the reals, the only k-block ($k \geq 4$) which contains no 3-block is the* *circuit* C_{k+2}.

The main contribution of this paper is to show the validity of Conjecture 2 (cf. Theorem 9); we also prove an analogous characterization for the graphs having sparsity order ≤ 2 over $\mathbb{C}$ (cf. Theorem 13). The essential ingredient in our proof is a decomposition result (in terms of clique sums) for the class of graphs having no 3-block and no circuit of length ≥ 5 as an induced subgraph (cf. Theorem 8). Section 5 is devoted to the proof of this result which is quite technical.

As an application, one can recognize in polynomial time whether a graph has order 1 or 2. As another application, we can characterize the graphs whose 'powers' all have order ≤ 2 (cf. Theorems 11 and 15); in the complex case we derive a result of McCullough [McC88, McC93]. Moreover, we obtain the classification of the 3-blocks, which was not known in the complex case (cf. Corollary 14).

The *minimum fill-in* $\mathrm{fill}(G)$ of a graph G is the minimum number of edges that need to be added to G in order to obtain a chordal graph; this parameter has been studied, in particular, in connection with the Gaussian elimination process for real symmetric positive definite matrices (cf. Rose [Ro70]). The following inequalities relating the sparsity order and the minimum fill-in have been shown in [SM94], solving a conjecture of [HPR89].

Proposition 3. $\mathrm{ord}_{\mathbb{R}}(G) \leq \mathrm{fill}(G) + 1, \quad \mathrm{ord}_{\mathbb{C}}(G) \leq 2 \cdot \mathrm{fill}(G) + 1.$ ∎

We will see in Section 2 that these inequalities follow as an easy application of a result (cf. Theorem 6) about the dimension of faces of the cone $\mathcal{P}_G$.

Note that the difference between the minimum fill-in and the sparsity order can be arbitrarily large. Indeed, if G is the clique sum of two graphs G_1 and G_2, then $\mathrm{ord}_{\mathbb{F}}(G) = \max(\mathrm{ord}_{\mathbb{F}}(G_1), \mathrm{ord}_{\mathbb{F}}(G)_2)$ while $\mathrm{fill}(G) = \mathrm{fill}(G_1) + \mathrm{fill}(G_2)$. We will see in Section 3 examples of graphs (those in class $\mathcal{G}_4$ - they are not clique sums) having order 2 and an arbitrarily large minimum fill-in. The complexity of computing the order of a graph is not known. On the other hand, evaluating the upper bound given by Proposition 3 is hard, since computing the minimum fill-in is NP-complete ([Ya81]).

Application to the completion problem. Let us now explain the link existing between the cone $\mathcal{P}_G$ and the completion problem for positive semidefinite matrices.

The matrix completion problem asks whether a given partial matrix can be completed so as to obtain a matrix satisfying a prescribed matrix property, in our case, being positive semidefinite. This problem has received a lot of attention in the literature; this is due, in particular, to its many applications, e.g., to statistics, molecular chemistry, distance geometry, etc. (Cf. the surveys by Johnson [Jo90], Laurent [La97] and further references there.)

Given a graph $G = (V, E)$ and a partial matrix $A = (a_{ij})$ whose entries are specified only on the diagonal positions and on the off-diagonal positions corresponding to the edges of G, A is said to be *completable to a positive semidefinite matrix* if there exist a positive semidefinite matrix B such that $b_{ii} = a_{ii}$ $(i \in V)$ and $b_{ij} = a_{ij}$ $(i \neq j \in V$ and $ij \in E)$. Then, $\mathcal{C}_G$ denotes the subset of $\mathbb{R}^{V \cup E}$ consisting of all such completable partial matrices A. When G is the complete graph, $\mathcal{C}_G$ is the cone of all positive semidefinite matrices, which is well known to be self polar. From this follows that the polar cone[1] of $\mathcal{C}_G$ is equal to the projection on $\mathbb{R}^{V \cup E}$ of the cone $\mathcal{P}_G$.

Therefore, a partial matrix is completable to a positive semidefinite matrix if and only if its inner product with any extremal matrix in $\mathcal{P}_G$ is nonnegative. Hence, knowledge about the extremal matrices in $\mathcal{P}_G$ is useful for deciding completability of partial matrices. This fact motivates the study of extremal matrices in $\mathcal{P}_G$ and of the order of graph G. In view of the polarity relation between the cones $\mathcal{C}_G$ and $\mathcal{P}_G$, Theorem 1 is equivalent to the following result of [GJSW84] concerning the cone $\mathcal{C}_G$.

Theorem 4. *A graph G is chordal if and only if every partial matrix whose entries are specified on G (and on the main diagonal) and for which all fully specified principal submatrices are positive semidefinite can be completed to a positive semidefinite matrix.* ∎

We now give some preliminaries about matrices and vector representations, leading to a reformulation for the sparsity order, and some notation on graphs.

Matrices and vector representations. An $n \times n$ matrix $X = (x_{ij})$ with entries in $\mathbb{F}$ $(= \mathbb{R}$ or $\mathbb{C})$ is *Hermitian* if $X^* = X$ and *positive semidefinite* (then, we write: $X \succeq 0$) if X is Hermitian and $x^*Xx \geq 0$ for all $x \in \mathbb{F}^n$. Equivalently, $X \succeq 0$ if there exist vectors $u_1, \ldots, u_n \in \mathbb{F}^k$ $(k \geq 1)$ such that $x_{ij} = u_i^* u_j$ for all $i, j = 1, \ldots, n$; the sequence $(u_1, \ldots, u_n)$ is then called a *Gram representation* of X and X is called the *Gram matrix* of $u_1, \ldots, u_n$. Note that $\{u_1, \ldots, u_n\}$ and X have the same rank. If X has rank k, then it has a unique (up to orthogonal transformation) Gram representation in the k-dimensional space $\mathbb{F}^k$.

Let $G = (V, E)$ be a graph with $V = \{1, \ldots, n\}$ and let X be an $n \times n$ positive semidefinite matrix with Gram representation $(u_1, \ldots, u_n)$. Then, $X \in \mathcal{P}_G$ if and only if the vectors $u_1, \ldots, u_n$ satisfy:

(2) $\qquad u_i^* u_j = 0$ for all $ij \in \overline{E}$,

[1] Recall that the *polar* C° of a cone $C \subseteq \mathbb{F}^d$ is the set of all $y \in \mathbb{F}^d$ such that $y^*x \geq 0$ $\forall x \in C$. When the cone C consists of $n \times n$ matrices, we view C as a subset of $\mathbb{F}^{n^2}$ equipped with the usual inner product. That is, for two $n \times n$ matrices A and B, their inner product $\langle A, B \rangle$ is defined as $\mathrm{Tr}(A^*B) = \sum_{i,j=1}^n a_{ij}^* b_{ij}$. Here, z^*, a^*, A^* denote, respectively, the conjugate of $z \in \mathbb{F}$ (equal to z if $\mathbb{F} = \mathbb{R}$), and the conjugate transpose of vector a or matrix A.

i.e., they form an *orthogonal representation* of G. Given a subset $A \subseteq E \cup \overline{E}$, set

$$
\textbf{(3)} \qquad \begin{array}{ll} U_A := \{u_i u_j^* + u_j u_i^* \mid ij \in A\} & \text{if } \mathbb{F} = \mathbb{R}, \\ U_A := \{u_i u_j^*, u_j u_i^* \mid ij \in A\} & \text{if } \mathbb{F} = \mathbb{C}. \end{array}
$$

If $X \in \mathcal{P}_G$, then all matrices in $U_{\overline{E}}$ are orthogonal to the identity matrix. The following is shown in [AHMR88].

Proposition 5. *Let $X \in \mathcal{P}_G$ with rank k and Gram representation $(u_1, \ldots, u_n)$ in $\mathbb{F}^k$. Then, X is extremal if and only if the identity matrix is (up to scalar multiple) the only $k \times k$ real symmetric matrix (resp. complex matrix) which is orthogonal to all matrices in $U_{\overline{E}}$ in the real case (resp. complex case). Equivalently, X is extremal if and only if the following holds:*

$$
\textbf{(4)} \qquad \operatorname{rank}(U_{\overline{E}}) = \binom{k+1}{2} - 1 (\mathbb{F} = \mathbb{R}), \ k^2 - 1 (\mathbb{F} = \mathbb{C}).
$$

Therefore,

$$
\textbf{(5)} \qquad \operatorname{ord}_{\mathbb{F}}(G) = k \Longrightarrow |\overline{E}| \geq \binom{k+1}{2} - 1 \ (\mathbb{F} = \mathbb{R}), \frac{1}{2}(k^2 - 1) \ (\mathbb{F} = \mathbb{C}).
$$

A set of vectors $u_1, \ldots, u_n \in \mathbb{F}^k$ with rank k and satisfying (2) and (4) is called *a k-dimensional extremal orthogonal representation* of G. Hence, the sparsity order of G is equal to the largest k for which there exists a k-dimensional extremal orthogonal representation of G. Proposition 5 turns out to be a direct consequence of the result from Theorem 6 about dimensions of faces of $\mathcal{P}_G$.

Graphs. All graphs are assumed to be simple (i.e., without loops and parallel edges). Given a graph $G = (V, E)$ with $\overline{E}$ as set of non edges, $\overline{G} := (V, \overline{E})$ is the *complementary graph* of G. As usual, K_n denotes the complete graph on n nodes, $K_{n,m}$ denotes the complete bipartite graph with colour classes of cardinalities n and m, and C_n denotes the circuit on n nodes. Given a subset U of V, $G[U]$ denotes the *subgraph of G induced by U*, with node set U and edge set $\{ij \in E \mid i, j \in U\}$. A subset $S \subseteq V$ is called a *stable set* of G if $ij \notin E$ for all $i \neq j \in S$, and a *clique* if $ij \in E$ for all $i \neq j \in S$. The *stability number* $\alpha(G)$ is the maximum cardinality of a stable set in G. A subset $F \subseteq E$ is called a *matching* in G if no two edges of F have a common endnode.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that the set $K := V_1 \cap V_2$ is a clique in both G_1, G_2 and there is no edge between a node of $V_1 \setminus V_2$ and a node of $V_2 \setminus V_1$. Then, the graph $G := (V_1 \cup V_2, E_1 \cup E_2)$ is called the *clique sum* of the graphs G_1 and G_2 and the set K is called a *clique cutset* of G.

