Semidefinite Relaxations for Max-Cut

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Abstract. We compare several semidefinite relaxations for the cut polytope obtained by applying the lift and project methods of Lovász and Schrijver and of Lasserre. We show that the tightest relaxation is obtained when aplying the Lasserre construction to the node formulation of the max-cut problem. This relaxation $Q_t(G)$ can be defined as the projection on the edge subspace of the set $\mathcal{F}_t(n)$, which consists of the matrices indexed by all subsets of $\{1,\dots,n\}$ of cardinality $\leq t+1$ with the same parity as $t+1$ and having the property that their (I,J)-th entry depends only on the symmetric difference of the sets I and J. The set $\mathcal{F}_0(n)$ is the basic semidefinite relaxation of max-cut consisting of the semidefinite matrices of order n with an all ones diagonal, while $\mathcal{F}_{n-2}(n)$ is the 2^{n-1}-dimensional simplex with the cut matrices as vertices. We show the following geometric properties. Let $Y \in \mathcal{F}_t(n)$ and let X be its principal submatrix indexed by the first n rows and columns; if rank $X \leq t+1$, then Y can be written as a convex combination of at most 2^t cut matrices; this extends a result of Anjos and Wolkowicz for the case $t=1$. Any 2^{t+1} cut matrices form a face of $\mathcal{F}_t(n)$ for $t=0,1,n-2$. The class $\mathcal{L}_t$ of the graphs G for which $Q_t(G)$ is equal to the cut polytope of G is shown to be closed under taking minors. The graph K_7 is a forbidden minor for membership in $\mathcal{L}_2$, while K_3 and K_5 are the only minimal forbidden minors for the classes $\mathcal{L}_0$ and $\mathcal{L}_1$, respectively.

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1 Introduction

1.1 Preamble

A basic problem in integer programming is to find the linear description of the convex hull P of the 0/1 valued points lying in a polytope $K \subseteq \mathbb{R}^d$ defined by an

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explicitely given linear system $Ax \leq b$, or at least to find linear relaxations of P that are good and efficient, meaning that they approximate well P and that one can optimize over them a linear objective function in polynomial time. If all the vertices of K are 0/1 valued, then $P = K$ and we are done. Otherwise one has to find "cutting planes" permitting to strengthen the relaxation K by cutting off its fractional vertices. Extensive research has been done for finding (partial) linear relaxations for many 0/1 polytopes arising from specific combinatorial optimization problems by exploiting the combinatorial structure of the problem at hand. Next to that research has also focused on developing general purpuse methods applying to arbitrary 0/1 (or integer) problems.

One of the first such methods (which applies to general integer polyhedra) is the method of Gomory, which constructs cutting planes from the linear system $Ax \leq b$ defining K using integer rounding. Namely, it constructs the *Chvátal closure*

$$K' := \{x \in \mathbb{R}^d \mid u^T Ax \leq \lfloor u^T b \rfloor \text{ for all } u \geq 0 \text{ such that } u^T A \text{ integer}\},$$

which satisfies $P \subseteq K' \subseteq K$. Define iteratively $K^{(1)} := K'$ and $K^{(t+1)} := (K^{(t)})'$ for $t \geq 1$. Then, $K^{(t)} = P$ for some integer t [7]; the smallest such t is the *Chvátal rank* of K. Although the Chvátal rank can be very large in general (as it depends on the dimension d and on the coefficients of A), it is bounded by $O(d^2 \log d)$ when K is contained in the cube $[0,1]^d$ [13]. ¿From an algorithmic point of view, the first Chvátal closure does not yield an efficient relaxation in general, since optimizing a linear objective function over K' is a co-NP-hard problem [12].

Another idea has been investigated for constructing cutting planes in an implicit way, which consists of trying to represent P as the projection of another polytope Q lying in a higher dimensional space. The rationale behind being that the projection of a polytope Q may have a much more complicated facial structure than Q itself. As a matter of fact, any d-dimensional 0/1 polytope can be realized as the projection of a $(2^d - 1)$-dimensional simplex! Several general purpose methods have been proposed for constructing projection representations for 0/1 polytopes; in particular, by Balas, Ceria and Cornuéjols [3], Sherali and Adams [29], Lovász and Schrijver [27] and, recently, by Lasserre [19, 20]. A common feature of these methods is the construction of a hierarchy

$$K \supseteq K_1 \supseteq \ldots \supseteq K_d \supseteq P$$

of relaxations of P obtained as projections of higher dimensional polyhedra, that finds P in d steps; that is, $K_d = P$. An important algorithmic property is that, if one can optimize a linear objective function over K in polynomial time, then the same holds for K_t for any *fixed* t (assuming that the number of rows of A is part of the input data in the case of Lasserre). The relaxations are linear or, in the case of Lovász-Schrijver and of Lasserre, semidefinite.

This idea of using semidefinite relaxations for a combinatorial 0/1 problem goes back to the seminal work of Lovász [26] who introduced the theta function $\vartheta(G)$ as bound for the stability number of a graph G, obtained by optimizing over the semidefinite relaxation $\mathrm{TH}(G)$ of the stable set polytope. This idea was again used successfully by Goemans and Williamson [16] who could prove the first non-trivial approximation algorithm for max-cut using a semidefinite relaxation of the cut polytope. Since then semidefinite relaxations have been widely used for approximating combinatorial problems (see, e.g., [25] for a survey).

A comparison of the various lift and project methods can be found in [22]. In particular, if we denote the t-th iterate in the Lovász-Schrijver hierarchy by $N_+^t(K)$ and the t-th iterate in the Lasserre hierarchy by $Q_t(K)$, it is shown there that

$$Q_t(K) \subseteq N_+^t(K). \tag{1}$$

In this paper we study how the Lovász-Schrijver and Lasserre procedures apply to the max-cut problem. There are, in fact, two possible ways in which they can be applied, either to the edge formulation of the problem or to its node formulation. We examine the relationships existing between the various semidefinite relaxations obtained for the cut polytope. It turns out that the best relaxation is obtained when using the Lasserre construction applied to the node model. Its definition involves an interesting set of matrices (*moment matrices*) having nice geometric properties; in particular, about adjacencies of cuts and matrices of small rank versus exact resolution of max-cut.

1.2 The max-cut problem

Let $G = (V, E)$ be a graph with node set $V = \{1, \dots, n\}$; its edge set E being viewed as a set of unordered pairs of distinct elements of V. Given a subset $S \subseteq V$, the *cut* $\delta(S)$ defined by S is the set of edges $ij \in E$ with $|S \cap \{i, j\}| = 1$. Given edge weights $w \in \mathbb{Q}^E$, the *max-cut problem* consists of finding a cut $\delta(S)$ whose weight $\sum_{ij\in\delta(S)} w_{ij}$ is maximum. It can be formulated as the problem

$$\max\ \frac{1}{2}\sum_{ij\in E} w_{ij}(1 - x_i x_j) \text{ subject to } x \in \{\pm 1\}^n. \tag{2}$$

For a cut $\delta(S)$, we use the same symbol $\delta(S)$ for denoting its ± 1 incidence vector in $\mathbb{R}^E$, with ij-th entry -1 if $ij \in \delta(S)$ and 1 otherwise. The *cut polytope* $\mathrm{CUT}(G)$ is the polytope in $\mathbb{R}^E$ defined as the convex hull of all cuts $\delta(S)$ ($S \subseteq V$). Thus the max-cut problem can be expressed as a linear programming problem over the polytope $\mathrm{CUT}(G)$:

$$\max\ \frac{1}{2}w(E) - \frac{1}{2}w^T z \ \text{ subject to } \ z \in \mathrm{CUT}(G). \tag{3}$$

In view of (2) and (3), there are two possible ways in which the lift and project methods can be applied to the max-cut problem.

The edge model. A first possibility is to work in the edge space and to apply the constructions to a linear relaxation K of the cut polytope $\mathrm{CUT}(G)$. As linear relaxation one can choose the *metric polytope* $\mathrm{MET}(G)$ which is the polytope in $\mathbb{R}^E$ defined by the bound constraints: $-1 \le x_{ij} \le 1$ $(ij \in E)$ and the *cycle inequalities*:

$$\sum_{ij \in F} x_{ij} - \sum_{ij \in E(C) \setminus F} x_{ij} \ge 2 - |C| \tag{4}$$

(for C cycle in G and $F \subseteq E(C)$ with an odd cardinality). Applying the Lovász-Schrijver and Lasserre constructions to the pair $K = \mathrm{MET}(G)$, $P = \mathrm{CUT}(G)$, one obtains respectively the semidefinite relaxations $N_+^t(\mathrm{MET}(G))$ and $Q_t(\mathrm{MET}(G))$ satisfying

$$\mathrm{CUT}(G) \subseteq Q_t(\mathrm{MET}(G)) \subseteq N_+^t(\mathrm{MET}(G))$$

(recall (1)). We will consider mainly the relaxation $N_+^t(\mathrm{MET}(G))$ obtained using the Lovász-Schrijver N_+ operator, since the definition of the relaxation $Q_t(\mathrm{MET}(G))$ involves a large number of semidefinite constraints; precise definitions are given in Section 2.

Let $E_n := \{ij \mid 1 \le i < j \le n\}$ denote the edge set of the complete graph K_n and let π_E denote the projection from $\mathbb{R}^{E_n}$ onto $\mathbb{R}^E$. Obviously, $\mathrm{CUT}(G) = \pi_E(\mathrm{CUT}(K_n))$ and Barahona [4] shows that $\mathrm{MET}(G) = \pi_E(\mathrm{MET}(K_n))$. In the linear description of $\mathrm{MET}(G)$, it suffices to consider the cycle inequalities (4) for chordless circuits [6]; therefore, $\mathrm{MET}(K_n)$ is defined by the $4\binom{n}{3}$ *triangle inequalities*:

$$x_{ij} + x_{ik} + x_{jk} \ge -1, \;\; x_{ij} - x_{ik} - x_{jk} \ge -1 \tag{5}$$

for all distinct $i, j, k \in V$. As a consequence, one can alternatively obtain a semidefinite relaxation of $\mathrm{CUT}(G)$ by first applying the N_+ operator to $\mathrm{MET}(K_n)$ and then projecting onto $\mathbb{R}^E$; namely define

$$N_+^t(G) := \pi_E(N_+^t(\mathrm{MET}(K_n))). \tag{6}$$

It can be verified (see [21]) that

$$N_+^t(G) \subseteq N_+^t(\mathrm{MET}(G)); \tag{7}$$

it is not known whether equality holds, i.e., whether the two operators N_+ and π_E commute.

The node model. A second possibility is to apply the lift and project constructions to the set $K = [-1,1]^n$ (lying in the node space) and to take projections onto the edge space $\mathbb{R}^E$ (instead of projections onto the node space $\mathbb{R}^n$). When applying the Lasserre construction in this framework, we obtain the semidefinite relaxation $Q_t(G)$ of $\mathrm{CUT}(G)$, which is defined as the projection on $\mathbb{R}^E$ of the set of vectors $y = (y_I)_{\substack{I \subseteq V \\ |I| \le 2t+2}}$ for which $y_\emptyset = 1$ and the matrix

$$M_{t+1}(y) := (y_{I \Delta J})_{\substack{I,J \subseteq V \\ |I|,|J| \le t+1}} \tag{8}$$

is positive semidefinite (see Section 2.3). The first member $Q_0(K_n)$ in this hierarchy corresponds to the basic semidefinite relaxation of $\mathrm{CUT}(K_n)$ considered in [16], while the second member $Q_1(K_n)$ tightens the semidefinite relaxation F_n introduced by Anjos and Wolkowicz [1].