A graph G is said to be *chordal* (or *triangulated*) if it does not contain a circuit

C_n ($n \geq 4$) as an induced subgraph. Equivalently, G is chordal if and only if G is a clique sum of cliques (Dirac [Di61]).

Given an integer $m \geq 1$, let $G^{(m)}$ denote the graph obtained from G by replacing every node $v \in V$ by a clique K_v of cardinality m and making any two nodes $i \in K_u$, $j \in K_v$ adjacent in $G^{(m)}$ if and only if the nodes u and v are adjacent in G.

2 Relating the sparsity order of a graph and its deficiency in chordality

In this section we show that the results from Propositions 3 and 5 follow as applications of a result concerning the structure of the faces of the cone $\mathcal{P}_G$.

Let $G = (V, E)$ be a graph with $V = \{1, \ldots, n\}$. A subset $F \subseteq \mathcal{P}_G$ is called a *face* of $\mathcal{P}_G$ if $X = Y + Z$ with $X \in F$, $Y, Z \in \mathcal{P}_G$ implies that $Y, Z \in F$. The *extreme rays* of $\mathcal{P}_G$ are its faces of dimension 1. Given $X \in \mathcal{P}_G$, let $F_{\mathcal{P}_G}(X)$ denote the smallest (with respect to inclusion) face of $\mathcal{P}_G$ that contains X. Then,

$$F_{\mathcal{P}_G}(X) = \{Y \in \mathcal{P}_G \mid \mathrm{Ker}X \subseteq \mathrm{Ker}Y\}$$

(where $\mathrm{Ker}X = \{x \in \mathbb{R}^n \mid Xx = 0\}$); this relation was shown in [HW87] in the case when $G = K_n$ and the general case follows easily. Moreover, one can compute the dimension of the face $F_{\mathcal{P}_G}(X)$ in terms of parameters of X (cf. Theorem 31.5.3 in [DL97]).

Theorem 6. *Let $G = (V, E)$ be a graph, let $X \in \mathcal{P}_G$ with rank k and Gram representation $(u_1, \ldots, u_n)$ in $\mathbb{F}^k$, and let $U_{\overline{E}}$ be defined by* (3)*. Then,*

$$\textbf{(6)} \qquad \dim F_{\mathcal{P}_G}(X) = \binom{k+1}{2} - \mathrm{rank}_{\mathbb{R}}(U_{\overline{E}}) \ (\mathbb{F} = \mathbb{R}), \ \ k^2 - \mathrm{rank}_{\mathbb{C}}(U_{\overline{E}}) \ (\mathbb{F} = \mathbb{C}).$$

In particular, X is extremal if and only if (4) *holds.*

Proof. Let U denote the $k \times n$ matrix whose columns are the vectors $u_1, \ldots, u_n$. Then, $X = U^*U$. Call a $k \times k$ matrix B a *perturbation* of X if $X \pm \lambda B \in \mathcal{P}_G$ for some $\lambda > 0$. Then, dim $F_{\mathcal{P}_G}(X)$ is equal to the dimension of the set of perturbations of X. One can verify that B is a perturbation of X if and only if $B = U^*RU$ for some $k \times k$ Hermitian matrix R satisfying

$$\langle R, u_i u_j^* \rangle = \langle R, u_j u_i^* \rangle = 0 \ \text{ for all } ij \in \overline{E}.$$

Let $\mathcal{U}$ denote the subspace of $\mathbb{F}^{k^2}$ (the set of $k \times k$ matrices) spanned by $U_{\overline{E}}$. In the real case, dim $F_{\mathcal{P}_G}(X)$ is equal to the dimension of the orthogonal complement of $\mathcal{U}$ in the space $\mathcal{S}_k$ of real symmetric matrices and, thus, to $\binom{k+1}{2} - \mathrm{rank}_{\mathbb{R}}(U_{\overline{E}})$. (We have used here the fact that, for a symmetric matrix B, $\langle B, u_i u_j^* \rangle = 0 \Longleftrightarrow$

$\langle B, u_i u_j^* + u_j u_i^* \rangle = 0$.) In the complex case, dim $F_{\mathcal{P}_G}(X)$ is equal to the dimension of the set $\mathcal{H}_k \cap \mathcal{U}^\perp$, the orthogonal complement of $\mathcal{U}$ in the space $\mathcal{H}_k$ of $k \times k$ complex Hermitian matrices. However, the set $\mathcal{H}_k \cap \mathcal{U}^\perp$ has the same dimension as its superset $\mathbb{C}^{k^2} \cap \mathcal{U}^\perp$, which implies that dim $F_{\mathcal{P}_G}(X) = k^2 - \mathrm{rank}_{\mathbb{C}}(U_{\overline{E}})$. Indeed, suppose that $\{R_1, \ldots, R_p\}$ is a set of linearly independent matrices in $\mathbb{C}^{k^2} \cap \mathcal{U}^\perp$. Note that for $R \in \mathbb{C}^{k^2} \cap \mathcal{U}^\perp$, both matrices $R + R^*$ and $i(R - R^*)$ belong to $\mathcal{H}_k \cap \mathcal{U}^\perp$. Moreover, at least one of the two systems $\{R_1 + R_1^*, R_2, \ldots, R_p\}$ and $\{i(R_1 - R_1^*), R_2, \ldots, R_p\}$ is linearly independent. Therefore, we can iteratively construct from $\{R_1, \ldots, R_p\}$ a set of p linearly independent matrices in $\mathcal{H}_k \cap \mathcal{U}^\perp$. ∎

We now indicate how Proposition 3 can be derived from Theorem 6.

Let $G = (V, E)$ be a graph and set $k := \mathrm{ord}_{\mathbb{F}}(G)$ and $p := \mathrm{fill}(G)$. There exists a subset F of $\overline{E}$ of cardinality p such that the graph $H := (V, E \cup F)$ is chordal. Let X be an extremal matrix in $\mathcal{P}_G$ of rank k and with Gram representation $(u_1, \ldots, u_n)$ in $\mathbb{F}^k$; thus, $X \in \mathcal{P}_H$. Set $\rho_{\mathbb{R}} := \binom{k+1}{2}$ and $\rho_{\mathbb{C}} := k^2$. By relation (6), we have:

$$\mathrm{rank}_{\mathbb{F}}(U_{\overline{E}}) = \rho_{\mathbb{F}} - \dim F_{\mathcal{P}_G}(X) = \rho_{\mathbb{F}} - 1,$$

$$\mathrm{rank}_{\mathbb{F}}(U_{\overline{E}\setminus F}) = \rho_{\mathbb{F}} - \dim F_{\mathcal{P}_H}(X).$$

On the other hand,

$$\mathrm{rank}_{\mathbb{F}}(U_{\overline{E}}) \leq \mathrm{rank}_{\mathbb{F}}(U_{\overline{E}\setminus F}) + \mathrm{rank}_{\mathbb{F}}(U_F) \leq \mathrm{rank}_{\mathbb{F}}(U_{\overline{E}\setminus F}) + \epsilon_{\mathbb{F}} \cdot |F|,$$

setting $\epsilon_{\mathbb{R}} = 1$ and $\epsilon_{\mathbb{C}} := 2$. This implies that

$$\dim F_{\mathcal{P}_H}(X) \leq \epsilon_{\mathbb{F}} \cdot |F| + 1 = \epsilon_{\mathbb{F}} \cdot p + 1.$$

There exist $d \leq \dim F_{\mathcal{P}_H}(X)$ extremal matrices $X_1, \ldots, X_d \in \mathcal{P}_H$ such that $X = X_1 + \ldots + X_d$. This implies that $\mathrm{rank}X \leq \mathrm{rank}X_1 + \ldots + \mathrm{rank}X_d$. Each matrix X_i has rank 1 since H is chordal. Therefore, $\mathrm{rank}X \leq d$ which, combined with the inequality: $d \leq \epsilon_{\mathbb{F}} \cdot p + 1$, implies that $k = \mathrm{rank}X \leq \epsilon_{\mathbb{F}} \cdot p + 1$. That is, $\mathrm{ord}_{\mathbb{F}}(G) \leq \epsilon_{\mathbb{F}} \cdot \mathrm{fill}(G) + 1$.

3 Graphs with sparsity order 2: The real case

In this section, we characterize the graphs having sparsity order ≤ 2 in the real case. The main result is Theorem 9 which gives two equivalent descriptions for these graphs; one is in terms of forbidden induced subgraphs and the other one shows how such graphs can be decomposed by means of clique sums using four basic classes of graphs.

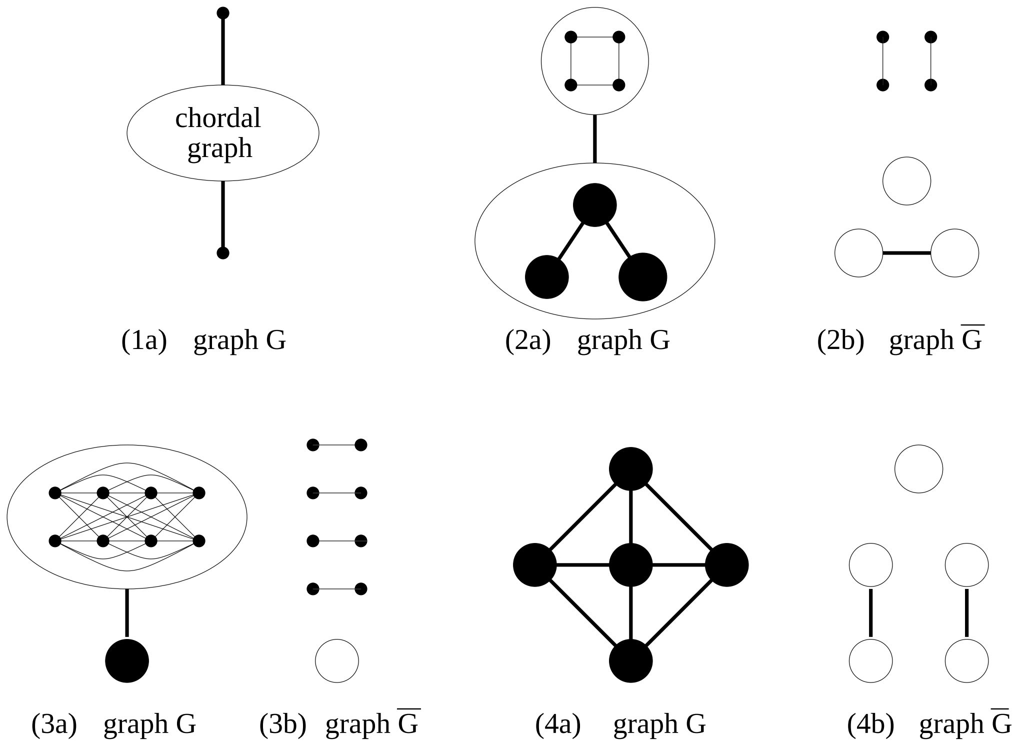

Figure 1: Classes $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{G}_4$

We begin with introducing four classes of graphs having order ≤ 2 over $\mathbb{R}$. These graphs are shown in Figure 1. For $i = 1, 2, 3, 4$, let $\mathcal{G}_i$ denote the class consisting of the graphs having the form shown in Figure 1 (ia) and of their induced subgraphs. We use the following convention in Figure 1: A small dark dot indicates a node, a big dark sphere indicates a clique, while a big white sphere indicates a stable set; edges are indicated by lines, while a thick line between two spheres or between two sets of nodes shows that every node in one set is adjacent to every node in the other set.

For each class $\mathcal{G}_i$ with $i = 2, 3, 4$, we picture not only graph $G \in \mathcal{G}_i$ but also its complementary graph $\overline{G}$, because the latter graph has a very simple form which will be used in the proof of Proposition 7. Note that a graph $G \in \mathcal{G}_1$ is obtained by adding two non adjacent nodes to a chordal graph H and making them adjacent to all nodes of H (and taking an induced subgraph of the resulting graph).

Remark that a graph in class $\mathcal{G}_i$ has minimum fill-in at most i for $i = 1, 2, 3$ while graphs in $\mathcal{G}_4$ may have an arbitrary large minimum fill-in.

Proposition 7. *If* $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$, *then* $\mathrm{ord}_{\mathbb{R}}(G) \leq 2$.

Proof. If $G \in \mathcal{G}_1$, then $\mathrm{ord}_{\mathbb{R}}(G) \leq 2$ follows from Proposition 3, since $\mathrm{fill}(G) \leq 1$. Let $G \in \mathcal{G}_i$ for $i = 2, 3, 4$. Let X be an extremal matrix in the cone $\mathcal{P}_G$ having rank $k := \mathrm{ord}_{\mathbb{R}}(G)$ and with Gram representation $(u_1, \ldots, u_n)$ in $\mathbb{R}^k$. Then, by Theorem 6,

$$\mathrm{rank}(U_{\overline{E}}) = \frac{1}{2}(k^2 + k - 2).$$

We compute in each case the rank of the set $U_{\overline{E}}$. Consider first the case when $G \in \mathcal{G}_2$. Let A, B denote the node sets corresponding to the two stable sets that are connected in $\overline{G}$ (cf. Figure 1 (2b)) and set $a := \mathrm{rank}\{u_i \mid i \in A\}$, $b := \mathrm{rank}\{u_i \mid i \in B\}$. Then,

$$\mathrm{rank}(U_{\overline{E}}) \leq 2 + ab.$$

We have that $a + b \leq k$ since every u_i $(i \in A)$ is orthogonal to every u_j $(j \in B)$; this implies that $ab \leq \frac{1}{4}k^2$. Therefore, we have:

$$\frac{1}{2}(k^2 + k - 2) \leq 2 + \frac{1}{4}k^2,$$

from which it follows that $k \leq 2$. If $G \in \mathcal{G}_3$, then we have:

$$\frac{1}{2}(k^2 + k - 2) \leq 4,$$

implying again that $k \leq 2$. Finally, if $G \in \mathcal{G}_4$, then we obtain in the same way as above that:

$$\frac{1}{2}(k^2 + k - 2) \leq \frac{1}{4}k^2 + \frac{1}{4}k^2,$$

which also implies that $k \leq 2$. ∎

We now characterize the graphs having order ≤ 2 over $\mathbb{R}$. The result relies essentially on a graph-theoretic result concerning the characterization in terms of forbidden induced subgraphs of the graphs in the classes $\mathcal{G}_i$ $(i = 1, 2, 3, 4)$ and their clique sums. We first formulate this graph-theoretic result whose proof, in view of its length, is delayed till Section 5.

Theorem 8. *The following assertions are equivalent for a graph G.*

(i) *G does not contain as an induced subgraph a circuit C_n $(n \geq 5)$ nor any of the graphs A_2-A_{10} and B_1-B_6 (cf. Figure 2).*

(ii) *G is a clique sum of a set of graphs belonging to $\bigcup_{i=1}^{4} \mathcal{G}_i$ (cf. Figure 1).*

We show in Figure 2 the complementary graphs of a list of sixteen graphs $A_1 - A_{10}$, $B_1 - B_6$. Note that A_1, B_2, A_2, B_4, B_5 are, respectively, the circuit C_5, the complete bipartite graph $K_{3,3}$, $K_{3,3} + e$ (add an edge to $K_{3,3}$), $K_{3,3} \backslash f$ (delete an edge from $K_{3,3}$), $K_{3,3} + e \backslash f$.

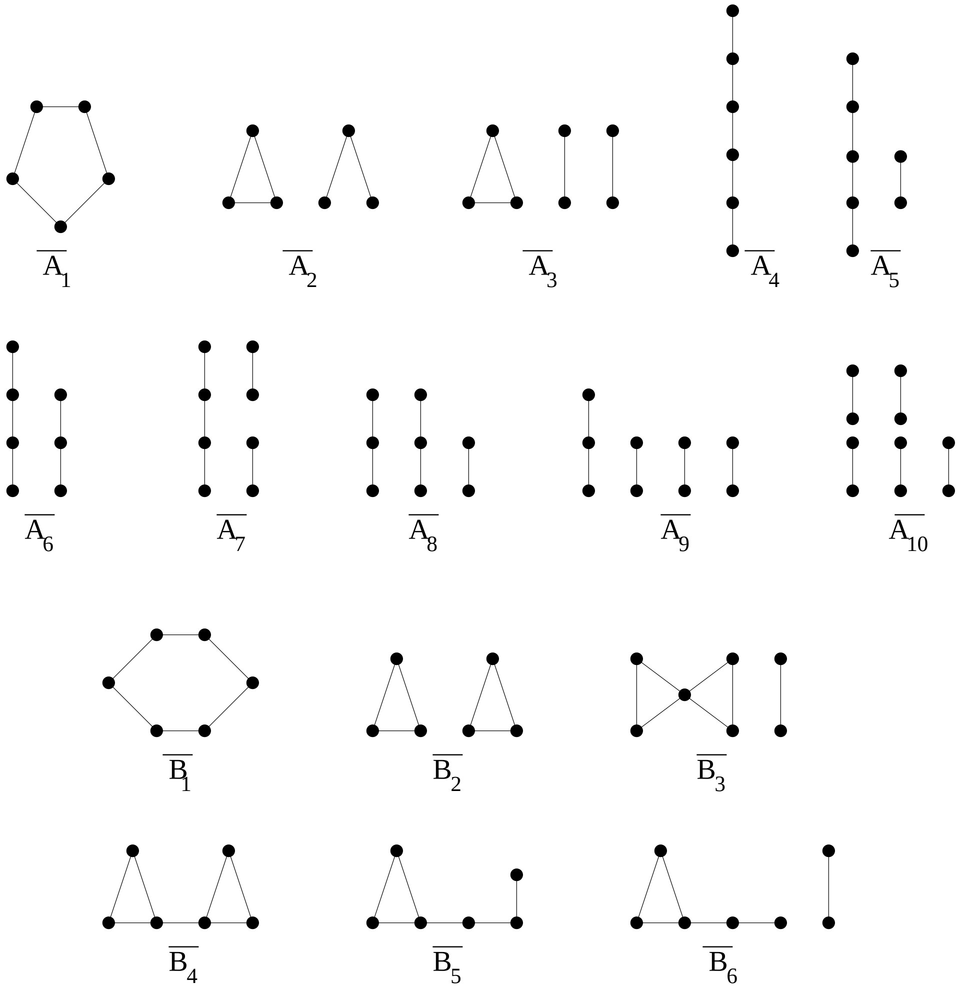

Figure 2: Complements of graphs $A_1 - A_{10}$, $B_1 - B_6$ (the 3-blocks over $\mathbb{R}$)

Theorem 9. *The following assertions are equivalent for a graph G.*

(i) $\mathrm{ord}_{\mathbb{R}}(G) \leq 2$.