1.3 Contents of the paper

The paper is organized as follows. In Section 2 we present the Lovász-Schrijver and Lasserre constructions and indicate how they apply to the max-cut problem. Our main result there is that $Q_t(G)$ (the relaxation obtained by applying the Lasserre construction to the node model) is contained in $N_+^{t-1}(G)$ (the relaxation obtained using the Lovász-Schrijver procedure in the edge model). In Section 3 we consider the class $\mathcal{L}_t$ consisting of the graphs G for which $\mathrm{CUT}(G) = Q_t(G)$. We show that $\mathcal{L}_t$ is closed under taking minors and that a graph G belongs to $\mathcal{L}_t$ if it contains an edge whose contraction produces a graph in $\mathcal{L}_{t-1}$. Section 7 contains a numerical comparison of the relaxations $Q_t(K_n)$ for small $n \leq 7$ and $t \leq 2$.

Section 4 is devoted to the study of some geometric properties of the matrix set $\mathcal{F}_t(n)$ underlying the relaxation $Q_t(K_n)$, which consists of the matrices of the form (8) (or rather of their principal submatrices indexed by the sets having the same parity as $t+1$; cf. (22)). Thus $\mathcal{F}_0(n)$ is the basic SDP relaxation of $\mathrm{CUT}(K_n)$ consisting of the semidefinite matrices of order n with an all ones diagonal while $\mathcal{F}_{n-2}(n)$ is the 2^{n-1}-dimensional simplex with the cut matrices as vertices. We study adjacency properties of cuts on $\mathcal{F}_t(n)$. Padberg [28] showed that any two cuts form a face of the metric polytope and Laurent and Poljak [23] showed the analogous result for $\mathcal{F}_0(n)$. We address here the question whether, more generally, any 2^{t+1} cuts form a face of $\mathcal{F}_t(n)$; we show that this property holds for $t = 1$ and $n-2$.

The matrix set $\mathcal{F}_t(n)$ permits to formulate the following upper bound for the max-cut problem:

$$\max \langle Q_w, Y \rangle \ \text{ subject to } Y \in \mathcal{F}_t(n)$$

(defining suitably Q_w). An interesting question is to find some conditions on the rank of an optimal solution Y ensuring that the above program solves the max-cut problem exactly. For $Y \in \mathcal{F}_t(n)$ let X be its principal submatrix indexed by the first n rows and columns. We show the following result: If rank $X \leq t+1$, then Y can be written as a convex combination of 2^t cut matrices (and thus the above relaxed program solves the max-cut problem exactly). The case $t = 0$ is obvious and the case $t = 1$ has been settled earlier by Anjos and Wolkowicz [2].

Notation. The notation $X \succeq 0$ means that X is a (symmetric) positive semidefinite matrix and PSD_n denotes the set of positive semidefinite matrices of order n. For a matrix X, $\ker X := \{u \mid Xu = 0\}$. We let $e_1, \ldots, e_n$ denote the standard unit vectors in $\mathbb{R}^n$. The following (easy to verify) properties of positive semidefinite matrices will be frequently used throughout the paper.

Lemma 1. *Let X be a positive semidefinite matrix of order n.*

(i) *Write X as $X = \begin{pmatrix} B & C \\ C^T & D \end{pmatrix}$. Then, $u \in \ker B \Longleftrightarrow \begin{pmatrix} u \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \ker X$.*

(ii) *If X has an all ones diagonal and $\epsilon = \pm 1$, then for distinct $i, j \in \{1, \ldots, n\}$, $X_{ij} = \epsilon \iff e_i - \epsilon e_j \in \ker X$.*

2 Comparing the Lovász-Schrijver and Lasserre Relaxations for Max-Cut

Let $K := \{x \in [-1,1]^d \mid Ax \geq b\}$ be an explicitely given polytope and let $P := \operatorname{conv}(K \cap \{\pm 1\}^d)$ be the polytope whose linear description is to be found. (As we want to treat max-cut, it is more convenient for us to work with ± 1 polytopes rather than 0/1 polytopes.) The following notation will be used throughout. Write K as

$$K = \{x \in [-1,1]^d \mid g_\ell^T \binom{1}{x} \geq 0 \mid \ell = 1, \ldots, m\} \tag{9}$$

where $g_1^T, \ldots, g_m^T \in \mathbb{R}^{d+1}$ are the rows of the matrix $(-b \; A)$. For a polytope $Q \subseteq \mathbb{R}^d$, the set

$$\tilde{Q} := \{\lambda \binom{1}{x} \mid x \in Q, \; \lambda \geq 0\} \tag{10}$$

denotes the *homogenization* of Q; $\tilde{Q}$ is a cone in $\mathbb{R}^{d+1}$ (the additional coordinate is indexed by 0) and $Q = \{x \in \mathbb{R}^d \mid \binom{1}{x} \in \tilde{Q}\}$.

2.1 The Lovász-Schrijver construction

Let $M(K)$ denote the set of symmetric matrices $Y = (y_{ij})_{i,j=0}^d$ satisfying

$$y_{jj} = y_{00} \text{ for } j = 1, \ldots, d, \tag{11}$$

$$Y(e_0 + e_j), \; Y(e_0 - e_j) \in \tilde{K} \text{ for } j = 1, \ldots, d \tag{12}$$

and let $M_+(K) := \{Y \in M(K) \mid Y \succeq 0\}$. Set

$$N(K) := \{x \in \mathbb{R}^d \mid \binom{1}{x} = Ye_0 \text{ for some } Y \in M(K)\},$$

$$N_+(K) := \{x \in \mathbb{R}^d \mid \binom{1}{x} = Ye_0 \text{ for some } Y \in M_+(K)\}.$$

Then,

$$P \subseteq N_+(K) \subseteq N(K) \subseteq K.$$

The inclusion $P \subseteq N_+(K)$ follows from the fact that the matrix $Y := \binom{1}{x}\binom{1}{x}^T$ belongs to $M_+(K)$ for all $x \in K \cap \{\pm 1\}^d$, the inclusion $N_+(K) \subseteq N(K)$ is obvious, and the inclusion $N(K) \subseteq K$ follows from property (12). The following consequence of (12) will be used in Section 3:

$$N(K) \subseteq \operatorname{conv}(K \cap \{x \mid x_j = \pm 1\}) \quad \text{for all } j = 1, \ldots, d. \tag{13}$$

The sets $N(K)$ and $N_+(K)$ are, respectively, linear and semidefinite relaxations of P. Define iteratively $N^1(K) := N(K)$, $N_+^1(K) := N_+(K)$ and, for $t \geq 2$, $N^t(K) := N(N^{t-1}(K))$, $N_+^t(K) := N_+(N_+^{t-1}(K))$. Then,

$$P \subseteq N^d(K) \subseteq \ldots \subseteq N^{t+1}(K) \subseteq N^t(K) \subseteq \ldots \subseteq N(K) \subseteq K.$$

Lovász and Schrijver [27] show that

$$N^d(K) = P.$$

Hence the sequence $N_+^t(K)$ also converges to P in d steps. There are instances where it converges faster to P than the sequence $N^t(K)$. This is the case, for example, when $P = \mathrm{ST}(G)$ is the stable set polytope of a graph G and $K = \mathrm{FR}(G)$ is its fractional stable set polytope defined by

$$\mathrm{FR}(G) := \{x \in \mathbb{R}_+^V \mid x_i + x_j \leq 1 \ (ij \in E)\}. \tag{14}$$

Lovász and Schrijver [27] show that $N_+(\mathrm{FR}(K_n)) = \mathrm{ST}(K_n)$ while the smallest t for which $N^t(\mathrm{FR}(K_n)) = \mathrm{ST}(K_n)$ is $t = n - 2$. On the other hand, there are also cases where the N_+ operator does not help. This is the case, for instance, for the polytope[1] $P = \{x \in \mathbb{R}^d \mid \sum_{i=1}^d x_i \geq 1\}$ if we start from its relaxation $K = \{x \in \mathbb{R}^d \mid \sum_{i=1}^d x_i \geq \frac{1}{2}\}$; then the same number d of iterations is needed for finding P using the N or the N_+ operator [8]. Other examples are given in [8], [15]. Moreover, geometric conditions are studied in [15] under which the N_+ operator yields a tighter relaxation than the N operator.

If we apply the Lovász-Schrijver construction to the pair $P = \mathrm{CUT}(G)$, $K = \mathrm{MET}(G)$, we obtain the sequence of linear and semidefinite relaxations $N^t(\mathrm{MET}(G))$ and $N_+^t(\mathrm{MET}(G))$ for the cut polytope. As mentioned in (7), one can obtain at least as good relaxations by applying the Lovász-Schrijver construction to the metric polytope of K_n and then projecting back on the edge set of the graph G; namely, set

$$N(G) := \pi_E(N(\mathrm{MET}(K_n)))$$

and define $N_+(G)$ as in (6). These relaxations are studied in detail in [21] where the following results are shown: $N(G) \subseteq N(\mathrm{MET}(G))$, $N_+(G) \subseteq N_+(\mathrm{MET}(G))$ (with equality if $G = K_n$). If the graph G has t edges whose contraction produces a graph with no K_5 minor, then $N^t(\mathrm{MET}(G)) = \mathrm{CUT}(G)$. In particular, $N^{n-\alpha(G)-3}(\mathrm{MET}(G)) = \mathrm{CUT}(G)$ if G has a maximum stable set whose deletion leaves a graph with at most three connected components; $N^{n-\alpha(G)-3}(G) = \mathrm{CUT}(G)$ for a graph G on n nodes with stability number $\alpha(G)$. No graph is known for which the sequence of relaxations converges faster to $\mathrm{CUT}(G)$ when using the N_+ operator than when using the N operator. Note that optimizing over the relaxation $N_+(K_n)$ amounts to solving a semidefinite program having a matrix variable Y of order $1 + \binom{n}{2}$ and $2\binom{n}{2} \cdot 4\binom{n}{3}$ linear inequalities.

[1] In this example and in the previous one of the stable set problem, P is a 0/1 polytope and thus one should use the corresponding definition of the N and N_+ operators for the 0/1 context; namely, replace (11) by $y_{jj} = y_{0j}$ and (12) by Ye_j, $Y(e_0 - e_j) \in \tilde{K}$ for $j = 1, \ldots, d$.

2.2 The Lasserre construction - General presentation

We first introduce some notation. In this subsection, we let $V = \{1, \ldots, d\}$ since we are looking for relaxations of the polytope P lying in $\mathbb{R}^d$. Let $\mathcal{P}(V)$ denote the collection of all subsets of V and, given an integer $t \geq 0$, let $\mathcal{P}_t(V)$ denote the collection of subsets of V with cardinality $\leq t$. The components of a vector $y \in \mathbb{R}^{\mathcal{P}(V)}$ are denoted as y_I or $y(I)$; we also set $y_0 = y_\emptyset$, $y_{i_1 \ldots i_k} = y_{\{i_1, \ldots, i_k\}}$. Given $y \in \mathbb{R}^{\mathcal{P}(V)}$ and an integer $t \geq 0$, the matrices

$$M(y) := (y(I \Delta J))_{I,J \in \mathcal{P}(V)}, \;\; M_t(y) := (y(I \Delta J))_{I,J \in \mathcal{P}_t(V)} \tag{15}$$

are known as the *moment matrices* of y (where $I \Delta J$ denotes the symmetric difference of the sets I, J). It is useful to realize that the principal submatrix of $M(y)$ indexed by a set $\mathcal{I} \subseteq \mathcal{P}(V)$ coincides with the principal submatrix of $M(y)$ indexed by the set $\mathcal{I} \Delta A := \{I \Delta A \mid I \in \mathcal{I}\}$ for any set $A \subseteq V$. Given $g, y \in \mathbb{R}^{\mathcal{P}(V)}$, define the vector $g * y \in \mathbb{R}^{\mathcal{P}(V)}$ with entries

$$g * y(I) := \sum_{J \in \mathcal{P}(V)} g_J y_{I \Delta J} \text{ (for } I \subseteq V\text{);}$$

that is, $g * y = M(y)g$. Given a subset $A \subseteq V$, let $\psi^A \in \{\pm 1\}^V$ denote its ± 1 incidence vector with entries $\psi^A(i) := -1$ if $i \in A$ and $\psi^A(i) := 1$ if $i \in V \setminus A$. The vectors e_A $(A \subseteq V)$ denote the standard unit vectors in $\mathbb{R}^{\mathcal{P}(V)}$. We use the representation of K given in (9); it will be convenient to consider g_ℓ as a vector in $\mathbb{R}^{\mathcal{P}(V)}$, setting $g_\ell(I) := 0$ if $|I| \geq 2$.

Lemma 2. *Given $A \subseteq V$, if $\psi^A \in K$ then*

$$M(y) \succeq 0, \;\; M(g_\ell * y) \succeq 0 \;\; \text{for } \ell = 1, \ldots, m, \tag{16}$$

where $y := ((-1)^{|A \cap I|})_{I \subseteq V}$.

Proof. Indeed, $M(y) = yy^T$ and $M(g_\ell * y) = g_\ell^T y \cdot yy^T$, since $y(I \Delta J) = y(I) \cdot y(J)$ for all $I, J \subseteq V$. Moreover, $g_\ell^T y \geq 0$ for all ℓ, since $\psi^A \in K$ and the projection of y on the subspace indexed by the singletons is equal to ψ^A. □

Let Z denote the symmetric matrix indexed by $\mathcal{P}(V)$ with entries

$$Z_{I,J} = (-1)^{|I \cap J|} \;\; \text{for } I, J \subseteq V. \tag{17}$$

This is the ± 1 analog of the zeta matrix of the lattice $\mathcal{P}(V)$ considered in [27] in the 0/1 case. The next result is the ± 1 analog of a result from [22] for the 0/1 case; we include the proof for completeness.

Lemma 3. *For $y \in \mathbb{R}^{\mathcal{P}(V)}$, equality $Z \operatorname{diag}(Zy) Z = 2^{|V|} M(y)$ holds. More generally, given $g \in \mathbb{R}^{\mathcal{P}(V)}$, $Z \operatorname{diag}(u) Z = 2^{|V|} M(g * y)$, where $u \in \mathbb{R}^{\mathcal{P}(V)}$ has entries $u_A := (Zy)_A \cdot g^T Z e_A$ (for $A \subseteq V$) (Ze_A denoting the A-th column of Z).*

Proof. Given $I, J \subseteq V$, the (I,J)-th entry of the matrix $Z\,\mathrm{diag}(u)Z$ is equal to

$$\begin{array}{c}\sum_A Z_{IA}Z_{JA}u_A = \sum_A Z_{IA}Z_{JA}(Zy)_A \cdot g^T Ze_A \\ = \sum_A Z_{IA}Z_{JA}\left(\sum_R Z_{AR}y_R\right)\cdot\left(\sum_S Z_{AS}g_S\right) = \sum_{R,S} y_R g_S \left(\sum_A Z_{IA}Z_{JA}Z_{AR}Z_{AS}\right).\end{array}$$

Since $Z_{IA}Z_{JA}Z_{AR}Z_{AS} = (-1)^{|I\cap A|+|J\cap A|+|R\cap A|+|S\cap A|} = (-1)^{|A\cap(I\Delta J\Delta R\Delta S)|}$, the inner sum (over A) is equal to $2^{|V|}$ if $I\Delta J\Delta R\Delta S = \emptyset$ and to 0 otherwise. Therefore, the (I,J)-th entry of $Z\ \mathrm{diag}(u)Z$ is equal to $2^{|V|}\sum_S g_S y_{I\Delta J\Delta S} = 2^{|V|} g * y(I\Delta J) = 2^{|V|}M(g*y)(I,J)$. This concludes the proof. $\square$

For $t \geq 0$, let $P_t(K)$ denote the set of vectors $y \in \mathbb{R}^{\mathcal{P}_{2t+2}(V)}$ satisfying

$$M_{t+1}(y) \succeq 0,\ \ M_t(g_\ell * y) \succeq 0 \ \text{ for all } \ell = 1,\ldots,m \tag{18}$$

and let $Q_t(K)$ denote the projection of $P_t(K) \cap \{y \mid y_\emptyset = 1\}$ onto the subspace $\mathbb{R}^d$ indexed by the singletons. Then,

$$P \subseteq Q_t(K) \subseteq K;$$

the first inclusion follows from Lemma 2 and the second one follows from the fact that $g_\ell * y(\emptyset) \geq 0$ for all ℓ and $y \in Q_t(K)$. The hierarchy of relaxations $Q_t(K)$ was introduced by Lasserre [18, 20] who showed that P is found after d steps; that is, $P = Q_d(K)$. His construction is motivated by results about representations of positive polynomials as sums of squares and his original presentation involves moment matrices indexed by integer sequences (rather than subsets of V). The presentation given here is taken from [22] where the following elementary proof for the convergence result is given, based on the above lemmas.

Let $\mathcal{C}_K$ denote the cone in $\mathbb{R}^{\mathcal{P}(V)}$ generated by the columns of Z corresponding to points in K; that is, $\mathcal{C}_K$ is generated by the vectors Ze_A for all the sets $A \subseteq V$ for which $\psi^A \in K$. Then, $\mathcal{C}_K$ is a simplicial cone and P is equal to the projection of the polytope $\mathcal{C}_K \cap \{y \mid y_\emptyset = 1\}$ on the subspace $\mathbb{R}^n$ indexed by the singletons.

Lemma 4. $P_d(K) = \mathcal{C}_K$.

Proof. By the definition, $y \in P_d(K)$ if and only if $M(y) \succeq 0$ and $M(g_\ell * y) \succeq 0$ for all $\ell = 1,\ldots,m$. Using Lemma 3, this is equivalent to the conditions $Zy \geq 0$ and $(Zy)_A \cdot g_\ell^T Ze_A \geq 0$ for all $A \subseteq V$ and $\ell = 1,\ldots,m$; this in turn holds if and only if $Zy \geq 0$ and, for all $A \subseteq V$, $(Zy)_A = 0$ whenever the vector ψ^A does not belong to K. Therefore, $y \in P_d(K)$ if and only if y belongs to the cone $\mathcal{C}_K$. $\square$

Corollary 5. $Q_d(K) = P$.

If we apply the above construction to the pair $P = \mathrm{CUT}(G)$, $K = \mathrm{MET}(G)$, then we obtain the sequence of semidefinite relaxations $Q_t(\mathrm{MET}(G))$ $(t = 0,\ldots,|E|)$ for the cut polytope $\mathrm{CUT}(G)$. The definition of $Q_t(\mathrm{MET}(G))$ involves the semidefinite program (18) which contains as many semidefinite constraints as the number of circuits in G (which can therefore be exponentially large in terms of n); moreover, the program is in the variable $y \in \mathbb{R}^{\mathcal{P}_{2t+2}(E)}$ since $\mathrm{MET}(G)$ lies in the edge space $\mathbb{R}^E$. One way to go around the difficulty of the large number of constraints is to consider instead of $Q_t(\mathrm{MET}(G))$ the set $\pi_E(Q_t(\mathrm{MET}(K_n)))$ whose definition involves now $1 + 4\binom{n}{3}$ constraints (corresponding to the triangles in K_n) and the variable $y \in \mathbb{R}^{\mathcal{P}_{2t+2}(E_n)}$. Although the number of constraints is now polynomial in n, we will see in the next subsection that if we apply the Lasserre construction to the node model of max-cut we obtain a much simpler relaxation, involving only one semidefinite constraint.

2.3 The Lasserre construction - The node model for max-cut

If we consider the formulation (2) for the max-cut problem, we arrive naturally at the following relaxations introduced by Lasserre [18] and obtained in the following way: Apply the Lasserre construction to the polytope $K := [-1, 1]^n$ and project on the edge subspace $\mathbb{R}^E$ (instead of projecting on the subspace $\mathbb{R}^n$ in which the starting polytope K lies). Thus in this subsection we let $V = \{1, \ldots, n\}$ since our starting polytope $K = [-1, 1]^n$ lies in the space $\mathbb{R}^n$. For $t \geq 0$, we have:

$$P_t(K) = \{y \in \mathbb{R}^{\mathcal{P}_{2t+2}(V)} \mid M_{t+1}(y) \succeq 0\};$$

let $Q_t(G)$ denote the projection of the set $P_t(K) \cap \{y \mid y_\emptyset = 1\}$ on the edge subspace $\mathbb{R}^E$. Then,

$$\mathrm{CUT}(G) \subseteq Q_t(G).$$

We now mention a more concise formulation for the set $Q_t(G)$. For this let $\mathcal{E}(V)$ (resp. $\mathcal{E}_t(V)$) denote the collection of all even subsets (resp. all even subsets of size $\leq t$) of V; $\mathcal{O}(V)$ and $\mathcal{O}_t(V)$ are the analogous families of odd subsets of V. It is also convenient to use the symbol $\mathcal{U}_t(V)$ for denoting the collection of subsets of V whose cardinality is $\leq t$ and has the same parity as t. Given a vector $y \in \mathbb{R}^{\mathcal{E}_{2t}(V)}$, let us define its *reduced moment matrix* $\widetilde{M}_t(y)$ as

$$\widetilde{M}_t(y) := (y(I \Delta J))_{I,J \in \mathcal{U}_t(V)} \tag{19}$$

Note that $\widetilde{M}_{n-1}(y) = \widetilde{M}_n(y)$ and that $\widetilde{M}_{n-1}(y)$ can be assumed to be indexed by either $\mathcal{E}(V)$ or $\mathcal{O}(V)$, since $\mathcal{O}(V) = \mathcal{E}(V)\Delta\{1\} := \{I \Delta \{1\} \mid I \in \mathcal{E}(V)\}$.