(ii) *G does not contain as an induced subgraph a circuit C_n ($n \geq 5$) nor any of the graphs A_2-A_{10} and B_1-B_6 (cf. Figure 2).*

(iii) *G is a clique sum of a set of graphs belonging to $\bigcup_{i=1}^{4} \mathcal{G}_i$ (cf. Figure 1).*

Proof. The implication (i) $\Longrightarrow$ (ii) follows from the fact that the graphs C_n ($n \geq 5$), $A_1 - A_{10}$, $B_1 - B_6$ all have order ≥ 3 (for this, it suffices to exhibit for each of them a 3-dimensional extremal orthogonal representation; cf. [AHMR88]).

The implication (ii) $\Longrightarrow$ (iii) holds by Theorem 8, while (iii) $\Longrightarrow$ (i) follows from relation (1) and Proposition 7. ∎

The result from Theorem 9 can be seen as an analogue and generalization of the corresponding characterization for graphs of order 1, which states that the following assertions are equivalent for a graph G: (i) $\mathrm{ord}_{\mathbb{R}}(G) = 1$; (ii) G is chordal; (iii) G can be decomposed as a clique sum of cliques. We now mention some applications of Theorem 9. The first application is that one can test in polynomial time whether a given graph G has order ≤ 2 over the reals. Indeed, it suffices for this to first (i) decompose G into graphs without clique cutsets by means of clique sums and then to (ii) test whether all the graphs produced by step (i) belong to $\bigcup_{i=1}^{4} \mathcal{G}_i$. Step (i) can be performed in time $O(nm)$ if G has n nodes and m edges [Ta85] and step (ii) can obviously be executed in polynomial time.

The second application is the classification of the 3-blocks over the reals, which was obtained by Agler et al. [AHMR88]. The only fact from [AHMR88] that we have used concerning the graphs $A_1 - A_{10}$, $B_1 - B_6$ is that they have order ≥ 3 (which easily implies that they are 3-blocks). But, we obtain 'for free' the hard part, which consists of showing that $A_1 - A_{10}$, $B_1 - B_6$ are the *only* 3-blocks.

Theorem 10. *In the real case, the 3-blocks are the graphs $A_1 - A_{10}$ and $B_1 - B_6$.*

PROOF. If G is a 3-block then, by Theorem 9, G must contain one of the graphs $A_1 - A_{10}$, $B_1 - B_6$ as an induced subgraph and, thus, G is equal to it (by the definition of a block). ∎

The third application of Theorem 9 is the characterization of the graphs G whose powers $G^{(m)}$ all have order ≤ 2 over $\mathbb{R}$.

Theorem 11. *The following assertions are equivalent for a graph G.*

(o) $\mathrm{ord}_{\mathbb{R}}(G^{(m)}) \leq 2$ *for every integer* $m \geq 1$.

(i) $\mathrm{ord}_{\mathbb{R}}(G^{(2)}) \leq 2$.

(ii) *G does not contain as an induced subgraph a circuit C_n ($n \geq 5$) nor any of the graphs A_4, B_1, D_1, D_2, D_3 (cf. Figures 2 and 3).*

(iii) *G is a clique sum of a set of graphs belonging to the class $\mathcal{G}_4$ (cf. Figure 1).*

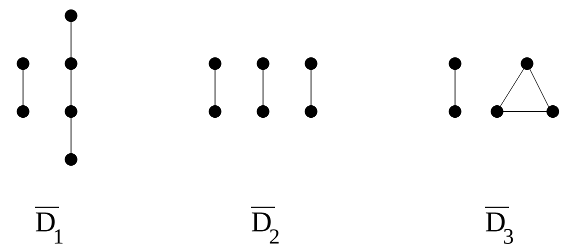

Figure 3: Complements of graphs D_1, D_2, and D_3

The next lemma will be used in the proof of Theorem 11 and later as well.

Lemma 12.

(i) *Let H be a chordal graph that does not contain an induced path of length 3 and with stability number $\alpha(H) = 2$. Then, its node set can be partitioned into $V_0 \cup V_1 \cup V_2$ in such a way that $V_0 \cup V_1$ and $V_0 \cup V_2$ are cliques and there is no edge between V_1 and V_2.*

(ii) *If $G \in \mathcal{G}_1$ contains neither D_1 nor D_3 as an induced subgraph, then $G \in \mathcal{G}_4$.*

Proof. (i) As H is chordal and is not a clique, there exists a clique cutset K in H. Hence, the node set V_H of H can be partitioned into $N_1 \cup N_2 \cup K$, in such a way that there is no edge between N_1 and N_2. Moreover, both N_1, N_2 are cliques (since $\alpha(H) = 2$). For $a = 1, 2$, set

$$K_a := \{k \in K \mid ik \not\in E \ \text{ for some } i \in N_a\}.$$

Then, $K_1 \neq \emptyset \Longrightarrow K_2 = \emptyset$, a node $k \in K_1$ is not adjacent to any node of N_1 (else, one would find a path of length 3 in H), and $k \in K_1$ is adjacent to all nodes in N_2 (since $\alpha(H) = 2$). Therefore, we can assume that $K_2 = \emptyset$ and, then,

$$V_H = N_1 \cup (K \setminus K_1) \cup (K_1 \cup N_2),$$

where the sets $N_1 \cup (K \setminus K_1)$ and $(K \setminus K_1) \cup K_1 \cup N_2$ are cliques and there is no edge between the sets N_1 and $K_1 \cup N_2$. Thus, (i) holds (setting $V_0 := K \setminus K_1$, $V_1 := N_1$, $V_2 := K_1 \cup N_2$).
(ii) Let $G \in \mathcal{G}_1$ and let H denote the chordal part in G (cf. Figure 1 (1a)). If G contains neither D_1 nor D_3 as an induced subgraph, then H does not contain an induced path of length 3 and $\alpha(H) \leq 2$. We may assume that H is not a clique (else we are done). Using (i) we conclude that G has indeed the form of a graph in $\mathcal{G}_4$. ∎

Proof of Theorem 11. (o) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (o) are obvious. The implication (i) $\Longrightarrow$ (ii) follows from the corresponding implication in Theorem 9; indeed, the graphs $H := D_1, D_2, D_3$ are forbidden as induced subgraphs of G since $H^{(2)}$ contains A_6, A_8, A_2, respectively. Similarly, (ii) $\Longrightarrow$ (iii) follows from the corresponding implication in Theorem 9. Indeed, each of the graphs $A_1 - A_{10}$, $B_1 - B_6$ contains one of A_4, B_1, D_1, D_2, D_3 as an induced subgraph. Hence, under assumption (ii), we know that G is a clique sum of a family of graphs belonging to $\bigcup_{i=1}^{4} \mathcal{G}_i$. In order to conclude the proof, it suffices now to verify that a graph $G \in \bigcup_{i=1}^{3} \mathcal{G}_i$ not containing D_1, D_2, D_3 necessarily belongs to $\mathcal{G}_4$. This is obvious for the classes $\mathcal{G}_2$ and $\mathcal{G}_3$ and Lemma 12 (ii) settles the case when $G \in \mathcal{G}_1$. ∎

4 Graphs with sparsity order 2: The complex case

In this section we characterize the graphs having sparsity order ≤ 2 over $\mathbb{C}$. As in the real case, we begin with exhibiting some basic classes of graphs having order ≤ 2 over $\mathbb{C}$ as well as some examples of graphs with order ≥ 3. Let $\mathcal{G}_5$ denote the class consisting of the graphs that can be obtained from a complete graph by deleting a matching of cardinality at most 3. It follows from (5) that $\text{ord}_{\mathbb{C}}(G) \leq 2$ if $G \in \mathcal{G}_5$ and McCullough (Prop. 2.6, [McC88]) has shown that $\text{ord}_{\mathbb{C}}(G) \leq 2$ if $G \in \mathcal{G}_4$.

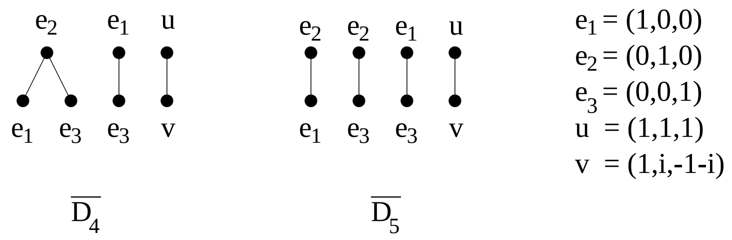

Figure 4: Complements of graphs D_4 and D_5

It is shown in [McC88] that $\text{ord}_{\mathbb{C}}(G) \geq 3$ if G is one of the graphs C_n ($n \geq 5$), A_4, B_1 (cf. Figure 2), D_1, D_3 (cf. Figure 3). We observe that the graphs D_4 and D_5 whose complements are shown in Figure 4 also have order ≥ 3. (The vector assignments indicated there provide a 3-dimensional extremal orthogonal representation.) In fact, the graphs $A_4, B_1, D_1, D_3, D_4, D_5$ all have order equal to 3 (by (5)). Therefore, the graphs C_n ($n \geq 5$), $A_4, B_1, D_1, D_3, D_4, D_5$ are forbidden induced subgraphs for the class of graphs having order ≤ 2 over $\mathbb{C}$. As we now see, there are no other minimal forbidden induced subgraphs. The proof of this result relies again on a decomposition result, which follows quite easily from Theorem 8.

Theorem 13. *The following assertions are equivalent for a graph G.*

(i) $\text{ord}_{\mathbb{C}}(G) \leq 2$.