Lemma 6. *For $t \geq 0$, $Q_t(G)$ is equal to the projection on $\mathbb{R}^E$ of the set of vectors $y \in \mathbb{R}^{\mathcal{E}_{2t+2}(V)}$ satisfying*

$$\widetilde{M}_{t+1}(y) \succeq 0, \ y_\emptyset = 1. \tag{20}$$

Proof. Let $\widetilde{Q}_t(G)$ denote the projection on $\mathbb{R}^E$ of the solution set to (20). The inclusion $Q_t(G) \subseteq \widetilde{Q}_t(G)$ is obvious. Conversely, assume that $\widetilde{M}_{t+1}(y) \succeq 0$ where $y \in \mathbb{R}^{\mathcal{E}_{2t+2}(V)}$. Extend y to a vector $z \in \mathbb{R}^{\mathcal{P}_{2t+2}(V)}$ by setting $z_I := y_I$ if $|I|$ is even and $z_I := 0$ if $|I|$ is odd. Then the matrix $M_{t+1}(z)$ has the block configuration

$$\begin{array}{cc} & \begin{array}{cc} \mathcal{E}_{t+1}(V) & \mathcal{O}_{t+1}(V) \end{array} \\ \begin{array}{c} \mathcal{E}_{t+1}(V) \\ \mathcal{O}_{t+1}(V) \end{array} & \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \end{array} \tag{21}$$

where $A = \widetilde{M}_{t+1}(y)$ and B is the principal submatrix of A indexed by $\{I\Delta\{1\} \mid I \in \mathcal{O}_{t+1}(V)\}$ if t is odd, and $B = \widetilde{M}_{t+1}(y)$ and A is the principal submatrix of B indexed by $\{I\Delta\{1\} \mid I \in \mathcal{E}_{t+1}(V)\}$ if t is even. From this follows that $M_{t+1}(z) \succeq 0$. Since y and z have the same even indexed entries, the reverse inclusion $\widetilde{Q}_t(G) \subseteq Q_t(G)$ follows. □

For $t = 0, 1, \ldots, n-2$ set

$$\mathcal{F}_t(n) := \{Y \succeq 0 \mid Y = \widetilde{M}_{t+1}(y) \text{ for some } y \in \mathbb{R}^{\mathcal{E}_{2t+2}(V)} \text{ with } y_\emptyset = 1\}. \tag{22}$$

Let us see what the matrix sets $\mathcal{F}_t(n)$ are for small values of t. For $t = 0$, $\mathcal{F}_0(n)$ is the basic semidefinite relaxation of the cut polytope $\mathrm{CUT}(K_n)$ consisting of the $n \times n$ symmetric positive semidefinite matrices with an all ones diagonal. For $t = 1$, $\mathcal{F}_1(n)$ is equal to the set of symmetric positive semidefinite matrices Y indexed by $\{\emptyset\} \cup E_n$ having an all ones diagonal and satisfying the two conditions

$$Y_{ij,ik} = Y_{\emptyset,jk} \text{ and } Y_{ij,rs} = Y_{ir,js} = Y_{is,jr}$$

for all distinct $i, j, k, r, s \in V$. If we remove in the definition of $\mathcal{F}_1(n)$ the second condition $Y_{ij,rs} = Y_{ir,js} = Y_{is,jr}$, then we obtain the larger matrix set $\mathcal{F}_n$ underlying the relaxation (SDP3) defined by Anjos and Wolkowicz [1] and their relaxation

$$F_n := \{x \in \mathbb{R}^{E_n} \mid \binom{1}{x} = Ye_\emptyset \text{ for some } Y \in \mathcal{F}_n\} \tag{23}$$

of the cut polytope $\mathrm{CUT}(K_n)$; thus

$$\mathcal{F}_1(n) \subseteq \mathcal{F}_n \text{ and } Q_1(K_n) \subseteq F_n.$$

A useful property of the matrix set $\mathcal{F}(n)$ (and thus of $\mathcal{F}_1(n)$) is that it implies the triangle inequalities.

Lemma 7. [1] $F_n \subseteq \mathrm{MET}(K_n)$.

Proof. Let $Y \in \mathcal{F}_n$ and set $y_{ij} := Y_{\emptyset,ij}$ for $ij \in E_n$. Given three nodes $1, 2, 3 \in V$, the principal submatrix X of Y indexed by the set $\{\emptyset, 12, 13, 23\}$ has the form

$$X = \begin{array}{c} \\ \emptyset \\ 12 \\ 13 \\ 23 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} \emptyset & 12 & 13 & 23 \end{array} \\ \begin{pmatrix} 1 & y_{12} & y_{13} & y_{23} \\ y_{12} & 1 & y_{23} & y_{13} \\ y_{13} & y_{23} & 1 & y_{12} \\ y_{23} & y_{13} & y_{12} & 1 \end{pmatrix} \end{array}.$$

As $X \succeq 0$, we have $e^T X e \geq 0$ which implies that $y_{12} + y_{13} + y_{23} \geq -1$. The other triangle inequalities are obtained by suitably flipping signs in X. □

The cuts of $\mathrm{CUT}(K_n)$ correspond to certain special matrices in $\mathcal{F}_t(n)$. Given $A \subseteq V$, the vector

$$y^A := ((-1)^{|I \cap A|})_{I \in \mathcal{E}(V)} \tag{24}$$

is called a *cut vector*; its projection on $\mathbb{R}^{E_n}$ is the cut $\delta(A)$. As $y^A = y^{V \setminus A}$, there are 2^{n-1} distinct cut vectors y^A obtained, for instance, for all $A \subseteq V \setminus \{n\}$. For convenience we often use below the same symbol y^A for denoting the cut vector in $\mathbb{R}^{\mathcal{E}(V)}$ or its projection on a subspace of it. For $t = 0, \ldots, n-2$, the reduced moment matrix $\widetilde{M}_{t+1}(y^A)$ is called a *cut matrix* of $\mathcal{F}_t(n)$ and is also denoted as $\widetilde{M}_{t+1}(A)$ for the sake of simplicity. Thus

$$\widetilde{M}_{n-1}(A) = y^A (y^A)^T$$

and every cut matrix $\widetilde{M}_{t+1}(A)$ has rank 1. The next lemma shows that the eigenvectors of any reduced moment matrix $\widetilde{M}_{n-1}(y)$ are the cut vectors, which permits to show that $\mathcal{F}_{n-2}(n)$ is a simplex with the cut matrices as vertices.

Lemma 8. *Let $Y = \widetilde{M}_{n-1}(y)$ where $y \in \mathbb{R}^{\mathcal{E}(V)}$. The eigenvectors of Y are the 2^{n-1} distinct vectors $y^A \in \mathbb{R}^{\mathcal{E}(V)}$ with respective eigenvalues $y^T y^A$.*

Proof. We verify that $Yy^A = (y^T y^A) y^A$; that is, $(Ye_S)^T y^A = y^T y^A \cdot (-1)^{|A \cap S|}$ for any $S \in \mathcal{E}(V)$. Indeed,

$$\begin{aligned}(Ye_S)^T y^A &= \textstyle\sum_{I \in \mathcal{E}(V)} y(S\Delta I)(-1)^{|A\cap I|} = \sum_{I\in\mathcal{E}(V)} y(S\Delta I)(-1)^{|A\cap S|}(-1)^{|A\cap(S\Delta I)|} \\ &= (-1)^{|A\cap S|} \textstyle\sum_{J\in\mathcal{E}(V)} y(J)(-1)^{|A\cap J|} = (-1)^{|A\cap S|}\cdot y^T y^A.\end{aligned}$$

□

Corollary 9. *The set $\mathcal{F}_{n-2}(n)$ is the simplex in $\mathbb{R}^{\mathcal{E}(V)}$ whose vertices are the 2^{n-1} distinct cut matrices $\widetilde{M}_{n-1}(A)$ $(A \subseteq V)$.*

Proof. By Lemma 8, any matrix $Y = \widetilde{M}_{n-1}(y) \in \mathcal{F}_{n-2}(n)$ can be written as $Y = \frac{1}{2^{n-1}} \sum_{A \subseteq V\setminus\{n\}} y^T y^A \cdot y^A (y^A)^T$, where $y^T y^A \geq 0$ for all A and $\frac{1}{2^{n-1}} \sum_A y^T y^A = 1$. □

2.4 Comparing the Lasserre relaxation $Q_t(G)$ and the Lovász-Schrijver relaxation $N_+^{t-1}(G)$.

We show here an inclusion relationship between the two semidefinite relaxations $N_+^{t-1}(G)$ and $Q_t(G)$ obtained earlier for the cut polytope using, respectively, the Lovász-Schrijver construction (applied to the edge model of max-cut) and the Lasserre relaxation (applied to the node model). As before, $G = (V, E)$ is a graph with $V = \{1, \ldots, n\}$. We begin with a preliminary result.

Lemma 10. *Given $y \in \mathbb{R}^{\mathcal{P}_{2t+2}(V)}$, $t \geq 0$, $M_{t+1}(y) \succeq 0 \Longrightarrow M_t((e_\emptyset + \epsilon e_{ij}) * y) \succeq 0$ for all $\epsilon = \pm 1$ and $ij \in E_n$.*

Proof. Let $ij = 12$ and set $u := (e_\emptyset + \epsilon e_{12}) * y$. Observe that $u(I) = \epsilon u(I\Delta\{1,2\})$ for all I. Let us consider the partition of $\mathcal{P}_t(V)$ into the following three sets: $\mathcal{I}_1 := \{I \in \mathcal{P}_t(V) \mid 1 \notin I, |\{I\Delta\{1,2\}| \leq t\}$, $\mathcal{I}_2 := \{I \in \mathcal{P}_t(V) \mid 1 \notin I, |\{I\Delta\{1,2\}| > t\}$, and $\mathcal{I}_3 := \{I\Delta\{1,2\} \mid I \in \mathcal{I}_1\}$. With respect to this partition, the matrix $M_t(u)$ has the block configuration $\begin{array}{c|ccc} & \mathcal{I}_1 & \mathcal{I}_2 & \mathcal{I}_3 \\ \hline \mathcal{I}_1 & A & B & \epsilon\cdot A \\ \mathcal{I}_2 & B^T & C & \epsilon\cdot B^T \\ \mathcal{I}_3 & \epsilon\cdot A & B & A \end{array}$. Hence, $M_t(u) \succeq 0$ if and only if its principal submatrix X indexed by $\mathcal{I} := \mathcal{I}_1 \cup \mathcal{I}_2$ is positive semidefinite. Therefore, it suffices to show that $M_{t+1}(y) \succeq 0 \Longrightarrow X \succeq 0$. For this consider the principal submatrix Y of $M_{t+1}(y)$ indexed by $\mathcal{I}'_1 \cup \mathcal{I}'_2$ where $\mathcal{I}'_i := \{I\Delta\{i\} \mid I \in \mathcal{I}\}$ for $i = 1, 2$. Then Y has the block configuration $\begin{array}{c|cc} & \mathcal{I}'_1 & \mathcal{I}'_2 \\ \hline \mathcal{I}'_1 & E & F \\ \mathcal{I}'_2 & F^T & E \end{array}$. Now $Y \succeq 0$ implies that $E + \epsilon F \succeq 0$ for $\epsilon = \pm 1$, since $(x^T \; \epsilon x^T) Y \binom{x}{\epsilon x} = 2 \cdot x^T(E + \epsilon F)x \geq 0$ for all x. Observe finally that $X = E + \epsilon F$. $\square$

Theorem 11. *For any graph G and $t \geq 1$, $Q_t(G) \subseteq N_+(Q_{t-1}(G))$.*

Proof. Let $x \in Q_t(G)$; that is, x is the projection on $\mathbb{R}^E$ of $y \in \mathbb{R}^{\mathcal{P}_{2t+2}(V)}$ satisfying $y_\emptyset = 1$ and $M_{t+1}(y) \succeq 0$. Let Y denote the principal submatrix of $M_{t+1}(y)$ indexed by $\{\emptyset\} \cup E$. Thus $\binom{1}{x} = Ye_0$ and $Y \succeq 0$. In order to show that $x \in N_+(Q_{t-1}(G))$ it suffices now to verify that the vector $z := Y(e_\emptyset + \epsilon e_{ij})$ belongs to $\widetilde{Q_{t-1}(G)}$ (the homogenization of $Q_{t-1}(G)$) for all $\epsilon = \pm 1$ and $ij \in E$. This follows from the fact that z is equal to the projection on $\mathbb{R}^{\{\emptyset\}\cup E}$ of the vector $u := (e_\emptyset + \epsilon e_{ij}) * y$ and that $M_t(u) \succeq 0$ by Lemma 10. $\square$

Corollary 12. *For any graph G and $t \geq 1$, $Q_t(G) \subseteq N_+^{t-1}(G)$.*

Proof. Directly from Theorem 11 and Lemma 7 using induction on $t \geq 1$. $\square$

3 Bounds on the Rank of the Lasserre Procedure

An interesting question is to determine the class $\mathcal{L}_t$ consisting of the graphs G for which $\mathrm{CUT}(G) = Q_t(G)$. Indeed the max-cut problem can be solved in polynomial time (with an arbitrary precision) over the class $\mathcal{L}_t$ for any fixed t. The same holds for the class $\mathcal{G}_t$ of the graphs G for which $\mathrm{CUT}(G) = N_+^t(G)$. By Corollary 12, we have that $\mathcal{G}_{t-1} \subseteq \mathcal{L}_t$ for $t \geq 1$; inclusion is strict for instance for $t = 2$. The class $\mathcal{G}_t$ is closed under taking minors [21]. We show that the same holds for $\mathcal{L}_t$. A crucial tool for showing that $\mathcal{G}_t$ is closed under taking contraction minors is the fact that validity of an inequality for the set $N_+^t(G) \cap \{x \mid x_e = \pm 1\}$ can be expressed in terms of validity of a transformed inequality for the set $N_+^t(G/e)$. An analogous idea will be used for showing that $\mathcal{L}_t$ is closed under taking contraction minors.