(ii) *G does not contain as an induced subgraph any of the graphs C_n ($n \geq 5$), A_4, B_1, D_1, D_3, D_4, and D_5.*

(iii) *G is a clique sum of a set of graphs belonging to $\mathcal{G}_4 \cup \mathcal{G}_5$.*

Proof. The implications (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (i) are clear. We now verify the implication (ii) $\Longrightarrow$ (iii). For this, let G be a graph satisfying Theorem 13 (ii). Then, G satisfies the condition (ii) from Theorem 9 (since the graphs A_2, A_3, B_2-B_6 all contain D_3; while A_5, A_6, A_7 contain D_1; A_8, A_9 contain D_4; and finally A_{10} contains D_5). Therefore, by Theorem 9, G is a clique sum of a set of graphs belonging to $\bigcup_{i=1}^{4} \mathcal{G}_i$. In order to conclude the proof, it suffices to verify that a graph $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ satisfying Theorem 13 (ii) belongs, in fact, to $\mathcal{G}_5$. This is easy to see when $G \in \mathcal{G}_2 \cup \mathcal{G}_3$ and Lemma 12 (ii) settles the case when $G \in \mathcal{G}_1$. ∎

Therefore, the graphs having order ≤ 2 over $\mathbb{C}$ can be recognized in polynomial time. Another application of Theorem 13 is the classification of the 3-blocks over $\mathbb{C}$.

Corollary 14. *The 3-blocks over the field $\mathbb{C}$ are the graphs C_5, B_1, A_4, D_1, D_3, D_4, and D_5.* ∎

As last application of Theorem 13, we obtain the following result of McCullough [McC88, McC93] characterizing the graphs G whose powers $G^{(m)}$ all have order ≤ 2 over $\mathbb{C}$. It turns out that we find the same graphs as in the real case.

Theorem 15. *The following assertions are equivalent for a graph G.*

(o) $\mathrm{ord}_{\mathbb{C}}(G^{(m)}) \leq 2$ *for every integer $m \geq 1$.*

(i) $\mathrm{ord}_{\mathbb{C}}(G^{(2)}) \leq 2$.

(ii) *G does not contain as an induced subgraph a circuit C_n ($n \geq 5$) nor any of the graphs A_4, B_1, D_1, D_2, D_3 (cf. Figures 2 and 3).*

(iii) *G is a clique sum of a set of graphs belonging to the class $\mathcal{G}_4$ (cf. Figure 1).*

Proof. The implications (o) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (o) are clear. The implication (i) $\Longrightarrow$ (ii) follows from Theorem 13 and the fact that $D_2^{(2)}$ contains D_4. We now verify the implication (ii) $\Longrightarrow$ (iii). For this, let G satisfy Theorem 15 (ii); then, G satisfies Theorem 13 (ii) and, thus, is a clique sum of a set of graphs belonging to $\mathcal{G}_4 \cup \mathcal{G}_5$. It suffices now to note that a graph belonging to $\mathcal{G}_5$ and satisfying Theorem 15 (ii) belongs, in fact, to $\mathcal{G}_4$. ∎

McCullough [McC88, McC93] has given an additional equivalent property for the graphs satisfying Theorem 15 (ii) (called by him *2-chordal*), in terms of existence of a certain linear ordering of the nodes involving the notion of 'simplicial pair of nodes'; this is in analogy with the existence a perfect elimination ordering for chordal graphs, that involve the notion of simplicial node. Note that the 'hard' part in his proof lies also in proving the decomposition result via clique sums (the original proof given in [McC88] for this decomposition result was not correct; it was later corrected in [McC93]).

5 Proof of Theorem 8

This section is devoted to the proof of Theorem 8. The implication (ii) $\Longrightarrow$ (i) follows from the fact that the graphs C_n $(n \geq 5)$, $A_2 - A_{10}$, $B_1 - B_6$ have no clique cutset and that they cannot occur as an induced subgraph of a graph in $\mathcal{G}_i$ $(i = 1, 2, 3, 4)$.

We now turn to the proof of the reverse implication: (i) $\Longrightarrow$ (ii). The starting point of our proof was inspired by the proof given in [Sc94] for the following result of Dirac [Di61]: Every chordal graph G which is not a clique has a clique cutset. The latter result can be shown in the following manner.

Let G be a chordal graph which is not a clique. Then, there exists a node u which is not adjacent to all nodes in V. Let $S \subseteq V$ be a maximal subset of V containing u such that $G[S]$ is connected and the set

$$N := \{i \in V \setminus S \mid i \text{ is adjacent to some node in } S\}$$

is strictly contained in $V \setminus S$. Setting $\overline{N} := V \setminus (S \cup N)$, we have partitioned V into

$$V = S \cup N \cup \overline{N},$$

where $G[S]$ is connected, $\overline{N} \neq \emptyset$, and there is no edge between the sets $\overline{N}$ and S. It follows from the maximality assumption on S that

(1) every node of N is adjacent to every node of $\overline{N}$.

Moreover, N is a clique and, thus, a clique cutset in G (cf. the proof of Claim 3).

We now return to the proof of Theorem 8 (i) $\Longrightarrow$ (ii). For this, we let G be a graph satisfying condition (i) (i.e., G does not contain C_n $(n \geq 5)$, $A_2 - A_{10}$, $B_1 - B_6$ as an induced subgraph) and we assume that G has no clique cutset. We show that G belongs to one of the classes $\mathcal{G}_i$, $i = 1, 2, 3, 4$. We can assume that G is not a clique (else, we are done). In the same manner as above, one can partition the node set V into

$$V = S \cup N \cup \overline{N},$$

where $S \neq \emptyset$, $G[S]$ is connected, $\overline{N} \neq \emptyset$, there is no edge between the sets $\overline{N}$ and S, N is not a clique, and relation (1) holds. The rest of the proof consists in a detailed analysis of the structure of the sets S, N and $\overline{N}$, so that one can finally reach the conclusion that G has indeed the form of a graph in $\bigcup_{i=1}^{4} \mathcal{G}_i$.

5.1 Preliminary results and sketch of proof

We group here a number of preliminary results on the structure of G which will lead to several distinct cases that we have to consider. In what follows, we let $\overline{n}$ denote a given element of $\overline{N}$. For $s \in S$, we set

$$N(s) := \{i \in N \mid is \in E\}.$$

Claim 2. *If st is an edge in S such that $N(s) \cup N(t)$ is not a clique, then $N(s) \subseteq N(t)$ or $N(t) \subseteq N(s)$.*

Proof. Assume that st is an edge in S and that $N(s) \setminus N(t)$, $N(t) \setminus N(s)$ are both non empty; we show that $N(s) \cup N(t)$ is a clique. For this, let $i \in N(s) \setminus N(t)$ and $j \in N(t) \setminus N(s)$; then, $ij \in E$ (else, $(\overline{n}, i, s, t, j)$ would be an induced C_5). Let i' be another node in $N(s) \setminus N(t)$; then, $ii' \in E$ (else, we find B_5 on $\{s, t, i, i', j, \overline{n}\}$). Let $k \in N(s) \cap N(t)$; then, $ki \in E$ (else, we find A_4 or B_1 on $\{\overline{n}, s, t, i, j, k\}$ depending whether $kj \in E$). Finally, if $k, k' \in N(s) \cap N(t)$, then $kk' \in E$ (else, we find A_5 on $\{s, t, k, k', i, j, \overline{n}\}$). ∎

Claim 3. *If $i, j \in N$ are two non adjacent nodes in N, then there exists $s \in S$ which is adjacent to both i and j.*

Proof. Let $i, j \in N$ be non adjacent and assume that no $s \in S$ is adjacent to both i, j. There exist $s, t \in S$ such that $si, tj \in E$ and $sj, ti \notin E$. Consider a shortest path P in S from s to t. Then, this path P together with the edges $si, i\overline{n}, \overline{n}j, jt$ yields a circuit of length ≥ 5 in G, a contradiction. ∎

As a consequence of Claim 3, we obtain that

(4) $G[\overline{N}]$ is chordal.

Indeed, suppose that $G[\overline{N}]$ contains an induced C_4; let $i, j \in N$ be nonadjacent, and let $s \in S$ be adjacent to i and j. Then, we find B_3 on the nodes of C_4 and i, j, s, a contradiction.

Claim 5. *Let $I \subseteq N$ be a stable set in N of cardinality $|I| \geq 3$. Then, there exists a unique node $s \in S$ which is adjacent to all nodes in I.*

Proof. We proceed by induction on $|I| \geq 3$. Suppose first that $|I| = 3$, $I = \{i, j, k\}$. If there exists no node in S adjacent to i, j, k then, by Claims 3 and 2, there exist pairwise nonadjacent nodes $r, s, t \in S$ such that $ri, rj, si, sk, tj, tk \in E$ and $rk, sj, ti \notin E$; this gives an induced circuit C_6 in G, a contradiction. Now, if s, t are two distinct nodes in S adjacent to i, j and k, then we find A_2 or B_2 on $\{i, j, k, s, t, \overline{n}\}$ depending whether s, t are adjacent or not. Hence, the result holds when $|I| = 3$. Suppose now that $I = \{i_1, \ldots, i_p\}$ with $p \geq 4$ and that no node of S is adjacent to all elements of I. By the induction assumption, we may assume that, for every $j = 1, \ldots, p$, there exists $s_j \in S$ adjacent to all nodes in $I \setminus \{i_j\}$; then, the subgraph of G induced by nodes $\overline{n}, s_1, s_p, i_{p-2}, i_{p-1}, i_p$ is B_4. Hence, there exists $s \in S$ adjacent to all nodes in I; uniqueness follows from the case $|I| = 3$. ∎

Claim 6. *Let $i, j, k \in N$ be distinct nodes such that $G[\{i, j, k\}]$ has exactly one edge. Then, there exists a node $s \in S$ which is adjacent to i, j, and k.*

Proof. Suppose the claim does not hold. Say, $ij \in E$ and $ik, jk \notin E$. Then, there exist $s, t \in S$ such that s is adjacent to i, k but not to j and t is adjacent to j, k but not to i. Then, $st \in E$ (else, we find C_5 on s, t, i, j, k) and we find B_1 on $\overline{n}, s, t, i, j, k$. ∎