We begin with some definitions and preliminary observations. Let $G = (V, E)$ be a graph with $V = \{1, \ldots, n\}$ and let $e := uv$ be a given edge of G. Deleting e produces the graph $G\backslash e := (V, E \setminus \{e\})$ while contracting e produces the graph $G \,/\, e := (V \setminus \{u, v\} \cup \{w\}, F)$ where w is the new node created by the contraction of the edge e and F is the resulting edge set (erasing multiple edges). A minor of G is any graph obtained from G by a sequence of deletions and contractions. For a node $i \in V$, $N_G(i)$ denotes the set of nodes that are adjacent to i in G. The following is a simple but powerful property of the metric polytope:

$$\text{If } y \in \mathrm{MET}(G) \text{ has } y_{uv} = \epsilon = \pm 1, \text{ then } y_{ui} = \epsilon y_{vi} \text{ for } i \in N_G(u) \cap N_G(v). \tag{25}$$

The same property holds if we replace $\mathrm{MET}(G)$ by its subset $\mathrm{CUT}(G)$ or by $Q_t(G)$ ($t \geq 0$) (in view of Lemma 1 (ii)). Based on this property, let us define for $x \in \mathbb{R}^F$ its *ϵ-extension* $y \in \mathbb{R}^E$ by

$$\begin{array}{l} y_{uv} := \epsilon,\ y_{ij} := x_{ij} \text{ for } ij \in E \text{ with } i,j \neq u,v, \\ y_{ui} := x_{wi} \text{ for } i \in N_G(u) \setminus \{v\},\ y_{vi} := \epsilon x_{wi} \text{ for } i \in N_G(v) \setminus \{u\}. \end{array} \tag{26}$$

One can easily verify that

$$x \in \mathrm{CUT}(G/e) \Longleftrightarrow y \in \mathrm{CUT}(G). \tag{27}$$

In order to establish the analogous result for $Q_t(G)$ we need to extend the notion of ϵ-extension to reduced moment matrices. For convenience we set below $u = w = 1$ and $v = n$.

Lemma 13. *Let $t \geq 0$, $X = \widetilde{M}_{t+1}(x)$ where $x \in \mathbb{R}^{\mathcal{E}_{2t+2}(V\setminus\{n\})}$, and $\epsilon = \pm 1$. Extend x to $y \in \mathbb{R}^{\mathcal{E}_{2t+2}(V)}$ by setting $y_I := \epsilon \cdot x_{I\Delta\{1,n\}}$ for all $I \in \mathcal{E}_{2t+2}(V)$ with $n \in I$ and set $Y := \widetilde{M}_{t+1}(y)$; Y is called an* ϵ-extension *of X. Then, $X \in \mathcal{F}_t(n-1) \Longleftrightarrow Y \in \mathcal{F}_t(n)$.*

Proof. We have to show that $X \succeq 0 \Longleftrightarrow Y \succeq 0$. The "only if" part is obvious since X is a principal submatrix of Y. Assume now that $X \succeq 0$. Partition the index set $\mathcal{U}_{t+1}(V)$ of Y into $\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3$, where $\mathcal{I}_1 := \{H \in \mathcal{U}_{t+1}(V\setminus\{n\}) \mid |H\Delta\{1,n\}| \leq t+1\}$, $\mathcal{I}_2 := \{H \in \mathcal{U}_{t+1}(V \setminus \{n\}) \mid |H\Delta\{1,n\}| > t+1\}$ and $\mathcal{I}_3 := \{H\Delta\{1,n\} \mid H \in \mathcal{I}_1\} = \{H \in \mathcal{U}_{t+1}(V) \mid n \in H\}$. Then X and Y have the following block configurations:

$$X = \begin{array}{c c} & \begin{array}{cc} \mathcal{I}_1 & \mathcal{I}_2 \end{array} \\ \begin{array}{c} \mathcal{I}_1 \\ \mathcal{I}_2 \end{array} & \begin{pmatrix} B & C \\ C^T & D \end{pmatrix} \end{array}, \quad Y = \begin{array}{c c} & \begin{array}{ccc} \mathcal{I}_1 & \mathcal{I}_2 & \mathcal{I}_3 \end{array} \\ \begin{array}{c} \mathcal{I}_1 \\ \mathcal{I}_2 \\ \mathcal{I}_3 \end{array} & \begin{pmatrix} B & C & \epsilon \cdot B \\ C^T & D & \epsilon \cdot C^T \\ \epsilon \cdot B & \epsilon \cdot C & B \end{pmatrix} \end{array}.$$

Indeed, for $H \in \mathcal{U}_{t+1}(V \setminus \{n\})$ and $K \in \mathcal{I}_3$, $Y(H,K) = y(H\Delta K)$ is equal to $\epsilon \cdot y(H\Delta K\Delta\{1,n\})$ (by the definition of y) and thus to $\epsilon \cdot Y(H, K\Delta\{1,n\})$ where $K\Delta\{1,n\} \in \mathcal{I}_1$. This implies that $Y \succeq 0$. $\square$

Lemma 14. *Let $t \geq 0$, $Y = \widetilde{M}_{t+1}(y) \in \mathcal{F}_t(n)$ and let X be the principal submatrix of Y indexed by $\mathcal{U}_{t+1}(V \setminus \{n\})$. If $y_{1n} = \epsilon = \pm 1$, then Y is the ϵ-extension of X.*

Proof. We have to show that $y_I = \epsilon \cdot y_{I\Delta\{1,n\}}$ for all $I \in \mathcal{E}_{2t+2}(V)$ containing n. For this, let $H \in \mathcal{U}_{t+1}(V)$ contain n; then $H\Delta\{1,n\} \in \mathcal{U}_{t+1}(V)$. As $Y(H, H\Delta\{1,n\}) = y_{1n} = \epsilon$, Lemma 1 (ii) implies that $Ye_H = \epsilon \cdot Ye_{H\Delta\{1,n\}}$ and thus for all $K \in \mathcal{U}_{t+1}(V)$, $Y(H,K) = \epsilon \cdot Y(H\Delta\{1,n\},K)$ which yields $y(H\Delta K) = \epsilon \cdot y(H\Delta K\Delta\{1,n\})$, concluding the proof. □

Corollary 15. *Let $t \geq 0$, $\epsilon = \pm 1$, $x \in \mathbb{R}^F$, and $y \in \mathbb{R}^E$ its ϵ-extension. Then, $x \in Q_t(G/e) \Longleftrightarrow y \in Q_t(G)$.*

Proof. Say, e is the edge $1n$ in order to match the notation in Lemma 13 and 14. Suppose first that $x \in Q_t(G/e)$ and let $X \in \mathcal{F}_t(n-1)$ whose projection on $\mathbb{R}^F$ is x. Then the ϵ-extension Y of X belongs to $\mathcal{F}_t(n)$ (by Lemma 13) and its projection on $\mathbb{R}^E$ is equal to y, which shows that $y \in Q_t(G)$. Conversely suppose that $y \in Q_t(G)$ and let $Y \in \mathcal{F}_t(n)$ whose projection on $\mathbb{R}^E$ is y. Then the principal submatrix X of Y indexed by $\mathcal{U}_{t+1}(V \setminus \{n\})$ belongs to $\mathcal{F}_t(n-1)$ and Y is the ϵ-extension of X by Lemma 14. As the projection of X on $\mathbb{R}^F$ is equal to x, we deduce that $x \in Q_t(G/e)$. □

As a first application, we can show that $\mathcal{L}_t$ is closed under taking minors.

Theorem 16. *Given $t \geq 0$, if $\mathrm{CUT}(G) = Q_t(G)$ then $\mathrm{CUT}(G/e) = Q_t(G/e)$ and $\mathrm{CUT}(G\backslash e) = Q_t(G\backslash e)$.*

Proof. Let $x \in Q_t(G/e)$ and let $y \in \mathbb{R}^E$ be its 1-extension. By Corollary 15, y belongs to $Q_t(G)$; hence $y \in \mathrm{CUT}(G)$ which, by (27), implies that x belongs to $\mathrm{CUT}(G/e)$. Equality $\mathrm{CUT}(G\backslash e) = Q_t(G\backslash e)$ follows from the fact that $\mathrm{CUT}(G\backslash e)$ (resp., $Q_t(G\backslash e)$) is the projection on $\mathbb{R}^{E\backslash\{e\}}$ of $\mathrm{CUT}(G)$ (resp., of $Q_t(G)$). □

As a second application, we can show the following result.

Theorem 17. *Given $t \geq 0$, if $\mathrm{CUT}(G/e) = Q_t(G/e)$, then $\mathrm{CUT}(G) = Q_{t+1}(G)$.*

Proof. By Theorem 11, $Q_{t+1}(G)$ is contained in $N_+(Q_t(G))$ which, in turn, is contained in $\mathrm{conv}(Q_t(G) \cap \{y \mid y_e = \pm 1\})$ by (13). Hence, it suffices to show that $Q_t(G) \cap \{y \mid y_e = \pm 1\} \subseteq \mathrm{CUT}(G)$. Let $y \in Q_t(G)$ with $y_e = \epsilon \in \{\pm 1\}$. By Corollary 15, y is the ϵ-extension of $x \in Q_t(G/e)$. By assumption, $x \in \mathrm{CUT}(G/e)$ which, by (27), implies that $y \in \mathrm{CUT}(G)$. □

We will see in Section 5 that

$$K_3 \in \mathcal{L}_1 \setminus \mathcal{L}_0,\ K_5 \in \mathcal{L}_2 \setminus \mathcal{L}_1,\ K_6 \in \mathcal{L}_2 \setminus \mathcal{G}_1,\ K_7 \notin \mathcal{L}_2.$$

The class $\mathcal{L}_0$ consists of the graphs with no K_3 minor (indeed $K_3 \notin \mathcal{L}_0$ and if G has no K_3 minor, then $\mathrm{CUT}(G) = [-1,1]^E$ is thus equal to $Q_0(G)$). The class $\mathcal{L}_1$ consists of the graphs having no K_5 minor (indeed $K_5 \notin \mathcal{L}_1$ and if G has no K_5 minor, then $\mathrm{CUT}(G) = \mathrm{MET}(G)$ is thus equal to $Q_1(G)$). The graph K_7 is a forbidden minor for the class $\mathcal{L}_2$ (we do not know whether K_7 is a *minimal* forbidden minor). There are other forbidden minors for the class $\mathcal{L}_2$ since the max-cut problem is known to be NP-hard for the class of graphs having no K_6 minor (in fact, also for the graphs having a node whose deletion results in a planar graph) [5].

One can show that the class $\mathcal{L}_t$ is closed under taking clique k-sums ($k = 0, 1, 2, 3$); the same holds for the class $\mathcal{G}_t$ [21] (the proof for $\mathcal{L}_t$ is analogous to that for $\mathcal{G}_t$). The next result follows as an application of Theorem 17, Corollary 12, and the fact from [21] that $\mathrm{CUT}(G) = N^t(G)$ for $t := \max(0, n - \alpha(G) - 3)$.

Corollary 18. *$\mathrm{CUT}(K_n) = Q_{n-4}(K_n)$ for $n \geq 6$ and, for a graph G on n nodes with stability number $\alpha(G)$, $\mathrm{CUT}(G) = Q_t(G)$ for $t := \max(1, n - \alpha(G) - 2)$.*

4 Geometric Properties of the Matrix Sets $\mathcal{F}_t(n)$

In this section we study some geometric properties of the matrix sets $\mathcal{F}_t(n)$ underlying the Lasserre relaxations $Q_t(K_n)$.

Let us first recall some definitions. A convex subset F of a convex set $\mathcal{K}$ is called a *face* of $\mathcal{K}$ if, for all $x \in F$, $y, z \in \mathcal{K}$, $0 < \alpha < 1$, $x = \alpha y + (1 - \alpha)z$ implies that $y, z \in F$. Given $x \in \mathcal{K}$, let $F(x)$ denote the smallest face of $\mathcal{K}$ containing x. A point $x \in \mathcal{K}$ is an *extreme point* if $F(x) = \{x\}$ and a *vertex* if its normal cone is full dimensional. One says that the points $x_1, \ldots, x_k \in \mathcal{K}$ *form a face* of $\mathcal{K}$ if the set $\mathrm{conv}(\{x_1, \ldots, x_k\})$ is a face of $\mathcal{K}$.

Consider a convex set $\mathcal{K}$ of the form

$$\mathcal{K} = \{X \in \mathrm{PSD}_n \mid \langle A_i, X \rangle = b_i \text{ for } i = 1, \ldots, m\}$$

where the A_i's are symmetric matrices and $b_i \in \mathbb{R}$. It follows from a result in [11] that the smallest face $F(A)$ of $\mathcal{K}$ containing a given element $A \in \mathcal{K}$ is given by

$$F(A) = \{X \in \mathcal{K} \mid \ker X \supseteq \ker A\}.$$

This description of the faces applies in particular to any set $\mathcal{F}_t(n)$.

Analogously to $\mathrm{CUT}(K_n)$, the set $\mathcal{F}_t(n)$ enjoys lots of symmetries. In particular, it is invariant under any permutation of the indices in $\{1, \ldots, n\}$ and under the following "switching" operation.

Lemma 19. *Given $A \subseteq V$, the* switching mapping*:*

$$r_A : Y \mapsto r_A(Y) := \mathrm{diag}(\zeta^A)\ Y\ \mathrm{diag}(\zeta^A)$$

leaves the set $\mathcal{F}_t(n)$ invariant, where $\zeta^A := ((-1)^{|A \cap H|})_{H \in \mathcal{U}_{t+1}(V)}$.

Proof. Let $Y = \widetilde{M}_{t+1}(y) \in \mathcal{F}_t(n)$. Then, $r_A(Y) = \widetilde{M}_{t+1}(z)$ where $z(I) := (-1)^{|A \cap I|} y(I)$ for all $I \in \mathcal{E}_{2t+2}(V)$. Therefore, $r_A(Y)$ is a reduced moment matrix and $r_A(Y) \succeq 0$ by the definition of r_A. $\square$

The matrix set $\mathcal{F}_t(n)$ permits to formulate the following semidefinite relaxation for the max-cut problem:

$$\max\ \frac{1}{2} w(E) - \frac{1}{2} \sum_{ij \in E} w_{ij} y_{ij} \ \text{ subject to } \ \widetilde{M}_{t+1}(y) \succeq 0,\ y_\emptyset = 1. \tag{28}$$

Let $Y := \widetilde{M}_{t+1}(y) \in \mathcal{F}_t(n)$ be an optimum solution to the program (28) and let $X := \widetilde{M}_1(y)$; hence the off-diagonal entries of X are y_{ij} ($ij \in E_n$). If the rank of X is equal to 1, then X is a cut matrix and, therefore, the program (28) solves the max-cut problem exactly. An interesting question is to find conditions on the rank of X ensuring that X can be written as a convex combination of cut matrices and, therefore, that the relaxation (28) solves the max-cut problem.

For $t = 0$, there exist matrices $X \in \mathcal{F}_0(n)$ with rank 2 that cannot be written as a convex combination of cut matrices. For $t = 1$, Anjos and Wolkowicz [2] show that if X has rank ≤ 2, then Y (and thus X) can be written as a convex combination of two cut matrices. This result can be generalized for any $t \geq 2$.

Theorem 20. *Let $t \geq 0$, $n \geq t+2$, $Y = \widetilde{M}_{t+1}(y) \in \mathcal{F}_t(n)$ and $X := \widetilde{M}_1(y)$. If* $\operatorname{rank}\ X \leq t+1$, *then Y can be written as a convex combination of 2^t cut matrices and, therefore, the vector $(y_{ij})_{ij \in E_n}$ belongs to* $\mathrm{CUT}(K_n)$.

The proof of Theorem 20 will be given in Section 4.1. We now examine some properties of the faces of the convex set $\mathcal{F}_t(n)$. All the 2^{n-1} cut matrices are vertices of $\mathcal{F}_t(n)$ (since they have rank 1). It is shown in [24] that the cut matrices are the *only* vertices of $\mathcal{F}_0(n)$. Moreover, it is shown in [23] that any two distinct cut matrices form a one dimensional face of $\mathcal{F}_0(n)$. This adjacency property extends to each of the matrix sets $\mathcal{F}_t(n)$.

Proposition 21. *Let R and S be distinct subsets of $\{1,\ldots,n\}$ and $t \geq 0$. Then the set* $\mathrm{conv}(\{\widetilde{M}_{t+1}(R), \widetilde{M}_{t+1}(S)\})$ *is a face of $\mathcal{F}_t(n)$.*

Proof. In view of the switching symmetry from Lemma 19, we can assume w.l.o.g. that $R = \emptyset$. Set $Y_0 := \widetilde{M}_{t+1}(\emptyset)$, $Y_1 := \widetilde{M}_{t+1}(S)$, and $A := \frac{1}{2}(Y_0 + Y_1)$. Then A has the block decomposition: $A = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$ with respect to the partition of its index set $\mathcal{P}_{2t+2}(V)$ into the sets $\mathcal{I}_o$ and $\mathcal{I}_e$ consisting, respectively, of the sets I having an odd or even intersection with S. We show that $F(A)$, the smallest face of $\mathcal{F}_t(n)$ containing A, is equal to the interval $[Y_0, Y_1]$. For this, let $Y \in F(A)$; that is, $Y \in \mathcal{F}_t(n)$ and $\ker A \subseteq \ker Y$. Then all the columns of Y indexed by sets in $\mathcal{I}_o$ (resp. in $\mathcal{I}_e$) are identical. As Y is symmetric positive semidefinite with an all ones diagonal, we find that Y has the block decomposition: $Y = \begin{pmatrix} J & \alpha J \\ \alpha J & J \end{pmatrix}$ for some scalar $\alpha \in [-1, 1]$. Therefore, $Y = \frac{1+\alpha}{2} Y_0 + \frac{1-\alpha}{2} Y_1$. This concludes the proof. □

Barahona and Mahjoub [6] had shown earlier that any two cuts form an edge of the cut polytope $\mathrm{CUT}(K_n)$. Padberg [28] showed moreover that any two cuts form an edge of the metric polytope $\mathrm{MET}(K_n)$; that is, $\mathrm{MET}(K_n)$ has the Trubin property with respect to $\mathrm{CUT}(K_n)$. This implies that any Lasserre relaxation $Q_t(K_n)$ also has the Trubin property. (For $t \geq 1$ this is true since $Q_t(K_n) \subseteq \mathrm{MET}(K_n)$ and, for $t = 0$, this is true by the above mentioned result of [23] since $Q_0(n)$ is a linear bijective image of $\mathcal{F}_0(n)$.) In fact, $\mathrm{CUT}(K_n)$ and $\mathrm{MET}(K_n)$ have lots of higher dimensional common faces; for instance, any three cuts or any set $\delta(S_1), \ldots, \delta(S_k)$

of cuts in general position (meaning that each cell in the Venn diagram of the sets $S_1, \ldots, S_k$ is non empty) form a face of $\mathrm{MET}(K_n)$ and thus of $\mathrm{CUT}(K_n)$ [10].

One may wonder whether some analogous result holds for the matrix set $\mathcal{F}_t(n)$. We saw above that any two cut matrices form a face of $\mathcal{F}_0(n)$; note that this does not extend to a set of three cut matrices (consider, e.g., $\mathcal{F}_0(3)$). On the other hand, Corollary 9 shows that $\mathcal{F}_{n-2}(n)$ is a simplex with the cut matrices as vertices. This suggests the following question.

Problem 22. Is it true that, for $t = 0, \ldots, n-2$, any set of 2^{t+1} cut matrices forms a face of $\mathcal{F}_t(n)$?

If this property holds, this suggests a "continuous" evolution from $\mathcal{F}_0(n)$ to the final simplex $\mathcal{F}_{n-2}(n)$. We saw above that the answer is yes for the two extreme values $t = 0$ and $t = n-2$ and the next result shows the answer is also positive for the next case $t = 1$. The proof of Theorem 23 is delayed till Section 4.2.

Theorem 23. *Any four cut matrices form a face of $\mathcal{F}_1(n)$.*

4.1 Proof of Theorem 20

A preliminary result

We begin with showing the auxiliary result from Proposition 25 that will play a central role in the proof of Theorem 20. The following lemma will be used as base of induction.

Lemma 24. *Let $Y = \widetilde{M}_2(y) \in \mathcal{F}_1(3)$ and $X := \widetilde{M}_1(y)$. If $\operatorname{rank} X \leq 2$, then $y_{ij} = \pm 1$ for some pair ij with $1 \leq i < j \leq 3$.*

Proof. As $\operatorname{rank} X \leq 2$, there exists a non zero vector $u \in \ker X$; thus $v := \binom{u}{0}$ belongs to $\ker Y$. Therefore, Y has at least two zero eigenvalues (since none of the eigenvectors y^A of Y has a zero coordinate). This implies that the vector (y_{12}, y_{13}, y_{23}) satisfies at least two of the equalities: $y_{12} + y_{13} + y_{23} = -1$, $y_{12} - y_{13} - y_{23} = -1$, $y_{13} - y_{12} - y_{23} = -1$, $y_{23} - y_{12} - y_{13} = -1$. Therefore, one of the components y_{ij} of y is equal to ± 1. $\square$

Proposition 25. *Let $Y = \widetilde{M}_{n-1}(y) \in \mathcal{F}_{n-2}(n)$ and $X := \widetilde{M}_1(y)$. If $\operatorname{rank} X \leq n-1$, then there exists a non empty set I of even cardinality for which $y_I = \pm 1$.*

Proof. The proof is by induction on $n \geq 3$. The result holds for $n = 3$ by Lemma 24. Let $n \geq 4$ and suppose that the result holds for all $p \leq n-1$; we show that it also holds for n. We know from Lemma 8 that the eigenvectors of Y are the cut vectors y^A for $A \subseteq \{1, \ldots, n-1\}$ with corresponding eigenvalues $y^T y^A$. Set $V := \{1, \ldots, n\}$, $U := \{1, \ldots, n-1\}$ and

$$\mathcal{A} := \{A \subseteq U \mid y^T y^A \neq 0\},\ \mathcal{B} := \{A \subseteq U \mid y^T y^A = 0\}. \tag{29}$$

Thus $\ker Y$ is spanned by the vectors y^A for $A \in \mathcal{B}$ and

$$y = \sum_{A \in \mathcal{A}} \lambda_A y^A \tag{30}$$

where $\lambda_A = \frac{1}{2^{n-1}} y^T y^A > 0$ for all $A \in \mathcal{A}$.