Claim 7. *Let $I, J \subseteq N$ be distinct maximal stable sets in N. If $I \cap J \neq \emptyset$, then any node $s \in S$ which is adjacent to all elements in I is adjacent to all elements in J.*

Proof. Suppose not. Let $s \in S$ be adjacent to all elements in I and let $j \in J \setminus I$ such that $sj \notin E$. By maximality of I, there exists $i \in I$ such that $ij \in E$. Let $k \in I \cap J$ and let $t \in S$ be adjacent to i, j, and k (which exists by Claim 6). Then, we find B_4 or B_5 on $\overline{n}, s, t, i, j, k$ (depending whether $st \in E$ or not). ∎

Claim 8. *Let (i, h, j, k) be an induced C_4 in N (i.e., $ih, jh, ik, jk \in E$, $ij, hk \notin E$). Then, any node $s \in S$ which is adjacent to i and j is adjacent to h and k. Moreover, every node $x \in N \setminus \{i, j, h, k\}$ is adjacent to at least three nodes in $\{i, j, h, k\}$.*

Proof. Let $s \in S$ be adjacent to i and j and suppose that s is not adjacent, say, to k. Let $t \in S$ be adjacent to h and k. Suppose in a first step that $sh \notin E$. If t is adjacent to both i, j, then we find B_3 or B_6 on the nodes $\overline{n}, s, t, i, j, h, k$ (depending whether s, t are adjacent); if t is adjacent to one of i, j, then we find A_2 or B_5 and, if t is not adjacent to i, j, then we find B_2 or B_4 on the nodes s, t, i, j, h, k. Therefore, we have that $sh \in E$ and, similarly, $ti \in E$. Then, we find A_5 or B_6 on $\{\overline{n}, s, t, i, j, h, k\}$ when $tj \in E$ and we find A_4 when $tj \notin E$ (on $\{s, t, i, j, h, k\}$ if $st \notin E$ and on $\{\overline{n}, s, t, i, j, k\}$ if $st \in E$).

We now prove the second assertion of the claim. For this, consider $x \in N \setminus \{i, j, h, k\}$ such that $xi \notin E$. Let $s \in S$ which is adjacent to x, i, j (which exists by Claim 6); then s is adjacent to h and k. Hence, the subgraph of G induced by

$\{\overline{n}, s, x, i, j, h, k\}$ is A_3, A_5, B_3, or B_6 if one of the edges xj, xh, xk is missing. ∎

Corollary 9. *If $\overline{N}$ is not a clique, then $G[N]$ is chordal and there are at least two edges among any three nodes in N.*

PROOF. Let $\overline{n}_1, \overline{n}_2$ be two non adjacent nodes in $\overline{N}$. Suppose first that (i, h, j, k) is an induced C_4 in N. Let $s \in S$ be adjacent to i, j, h, k (which exists by Claims 3 and 8); thus, we find A_3 on $\{\overline{n}_1, \overline{n}_2, s, i, j, h, k\}$. This shows that $G[N]$ is chordal. Suppose now that i, j, k are distinct nodes in N having at most one edge among them. Then, there exists $s \in S$ adjacent to i, j, k (by Claims 5 and 6); thus, we find B_2 or A_2 on $\{\overline{n}_1, \overline{n}_2, s, i, j, k\}$. ∎

Let ν denote the largest cardinality of an induced matching in $\overline{G}[N]$, the complementary graph of $G[N]$. Then, $\nu \geq 1$ since N is not a clique and

(10) $\quad \nu \leq 3.$

Indeed, suppose that $\nu \geq 4$ and let $\{i_a j_a \mid a = 1, 2, 3, 4\}$ be an induced matching in $\overline{G}[N]$. By Claims 3 and 8, there exists a node $s \in S$ which is adjacent to all nodes i_a, j_a, $a = 1, \ldots, 4$. Then, we find A_{10} on $\{\overline{n}, s, i_a, j_a \ (a = 1, 2, 3, 4)\}$.
Note that $\nu \geq 2$ if and only if $G[N]$ is not chordal.

We can now describe the overal structure of the proof. We will organize our discussion according to the value of the parameter $\nu = 1, 2, 3$ (by (10)). In the case when $\nu = 1$, i.e., when the graph $G[N]$ is chordal, it will be convenient to consider separately the two cases when $\alpha(G[N]) = 2$ and when $\alpha(G[N]) \geq 3$. To summarize, the proof will consist of examining the following disjoint cases:
Case A: $\nu = 1$ and $\alpha(G[N]) = 2$; then, we show that $G \in \mathcal{G}_1 \cup \mathcal{G}_4$.
Case B: $\nu = 1$ and $\alpha(G[N]) \geq 3$; then, we show that $G \in \mathcal{G}_1$.
Case C: $\nu \in \{2, 3\}$; then, we show that $G \in \mathcal{G}_\nu$.

5.2 Case A

We assume here that $G[N]$ is chordal with stability number $\alpha(G[N]) = 2$. Let $K \subseteq N$ be a clique cutset in $G[N]$. Then, N can be partitioned as

$$N = K \cup N_1 \cup N_2$$

where $N_1, N_2 \neq \emptyset$ and there is no edge between the sets N_1 and N_2. Moreover, both N_1 and N_2 are cliques (since $\alpha(G[N]) = 2$). We show that G belongs to $\mathcal{G}_1 \cup \mathcal{G}_4$. For convenience, we introduce the following sets:

$$S_1 := \{s \in S \mid s \text{ is adjacent to } N_1 \text{ but not to } N_2\},$$

$$S_2 := \{s \in S \mid s \text{ is adjacent to } N_2 \text{ but not to } N_1\},$$

$$S_{12} := \{s \in S \mid s \text{ is adjacent to } N_1 \text{ and } N_2\}, \quad S_0 := S \setminus (S_1 \cup S_2 \cup S_{12})$$

and, for $a = 1, 2$,

$$K_a := \{k \in K \mid ik \notin E \ \text{ for some } i \in N_a\} \text{ and } K_0 := K \setminus (K_1 \cup K_2).$$

Given a set $A \subseteq V$ and $u \in V \setminus A$, we say that u *is adjacent to* A if u is adjacent to some element in A. Moreover, a path connecting a node of S_1 to a node of S_2 whose set of internal nodes is contained in S_0 is called a *path from* S_1 *to* S_2 *via* S_0. We have:

$$K_1 \cap K_2 = \emptyset; N_1 \cup K_2 \text{ and } N_2 \cup K_1 \text{ are cliques,}$$

since $\alpha(G[N]) = 2$. Moreover,

$$S_{12} \neq \emptyset.$$

Indeed, given $i_1 \in N_1$, $i_2 \in N_2$, there exists (by Claim 3) a node $s \in S$ which is adjacent to i_1 and i_2; thus, $s \in S_{12}$. The following observation will be used repeatedly.

Claim 11. *There does not exist an induced path from* S_1 *to* S_2 *via* S_0*. Moreover, any induced path contained in* $S_0 \cup S_1 \cup S_2$ *is contained in* $S_0 \cup S_1$ *or in* $S_0 \cup S_2$*.*

Proof. Suppose that there exists an induced path $(s_1, u_1, \ldots, u_p, s_2)$ where $s_1 \in S_1$, $s_2 \in S_2$, and $u_1, \ldots, u_p \in S_0$ $(p \geq 0)$. Let $i_a \in N_a$ be adjacent to s_a, for $a = 1, 2$. Then, $(i_1, s_1, u_1, \ldots, u_p, s_2, i_2, \overline{n})$ is an induced circuit of length ≥ 5 in G, yielding a contradiction. The second assertion in the claim follows easily. ∎

By definition, every node of S_{12} is adjacent to at least one node in N_1 and in N_2. More strongly, we have:

(12) Every node $s \in S_{12}$ is adjacent to every node in $N_1 \cup N_2 \cup K_1 \cup K_2$.

This follows using Claim 7, since any two non adjacent nodes of N form a maximal stable set in $G[N]$. Indeed, let $s \in S_{12}$ and let $i_1 \in N_1$, $i_2 \in N_2$ be adjacent to s. Then, s is adjacent to every other node $j_1 \in N_1$ since $\{i_1, i_2\}$ and $\{j_1, i_2\}$ are two intersecting maximal stable sets. Moreover, if $k \in K_1$ is not adjacent to some $j_1 \in N_1$, then s is adjacent to k since $\{k, j_1\}$ is a maximal stable set meeting $\{j_1, i_2\}$.

Claim 13. *The graph* $G[S_{12} \cup K \cup \overline{N}]$ *is chordal.*

Proof. We already know that $G[K \cup \overline{N}]$ is chordal (using (4)); hence, a possible C_4 is necessarily contained in $K \cup S_{12}$ and has at least two nodes in S_{12}. Let $i_1 \in N_1$ and $i_2 \in N_2$. If (i, j, s, t) is an induced C_4 with $i, j \in K$ and $s, t \in S_{12}$, then $i, j \in K_0$ (by (12)) and we find A_5 on $\overline{n}, s, t, i, j, i_1, i_2$. In the case when (i, r, s, t) is an induced C_4 with $i \in K$ and $r, s, t \in S$, then $i \in K_0$ and we find B_6 on $\overline{n}, r, s, t, i, i_1, i_2$. Finally, if (r, s, t, u) is an induced C_4 contained in S_{12}, then we find B_3 on $\overline{n}, r, s, t, u, i_1, i_2$. ∎

Our next objective is to show that $S = S_{12}$, i.e., that the set $T := S_0 \cup S_1 \cup S_2$ is empty. Given $s \in T$, set

$$X_s := \{x \in S_{12} \cup N \mid sx \in E\}.$$

Claim 14. *X_s is a clique for every $s \in T$ and $X_s \cup X_t$ is a clique for every edge st in T.*

PROOF. Note first that, if x, y are two non adjacent nodes in $S_{12} \cup N$, then one of the following holds: either $x \in S_{12}$, $y \in S_{12} \cup K_0$; or $x \in K_1$, $y \in N_1$ (or the symmetric case: $x \in K_2$, $y \in N_2$); or $x \in N_1$, $y \in N_2$.