Let $R_{\mathcal{A}}$ denote the matrix of order $|\mathcal{A}| \times n$ whose rows are the ± 1 incidence vectors ψ^A of the sets $A \in \mathcal{A}$ (viewing A as a subset of V). We claim that

$$\ker X = \ker R_{\mathcal{A}}. \tag{31}$$

Indeed, let $u \in \mathbb{R}^n$. Then, u belongs to $\ker X$ if and only if the extended vector $v := \begin{pmatrix} u \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{\mathcal{E}(V)}$ belongs to $\ker Y$ which, in turn, is equivalent to the fact that $v^T y^A = 0$ for all $A \in \mathcal{A}$. As the projection of y^A on the subspace $\mathbb{R}^n$ indexed by $\emptyset$ and the pairs $12, \ldots, 1n$ is equal to $\pm\psi^A$ (the sign $\pm$ depending whether $1 \in A$), we have that $v^T y^A = \pm u^T \psi^A$. Therefore, $u \in \ker X$ if and only if $R_{\mathcal{A}} u = 0$ and thus (31) holds. We can assume that

$$\text{rank } X = n - 1.$$

Indeed, if rank $X \leq n-2$, then consider the principal submatrix Y' (resp., X') of Y (resp., of X) indexed by subsets of $\{1, \ldots, n-1\}$; as rank $X' \leq n-2$, the induction assumption implies that $y_I = \pm 1$ for some nonempty even set $I \subseteq \{1, \ldots, n-1\}$ and thus Proposition 25 holds. Together with (31), this implies that

$$\text{rank } R_{\mathcal{A}} = n - 1. \tag{32}$$

In view of (30), we have $y_I = 1$ (resp., -1) for a set $I \in \mathcal{E}(V)$ if and only if $|I \cap A|$ is even (resp., odd) for all $A \in \mathcal{A}$. Thus we are left with the task of showing the existence of a non empty set $I \in \mathcal{E}(V)$ and of $\epsilon \in \{0, 1\}$ satisfying $|I \cap A| \equiv \epsilon$ (modulo 2) for all $A \in \mathcal{A}$. Let $M_{\mathcal{A}}$ denote the matrix of order $|\mathcal{A}| \times (n-1)$ whose rows are the 0/1 incidence vectors $\chi^A \in \{0,1\}^{n-1}$ of the sets $A \in \mathcal{A}$ (viewed as subsets of $U = \{1, \ldots, n-1\}$). Equivalently, we have to show that at least one of the following two systems in the binary variable $x \in \text{GF}(2)^{n-1}$

$$\begin{cases} M_{\mathcal{A}} x = 0, \;\; x \neq 0 \\ M_{\mathcal{A}} x = e \end{cases}$$

has a solution; e denoting the all ones vector. Indeed, if x is a solution of one of the above two systems and $I_0 := \{i \in \{1, \ldots, n-1\} \mid x_i = 1\}$, then $I := I_0$ (resp., $I := I_0 \cup \{n\}$) is the required non empty even set with $y_I = \pm 1$ if $|I_0|$ is even (resp., odd).

If the matrix $M_{\mathcal{A}}$ has rank $\leq n-2$ over GF(2) then the system $M_{\mathcal{A}} x = 0$ has a non zero solution over GF(2) and we are done. A collection of sets $A_1, \ldots, A_p \in \mathcal{A}$ is said to form a GF(2)-dependency if $\Delta_{i=1}^p A_i = \emptyset$. We claim:

> If the rank of $M_{\mathcal{A}}$ over GF(2) is equal to $n-1$ and every $GF(2)-$dependency in $\mathcal{A}$ involves an even number of sets, then the system $M_{\mathcal{A}} x = e$ has a solution over $GF(2)$. (33)

To see it, let $A_1, \ldots, A_{n-1} \in \mathcal{A}$ whose incidence vectors are linearly independent over GF(2) and let x be a solution to the system

$$\chi^{A_i} x = 1 \mod 2 \ (i = 1, \ldots, n-1).$$

Then $M_{\mathcal{A}} x = e$ holds. Indeed, if $A \in \mathcal{A} \backslash \{A_1, \ldots, A_{n-1}\}$, then $A = \Delta_{i \in P} A_i$ where P is an odd subset of $\{1, \ldots, n-1\}$. Therefore, $\chi^A x \equiv \sum_{i \in P} \chi^{A_i} x \equiv |P| \equiv 1 \bmod 2$. Thus (33) holds. In order to conclude the proof of Proposition 25, it suffices now to show the following result.

Lemma 26. *If there exists a* GF(2)*-dependency in $\mathcal{A}$ involving an odd number of sets, then the rank of the matrix $M_{\mathcal{A}}$ over* GF(2) *is $\leq n-2$.*

Proof of Lemma 26. Let $A_0 = \Delta_{i=1}^p A_i$ be a smallest GF(2)-dependency in $\mathcal{A}$ involving an odd number of sets; that is, p is even. Hence the vectors $\chi^{A_1}, \ldots, \chi^{A_p}$ are linearly independent over GF(2). Suppose for a contradiction that $M_{\mathcal{A}}$ has rank $n-1$. Let $A_{p+1}, \ldots, A_{n-1} \in \mathcal{A}$ whose incidence vectors together with those of $A_1, \ldots, A_p$ are linearly independent over GF(2). We claim:

$$\text{The vectors } \chi^{A_i \cup \{n\}} \ (i = 0, 1, \ldots, n-1) \text{ are linearly independent over } GF(2). \tag{34}$$

Suppose not. Then $\Delta_{i \in H}(A_i \cup \{n\}) = \emptyset$ for some set $H \subseteq \{0, 1, \ldots, n-1\}$. Therefore, $|H|$ is even (in order to eliminate n) and $\Delta_{i \in H} A_i = \emptyset$, which implies that $0 \in H$ (since the vectors $\chi^{A_1}, \ldots, \chi^{A_{n-1}}$ are linearly independent over $GF(2)$). This combined with the fact that $A_0 = \Delta_{i=1}^p A_i$ implies that $H \setminus \{0\} = \{1, \ldots, p\}$. We reach a contradiction since p is even while $H \setminus \{0\}$ has an odd cardinality. Thus (34) holds. This implies:

$$\text{The vectors } \chi^{A_i \cup \{n\}} \ (i = 0, 1, \ldots, n-1) \text{ are linearly independent over } \mathbb{R}. \tag{35}$$

Indeed, suppose not. Then $\sum_{i=0}^{n-1} \lambda_i \chi^{A_i \cup \{n\}} = 0$ for some $\lambda_i \in \mathbb{R}$ not all equal to 0. Such λ_i exist that are rational valued and thus integer valued, not all of them even. Taking a reduction modulo 2 we find a linear dependency over $GF(2)$ contradicting (34). Finally we show:

$$\text{The vectors } \psi^{A_i} \ (i = 0, 1, \ldots, n-1) \ \text{ are linearly independent over } \mathbb{R}. \tag{36}$$

For this note that $\psi^{A_i} = \begin{pmatrix} -2I & e \\ 0 & 1 \end{pmatrix} \chi^{A_i \cup \{n\}}$ (where I is the identity matrix of order $n-1$ and e is the all ones vector of length $n-1$) and that the matrix $\begin{pmatrix} -2I & e \\ 0 & 1 \end{pmatrix}$ is nonsingular. We reach a contradiction since the matrix $R_{\mathcal{A}}$ has rank $n-1$ by (32). This concludes the proof of Lemma 26 and in turn the proof of Proposition 25. □

Corollary 27. *Let $Y = \widetilde{M}_{n-1}(y) \in \mathcal{F}_{n-2}(n)$ and $X := \widetilde{M}_1(y)$. If* rank $X \leq n-1$ *then Y can be written as a convex combination of at most 2^{n-2} cut matrices.*

Proof. As in the proof of Proposition 25, let $\mathcal{A}$ be defined by (29) and, as in (30), let $y = \sum_{A \in \mathcal{A}} \lambda_A y^A$, $Y = \sum_{A \in \mathcal{A}} \lambda_A \widetilde{M}_{n-1}(y^A)$ where $\lambda_A > 0$ for all $A \in \mathcal{A}$ and

$\sum_A \lambda_A = y_\emptyset = 1$. It suffices now to show that $|\mathcal{A}| \leq 2^{n-2}$. By Proposition 25, there exists a non empty set $I \in \mathcal{E}(V)$ for which $y_I = \pm 1$. Therefore, the 0/1 incidence vectors of the sets $A \in \mathcal{A}$ are solutions of the equation $\chi^I x = \epsilon \bmod 2$, where $\epsilon = 0$ if $y_I = 1$ and $\epsilon = 1$ if $y_I = -1$. This implies that $|\mathcal{A}| \leq 2^{n-2}$. □

Proof of Theorem 20

The proof of Theorem 20 is by induction on n. The case $n = t+2$ has been settled in Corollary 27. Hence we can assume that $n \geq t+3$ and that the result of Theorem 20 holds for $n-1$. We can also assume that $t \geq 1$ as the result holds obviously for $t = 0$. As before $V = \{1, \ldots, n\}$ and $\mathcal{U}_{t+1}(V)$ is the set indexing the matrix Y, which consists of the sets $H \subseteq V$ with $|H| \leq t+1$ and $|H| \equiv t+1 \bmod 2$. Set

$$\mathcal{I} := \{I \in \mathcal{E}_{2t+2}(V) \mid I \neq \emptyset \text{ and } y_I = \pm 1\}.$$

Consider the relation on $\mathcal{U}_{t+1}(V)$: Given $H, K \in \mathcal{U}_{t+1}(V)$,

$$\begin{array}{c} H \sim K \text{ if } y(H\Delta K) = \pm 1, \text{ i.e., } H\Delta K \in \mathcal{I} \cup \{\emptyset\} \\ \text{or, equivalently, if } Ye_H = \pm Ye_K, \end{array} \tag{37}$$

where Ye_H denotes the H-th column of Y (the last equivalence in (37) follows using Lemma 1 (ii)). This is obviously an equivalence relation on $\mathcal{U}_{t+1}(V)$. We begin with some preliminary results about the collection $\mathcal{I}$.

Lemma 28. *If $I \neq J \in \mathcal{I}$ with $|I \setminus J|, |J \setminus I| \leq t+1$, then $I\Delta J \in \mathcal{I}$.*

Proof. As I, J have an even cardinality, the three sets $I \setminus J$, $J \setminus I$ and $I \cap J$ have the same parity. Say, $|I \setminus J| \geq |J \setminus I|$. Suppose first that $|I \cap J| \leq t+1$. If $|I \cap J| \equiv t+1 \bmod 2$, then $I \setminus J,\ J \setminus I,\ I \cap J \in \mathcal{U}_{t+1}(V)$ with $I \setminus J \sim I \cap J$ (since $I \in \mathcal{I}$) and $I \cap J \sim J \setminus I$ (since $J \in \mathcal{I}$). Therefore, $I \setminus J \sim J \setminus I$ implying that $(I \setminus J)\Delta(J \setminus I) = I\Delta J \in \mathcal{I}$. If $|I \cap J| \not\equiv t+1 \bmod 2$, let $a \in I \setminus J$. Then, $I \setminus (J \cup \{a\})$, $(I \cap J) \cup \{a\}$, $(J \setminus I) \cup \{a\} \in \mathcal{U}_{t+1}(V)$ with $I \setminus (J \cup \{a\}) \sim (I \cap J) \cup \{a\}$ and $(I \cap J) \cup \{a\} \sim (J \setminus I) \cup \{a\}$, which implies that $I\Delta J \in \mathcal{I}$.