Suppose that X_s is not a clique for some $s \in T$ and let x, y be nonadjacent nodes in X_s. If $x \in S_{12}$, then $N(s) \subseteq N(x)$ (by Claim 2, since $N(x)$ is not a clique) and, thus, $y \notin N$ which, in view of the above observation, means that $y \in S_{12}$. But, then, we find B_4 or B_5 on $\{\overline{n}, s, x, y, i_1, i_2\}$ (where $i_1 \in N_1$, $i_2 \in N_2$). We cannot have: $x \in N_1, y \in N_2$; therefore, $x \in K_1$, $y \in N_1$ and, thus, $s \in S_1$. Then, s must be adjacent to any $i_2 \in N_2$ (by Claim 7, since $\{y, i_2\}$ is a maximal stable set in N meeting $\{x, y\}$), contradicting the fact that $s \in S_1$. Hence, X_s is a clique.

Suppose now that $X_s \cup X_t$ is not a clique for some edge st in T. Then, there exist two nonadjacent nodes x, y in $X_s \cup X_t$ with, say, $x \in X_s \setminus X_t$ and $y \in X_t \setminus X_s$. We can assume, e.g., that $s, t \in S_0 \cup S_1$. Again we cannot have: $x \in N_1, y \in N_2$. Hence, given $i_2 \in N_2$, (i_2, x, s, t, y) is an induced C_5 in the case when $x \in S_{12}$ and $y \in S_{12} \cup K_0$; in the case when $x \in K_1$, $y \in N_1$, then $(\overline{n}, x, s, t, y)$ is an induced C_5. ∎

If $T \neq \emptyset$, we let A denote a maximal subset of T for which $G[A]$ is connected and the set $X(A) := \bigcup_{a \in A} X_a$ is a clique.

Claim 15. *There is no edge between the sets A and $T \setminus A$.*

PROOF. Assume that $a \in A$ is adjacent to $b \in T \setminus A$. By maximality of A, we deduce that $X(A) \cup X_b$ is not a clique; hence, there exist two nonadjacent nodes $x \in X(A)$ and $y \in X_b$. Let $a_0 \in A$ be adjacent to x. As x, y are not adjacent, we deduce from Claim 14 that $a_0 \neq a$, $a_0 b, a_0 y, bx \notin E$. Let $(a_0, a_1, \ldots, a_p, a)$ be a shortest path connecting a_0 and a in A (possibly $p = 0$ if $a_0 a \in E$). Together with nodes x, y, this yields an induced path P of length ≥ 4 from x to y and whose internal nodes belong to T. Now, by Claim 11, we may assume that all internal nodes of P belong to $S_0 \cup S_1$. Therefore, the path P together with edges $i_2 x, i_2 y$

(resp. with edges $\overline{n}x, \overline{n}y$) yields an induced circuit of length ≥ 6 in G when $x \in S_{12}$ and $y \in S_{12} \cup K_0$ (resp. when $x \in K_1$ and $y \in N_1$). ∎

We can now deduce that

$$T = S_0 \cup S_1 \cup S_0 = \emptyset; \text{ that is}, S = S_{12}.$$

For, if $T \neq \emptyset$, then $A \neq \emptyset$ and $X(A)$ is a clique cutset in G. (To see it, note that there is no edge between A and the sets $T \setminus A$, $(S_{12} \cup N) \setminus X(A)$ and $\overline{N}$. Hence, if we delete the clique $X(A)$ in G, we obtain a graph in which A is disconnected from the rest of the graph.)

Corollary 16. *If $|N_1| = |N_2| = 1$ and $K_1 = K_2 = \emptyset$, then $G \in \mathcal{G}_1$.*

Proof. Indeed, under this assumption, we have that $N_1 = \{i_1\}$, $N_2 = \{i_2\}$ where both i_1 and i_2 are adjacent to all nodes in $V \setminus \{i_1, i_2\} = S \cup K \cup \overline{N}$. As $G[V \setminus \{i_1, i_2\}]$ is chordal (by Claim 13), we obtain that $G \in \mathcal{G}_1$. ∎

From now on, we can assume without loss of generality that the following holds:

(17) $\quad |N_1| \geq 2, \text{ or } K_1 \neq \emptyset.$

Then, $\overline{N}$ is a clique by Corollary 9. Moreover,

$$S \text{ is a clique and every node of } S \text{ is adjacent to every node of } N.$$

Indeed, it follows from assumption (17) that $\overline{G}[N]$ contains a path (i, j, k) of length 2 (choosing, either $i, k \in N_1$, $j \in N_2$; or $i \in K_1$, $j \in N_1$, $k \in N_2$ where $ij \notin E$). Therefore, if s, t are two non adjacent nodes of S, then we find A_2 on the set $\{i, j, k, s, t, \overline{n}\}$. Hence, S is a clique. Suppose now that $s \in S$ is not adjacent to some node $h \in N$; then, $h \in K_0$ by (12). Let $t \in S$ be adjacent to h. Then, we find A_6 on the set $\{\overline{n}, s, t, h, i, j, k\}$.

Hence, we know the following information about G: The sets S and $\overline{N}$ are cliques, every node of $S \cup \overline{N}$ is adjacent to every node of N, and $G[N]$ is chordal. This implies:

Corollary 18. *If $|S| = |\overline{N}| = 1$, then $G \in \mathcal{G}_1$.* ∎

Henceforth, we can now assume, moreover, that

(19) $\quad \max(|S|, |\overline{N}|) \geq 2.$

This implies that

$$G[N] \text{ does not contain an induced path of length 3.}$$

Indeed, $\overline{G}[S \cup \overline{N}]$ contains a path of length 2 by (19); hence, if $G[N]$ would contain an induced path of length 3, we would find graph A_6.

Therefore, by Lemma 12 (i), we know that N can be partitioned into

$$N = V_1 \cup V_0 \cup V_2$$

where $V_0 \cup V_1$ and $V_0 \cup V_2$ are cliques and there is no edge between V_1 and V_2. (Namely, $V_0 = K_0$, $V_1 = N_1$ and $V_2 = K_1 \cup N_2$.) We can now conclude that G belongs to class $\mathcal{G}_4$ (with the cliques $V_1, S, V_2, \overline{N}$ forming the outer circuit and V_0 as central clique). This concludes the proof in case A.

5.3 Case B

We assume here that $G[N]$ is chordal with stability number $\alpha(G[N]) \geq 3$. We show that $G \in \mathcal{G}_1$. For this, let I be a maximal stable set in N; $|I| \geq 3$. By Claim 5, there exists an element $s_I \in S$ which is adjacent to all nodes in I. A first easy observation is that

$$|\overline{N}| = 1.$$

Indeed, if $\overline{n}_1, \overline{n}_2 \in \overline{N}$ and $i, j, k \in I$, then we find A_2 or B_2 on $\{\overline{n}_1, \overline{n}_2, s_I, i, j, k\}$. Next, we observe that

$$s_I \text{ is adjacent to every node of } N.$$

Indeed, let $h \in N \setminus I$. If $hi \notin E$ for some $i \in I$, then s_I is adjacent to h by Claim 7. If $hi \in E$ for all $i \in I$, then $s_I h \in E$ for, if not, we find A_2 on $\{\overline{n}, s_I, h, i, j, k\}$ (where $i, j, k \in I$). At this point, we can already conclude that

(20) if $S = \{s_I\}$, then $G \in \mathcal{G}_1$.

Indeed, s_I and $\overline{n}$ are both adjacent to all elements of N and $G[N]$ is chordal. We now show that the set $T := S \setminus \{s_I\}$ is empty. For this, note first that

(21) every node $s \in T$ is adjacent to at most one node of I.

Indeed, we know from Claim 5 that s is adjacent to at most two nodes of I. If $s \in T$ is adjacent to $i, j \in I$ and if $k \in I \setminus \{i, j\}$, then we find B_4 or B_5 on $\{\overline{n}, s, s_I, i, j, k\}$. For $s \in T$, set

$$Y_s := \{x \in N \mid sx \in E\}.$$

Claim 22. *For every $s \in T$, Y_s is a clique and, for every edge st in T, $Y_s \cup Y_t$ is* *a clique.*

Proof. Suppose that $x, y \in Y_s$ are not adjacent where $s \in T$. Let $i, j \in I$ be both non adjacent to s (such i, j exist by (21)). Then, we have that $xi, yi \in E$ (for, if not, we find A_4, B_4 or B_5 on $\{\overline{n}, s, s_I, x, y, i\}$). Similarly, $xj, yj \in E$. Hence, we have found (x, i, y, j) as induced C_4 in $G[N]$, which contradicts our assumption that $G[N]$ is chordal.

Suppose now that $Y_s \cup Y_t$ is not a clique for some edge st in T; let $x \in Y_s \setminus Y_t$, $y \in Y_t \setminus Y_s$ be non adjacent. Then, $(\overline{n}, x, s, t, y)$ is an induced C_5. ∎

Let A be a maximal subset of T such that $G[A]$ is connected and the set $Y(A) := \bigcup_{a \in A} Y_a$ is a clique. One can verify that

$$\text{there is no edge between the sets } A \text{ and } T \setminus A.$$

(The proof is similar to that of Claim 15. Namely, if $a \in A$ is adjacent to $b \in T \setminus A$, then we find two non adjacent nodes $x \in Y(A)$, $y \in Y_b$ and an induced path P from x to y whose internal nodes belong to T. This path P together with edges $\overline{n}x$, $\overline{n}y$ yields an induced circuit of length ≥ 6.)

From this it follows that $A = \emptyset$ (otherwise, the clique $Y(A) \cup \{s_I\}$ would be a clique cutset in G). Therefore, $T = \emptyset$, which shows that $S = \{s_I\}$ and, thus, $G \in \mathcal{G}_1$ by (20). This concludes the proof in Case B.

5.4 Case C

We assume here that $\nu \in \{2, 3\}$; thus, $G[N]$ is not chordal. By Corollary 9, this implies that

$$\overline{N} \text{ is a clique.}$$

Let $\{i_a j_a \mid a = 1, \ldots, \nu\}$ be an induced matching of maximum cardinality in $\overline{G}[N]$. In view of Claim 8, every node of N is non adjacent to at most one of the i_a, j_a's. This leads us to defining the following sets:

$$I_a := \{i \in N \mid i \neq j_a \text{ and } ij_a \notin E\}, \; J_a := \{i \in N \mid i \neq i_a \text{ and } ii_a \notin E\}$$

for $a = 1, \ldots, \nu$. Thus, $i_a \in I_a$ and $j_a \in J_a$ for $a \leq \nu$. Set

$$I := \bigcup_{a=1}^{\nu} I_a \cup J_a, \; N_0 := N \setminus I,$$

$$S_I := \{s \in S \mid s \text{ is adjacent to all nodes of } I\}, \; \text{ and } T := S \setminus S_I.$$

We group several observations about the sets I, N_0, S_I. By Claim 8, $S_I \neq \emptyset$ and a node $s \in S$ belongs to S_I if and only if s is adjacent to i_a and j_a for some $a = 1, \ldots, \nu$. Moreover,

$$N_0 \text{ is a clique}$$

(else we find an induced matching of size $\nu + 1$ in $\overline{G}[N]$).

Every node of N_0 is adjacent to every node of I.

Indeed, suppose that $x \in I$ is not adjacent to $y \in N_0$; say, $x \in I_1$. Then, y is adjacent to i_1, j_1, i_2, j_2 and $x \neq i_1$. Given $s \in S_I$ we find A_7 on $\{i_1, j_1, i_2, j_2, x, y, s, \overline{n}\}$. Moreover,

(23) S_I is a clique.

Indeed, if $s, t \in S_I$ are not adjacent, then we find A_3 on $\{\overline{n}, s, t, i_1, j_1, i_2, j_2\}$. We also have:

(24) If $x \in I_a, j \in J_a, a \in \{1, \ldots, \nu\}$, then $xy \notin E$.

Say, $a = 1$. The statement is obvious if $x = i_1$ or $y = j_1$. Otherwise, we find A_7 on $\{i_1, j_1, i_2, j_2, x, y, s, \overline{n}\}$ where $s \in S_I$. The following statements (25) and (26) are direct consequences of Claim 8.

(25) If $x \in I_a \cup J_a$, then x is adjacent to i_b and j_b, for $a \neq b \in \{1, \ldots, \nu\}$.

(26) If $s \in T, x, y \in I_a \cup J_a$ and $xy \notin E$, then s is adjacent to at most one of x, y.

(27) If $x \in I_a \cup J_a, y \in I_b \cup J_b, a \neq b \in \{1, \ldots, \nu\}$, then $xy \in E$ and $x \in \{i_a, j_a\}$ or $y \in \{i_b, j_b\}$.

The statement follows from (25) if $x \in \{i_a, j_a\}$ or $y \in \{i_b, j_b\}$. Otherwise, we find A_4 on $\{x, y, i_a, j_a, i_b, j_b\}$ if $xy \notin E$ and we find A_8 on $\{x, y, i_a, j_a, i_b, j_b, s, \overline{n}\}$ if $xy \in E$ (where $s \in S_I$).

Our next objective is to show that $S = S_I$, i.e., $T = \emptyset$. For $s \in T$, set:

$$Z_s := \{x \in S_I \cup N \mid sx \in E\}.$$

Claim 28. *Z_s is a clique for every $s \in T$ and $Z_s \cup Z_t$ is a clique for every edge st in T.*

Proof. Suppose that $x, y \in Z_s$ are not adjacent for some $s \in T$. As S_I and N_0 are cliques and every node of I is adjacent to every node of N_0, there are two cases: either $x, y \in I$, or $x \in S_I$ and $y \in N$. The case $x, y \in I$ is excluded by relations (26) and (27). If $x \in S_I$ and $y \in N$, then $N(s) \cup N(y)$ is not a clique since $I \subseteq N(x)$. By Claim 2, this implies that $N(s) \subseteq N(x)$ and, thus, $xy \in E$, a contradiction.

Suppose now that $Z_s \cup Z_t$ is not a clique for some edge st in T; let $x \in Z_s \setminus Z_t$, $y \in Z_t \setminus Z_s$ be non adjacent. If $x, y \in N$ then $(\overline{n}, x, s, t, y)$ is an induced C_5 in G. Hence, $x \in S_I$ and $y \in N_0$. Consider $i_1 \in I_1, j_1 \in J_1$. By (26), we may assume that

$si_1 \notin E$. Then, $ti_1 \in E$ (else, (i_1, x, s, t, y) is an induced C_5) and, thus, $tj_1 \notin E$, $sj_1 \in E$. But, then, we have found B_1 on $\{i_1, j_1, x, y, s, t\}$. ∎

Let A be a maximal subset of T for which $G[A]$ is connected and the set $Z(A) := \bigcup_{a \in A} Z_a$ is a clique.

Claim 29. *There is no edge between the sets A and $T \setminus A$.*

Proof. Suppose that $a \in A$ is adjacent to $b \in T \setminus A$. By maximality of A, we deduce that $Z(A) \cup Z_b$ is not a clique; let $x \in Z(A)$, $y \in Z_b$ be non adjacent and let $a_0 \in A$ be adjacent to x. We know from Claim 28 that $a \neq a_0$ and $a_0y, xa, xb \notin E$. Considering a shortest path in $G[A]$ from a_0 to a, we find an induced path $(x, a_1, \ldots, a_p, y)$ where $a_1, \ldots, a_p \in T$ and $p \geq 3$. If $x, y \in N$, then this path together with edges $\overline{n}x, \overline{n}y$ yields an induced circuit in G. Hence, $x \in S_I$ and $y \in N_0$. Consider $i_1 \in I_1$, $j_1 \in J_1$. Then, i_1 is adjacent to one of a_1, a_2. (Indeed, i_1 is adjacent to some a_i for, if not, then $(i_1, x, a_1, \ldots, a_p, y)$ is an induced circuit. Let $k \geq 1$ be the smallest index such that $i_1a_k \in E$; then, $(i_1, x, a_1, \ldots, a_k)$ is an induced circuit which implies that $k \leq 2$.) Similarly, j_1 is adjacent to one of a_1, a_2. Hence, we can assume that $i_1a_1 \in E$ ($\Longrightarrow j_1a_1 \notin E$), $j_1a_2 \in E$ ($\Longrightarrow i_1a_2 \notin E$). Then, we find B_5 on $\{i_1, j_1, x, y, a_1, a_2\}$. ∎

From this it follows that

$$T = \emptyset;\ \text{that is}, S = S_I.$$

For, if not, then $A \neq \emptyset$ and $Z(A)$ would be a clique cutset in G.

Corollary 30. *If $\nu = 3$, or if $\nu = 2$ with $\max(|I_a \cup J_a| : a = 1, 2) \geq 3$, then $G \in \mathcal{G}_\nu$.*

Proof. By the assumption, we have that $|\overline{N}| = |S| = 1$ and $S \cup N_0$ is a clique (for, otherwise, one would find A_8 or A_9 in G). Moreover, if $\nu = 3$, then $\max(|I_a \cup J_a| : a = 1, 2, 3) = 2$ (else, one finds A_9). Using (24), (25), (27), one obtains that $G \in \mathcal{G}_\nu$. ∎

Therefore, we can now assume that

$$\nu = 2 \ \text{ and } \max(|I_a \cup J_a| : a = 1, 2) = 2.$$

Set

$$N_0' := \{i \in N_0 \mid is \notin E \text{ for some } s \in S\}.$$

The following holds:

$$si \notin E \text{ for every } s \in S, i \in N_0'.$$

Indeed, suppose that $si \in E$ for some $s \in S$, $i \in N_0'$ and let $t \in S$ such that $ti \notin E$. Then, we find A_7 on $\{\overline{n}, s, t, i, i_1, j_1, i_2, j_2\}$.

Hence, the node set of G can be partitioned into the sets S, $N_0 \setminus N_0'$, $N_0' \cup \overline{N}$, and $I = \{i_1, j_1, i_2, j_2\}$, in such a way that $S \cup (N_0 \setminus N_0')$ and $(N_0 \setminus N_0') \cup N_0' \cup \overline{N}$ are cliques and there is no edge between S and $N_0' \cup \overline{N}$. Therefore, G belongs to $\mathcal{G}_2$. This concludes the proof in Case C and, thus, of Theorem 8.

6 Conclusions

As mentioned in the Introduction, one of the motivations for the study of the cone $\mathcal{P}_G$ comes from its link with the positive semidefinite matrix completion problem. It has been shown in [La98] that the completion problem can be solved in polynomial time for the following classes of graphs: chordal graphs (i.e., graphs with sparsity order 1); graphs with fixed minimum fill-in; hence, for graphs in $\bigcup_{i=1}^{3} \mathcal{G}_i$; graphs in $\mathcal{G}_4$ when one of the sets on the 4-circuit is assumed to have a fixed size. Therefore, if one can show that the completion problem is polynomial time solvable over the *whole* class $\mathcal{G}_4$, then the problem would be polynomial time solvable for all graphs with sparsity order 2.

Acknowledgements

We thank a referee for his/her careful reading of the paper and for his/her suggestions that helped improve the presentation of the paper.

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