Suppose now that $|I \cap J| > t+1$. Let A be a subset of $I \cap J$ for which $|A| + |I \setminus J| = t+1$ and set $B := (I \cap J) \setminus A$. Then, $|A| + |J \setminus I| \leq t+1$, $|A| + |J \setminus I| \equiv t+1 \bmod 2$, $|B| = |I| - t - 1 \leq t+1$, and $|B| \equiv t+1 \bmod 2$. As $(I \setminus J) \cup A \sim B$ and $B \sim (J \setminus I) \cup A$, we deduce again that $I\Delta J \in \mathcal{I}$. □

As a consequence of Proposition 25 we have the following result.

Lemma 29. *For any subset $T \subseteq V$ with $|T| \geq t+2$, there exists a set $I \in \mathcal{I}$ which is contained in T; moreover, if $|T| \geq t+3$, such I exists having cardinality $\leq t+1$.*

Proof. The first part of the lemma is a direct application of Proposition 25. (Indeed let $S \subseteq T$ with $|S| = t+2$ and let Y' (resp., X') be the principal submatrix of Y (resp., of X) indexed by subsets of S; thus $Y' \in \mathcal{F}_t(t+2)$ with rank $X' \leq t+1$ and Proposition 25 implies the existence of a set $I \in \mathcal{I}$ contained in S.) Suppose now that $|T| \geq t+3$ and let $T_1, T_2 \subseteq T$ with $|T_1| = |T_2| = t+2$ and $|T_1 \Delta T_2| = 2$. If T_1 or T_2 contains a member of $\mathcal{I}$ of size $\leq t+1$, then we are done. Otherwise, both T_1 and T_2 belong to $\mathcal{I}$ which, using Lemma 28, implies that $T_1 \Delta T_2 \in \mathcal{I}$. Thus we have found a member of $\mathcal{I}$ contained in T of size $2 \leq t+1$. $\square$

Choose a set $I_0 \in \mathcal{I}$ having the minimum cardinality among all sets in $\mathcal{I}$. Then, $|I_0| \leq t+1$ by Lemma 29 (since $|V| \geq t+3$). We can assume without loss of generality that the element $n \in V$ belongs to I_0. Set

$$\mathcal{I}(n) := \{I \in \mathcal{I} \mid n \in I\},\ \mathcal{I}(\overline{n}) := \mathcal{I} \setminus \mathcal{I}(n).$$

The rest of the proof of Theorem 20 can be sketched as follows. Let Y_0 denote the principal submatrix of Y indexed by $\mathcal{U}_{t+1}(V \setminus \{n\})$ and let y_0 denote the projection of y onto the subspace $\mathbb{R}^{\mathcal{E}_{2t+2}(V\setminus\{n\})}$; thus $Y_0 = \widetilde{M}_{t+1}(y_0)$. Using the induction assumption, we know that Y_0 can be decomposed as a convex combination of at most 2^t cut matrices or, equivalently, that y_0 can be written as a convex combination of at most 2^t cut vectors; that is,

$$Y_0 = \sum_{A \in \mathcal{A}} \lambda_A \widetilde{M}_{t+1}(A),\ y_0 = \sum_{A \in \mathcal{A}} \lambda_A y^A, \tag{38}$$

where $\lambda_A > 0$ for $A \in \mathcal{A}$, $\sum_A \lambda_A = 1$, and $\mathcal{A}$ is a collection of subsets of $V \setminus \{n\}$ with $|\mathcal{A}| \leq 2^t$. Our goal is now to show that the above decomposition of y_0 can be extended to a decomposition of y. Namely, we will show that for each set $A \in \mathcal{A}$ one can define $A' := A$ or $A \cup \{n\}$ in such a way that the identity: $y = \sum_{A \in \mathcal{A}} \lambda_A y^{A'}$ holds, which implies then that Y can be written as a convex combination of at most 2^t cut matrices, concluding the proof.

Lemma 30. *For any $H \in \mathcal{U}_{t+1}(V)$, there exists a set $I \in \mathcal{I}(n)$ for which $|H \Delta I| \leq t+1$.*

Proof. Suppose not. Let $H \in \mathcal{U}_{t+1}(V)$ be a counterexample (that is, $|H \Delta I| \geq t+2$ for all $I \in \mathcal{I}(n)$) of minimum cardinality. Choose $I_1 \in \mathcal{I}(n)$ with $|I_1| \leq t+1$ and for which $|I_1 \Delta H|$ is minimum. Then, $|H \Delta I_1| \geq t+2$, implying $|H \Delta I_1| \geq t+3$ since $|H \Delta I_1| \equiv |H| \equiv t+1$ (modulo 2). By Lemma 29, there exists $I \in \mathcal{I}$ such that $I \subseteq (H \Delta I_1) \setminus \{n\}$. Then, the set $J := I \Delta I_1$ belongs to $\mathcal{I}$ by Lemma 28 (since $I \setminus I_1 \subseteq H$ and $I_1 \setminus I \subseteq I_1$ have size $\leq t+1$) and thus $J \in \mathcal{I}(n)$. Suppose first that $|I \setminus I_1| \leq |I \cap I_1|$. Then, $|J| = |I \setminus I_1| + |I_1 \setminus I| \leq |I_1| \leq t+1$ and $|H \Delta J| = |H \Delta I_1| - |I| < |H \Delta I_1|$ which contradicts the minimality assumption made about the set I_1. Therefore, $|I \setminus I_1| > |I \cap I_1|$. This implies that $|H \Delta I| = |H| - |I \setminus I_1| + |I \cap I_1| < |H|$ and thus $H \Delta I \in \mathcal{U}_{t+1}(V)$. Therefore, by the minimality assumption made about H, the set $H \Delta I$ is not a counterexample to the lemma and thus there exists a set $J \in \mathcal{I}(n)$ for which $|H \Delta I \Delta J| \leq t+1$. Now, $H \Delta I \Delta J \in \mathcal{U}_{t+1}(V)$, $H \Delta I \Delta J \sim H \Delta I$ (since $J \in \mathcal{I}$), $H \Delta I \sim H$ (since $I \in \mathcal{I}$) which implies that $H \Delta I \Delta J \sim H$ and thus $I \Delta J \in \mathcal{I}$. As $I \Delta J \in \mathcal{I}(n)$ with $|H \Delta I \Delta J| \leq t+1$, we contradict our assumption that H is a counterexample to the lemma. $\square$

Denote by $\mathcal{R}_1, \ldots, \mathcal{R}_m$ the equivalence classes of the equivalence relation $\sim$ from (37) on $\mathcal{U}_{t+1}(V)$. By Lemma 30, each class $\mathcal{R}_j$ contains sets $H, K \in \mathcal{U}_{t+1}(V)$ with $n \in H \setminus K$. For $j = 1, \ldots, m$ set

$$\mathcal{I}_j(n) := \{H\Delta K \mid H, K \in \mathcal{R}_j,\ n \in H \setminus K\}.$$

Thus, $\mathcal{I}(n) = \bigcup_{j=1}^m \mathcal{I}_j(n)$. Recall that I_0 is a member of $\mathcal{I}(n)$ having minimum cardinality. The following property of I_0 will play a crucial role in the rest of the proof.

Lemma 31. $I_0 \in \bigcap_{j=1}^{m} \mathcal{I}_j(n)$.

Proof. Fix $j = 1$; we show that $I_0 \in \mathcal{I}_1(n)$. Let H be a member of $\mathcal{R}_1$ containing n for which $|H\Delta I_0|$ is minimum. If $|H\Delta I_0| \leq t+1$, then $H\Delta I_0 \in \mathcal{U}_{t+1}(V)$, $H\Delta I_0 \sim H$ which implies that $I_0 \in \mathcal{I}_1(n)$ and we are done. Suppose now that $|H\Delta I_0| \geq t+2$ and thus $|H\Delta I_0| \geq t+3$. Let $I \in \mathcal{I}$ such that $I \subseteq H\Delta I_0$ and $|I| \leq t+1$ (apply Lemma 29). Then, $I\Delta I_0 \in \mathcal{I}(n)$ (by Lemma 28) and thus $|I\Delta I_0| \geq |I_0|$ by the choice of I_0. Therefore, $|I \setminus I_0| \geq |I \cap I_0|$. Then, $|H\Delta I| = |H| - |I \setminus I_0| + |I \cap I_0| \leq |H| \leq t+1$ and, hence, $H\Delta I \in \mathcal{U}_{t+1}(V)$. Now, $H\Delta I \sim H$ implying that $H\Delta I \in \mathcal{R}_1$. As $n \in H\Delta I$ and $|H\Delta I\Delta I_0| < |H\Delta I_0|$, we reach a contradiction with the way we have chosen H. $\square$

As a consequence of (38) and of the definition (24) of y^A, we have that

$$y_I = (-1)^{|A\cap I|} \text{ for all } A \in \mathcal{A} \text{ and } I \in \mathcal{I}(\overline{n}). \tag{39}$$

We now observe that we can assume without loss of generality that

$$y_I = 1 \text{ for all } I \in \mathcal{I}(\overline{n}). \tag{40}$$

Indeed, fix $A \in \mathcal{A}$ and consider the matrix $Y' := r_A(Y)$ obtained using the switching symmetry described in Lemma 19. Then, $Y' = \widetilde{M}_{t+1}(y')$ where $y'_I = (-1)^{|A\cap I|} y_I$ for all $I \in \mathcal{E}_{2t+2}(V)$. Therefore, in view of (39), $y'_I = 1$ for all $I \in \mathcal{I}(\overline{n})$. If we can show that Y' is a convex combination of 2^t cut matrices then the same holds for $Y = r_A(Y')$. Hence, replacing Y by Y', we can assume that (40) holds. Moreover, we can assume without loss of generality that

$$y_{I_0} = 1. \tag{41}$$

Indeed, if $y_{I_0} = -1$ then we can replace Y by its switching $r_{\{n\}}(Y)$.

Lemma 32. *For all $I \in \mathcal{I}(n)$, $y_I = 1$ and $|A\cap I| \equiv |A \cap I_0| \pmod 2$ for all $A \in \mathcal{A}$.*

Proof. Let $I \in \mathcal{I}(n)$; then $I, I_0 \in \mathcal{I}_j(n)$ for some $j = 1, \ldots, m$ by Lemma 31. Hence, $I_0 = H_0\Delta K_0$, $I = H\Delta K$ where $H, H_0, K, K_0 \in \mathcal{R}_j$ with $n \in H \cap H_0$ and $n \notin K \cup K_0$. As $H\Delta H_0, K\Delta K_0 \in \mathcal{I}(\overline{n})$, we deduce from (40) that $y(H\Delta H_0) = y(K\Delta K_0) = 1$. Together with $y(H_0\Delta K_0) = 1$ from (41), this implies that $Ye_H = Ye_{H_0}$, $Ye_K =$

Ye_{K_0}, $Ye_{H_0} = Ye_{K_0}$ and thus $Ye_H = Ye_K$ yielding $y_I = y_{H\Delta K} = 1$. This shows the first part of the lemma.

We have that $I\Delta I_0 = (H\Delta H_0)\Delta(K\Delta K_0)$ where $H\Delta H_0, K\Delta K_0 \in \mathcal{I}(\overline{n})$. As each $A \in \mathcal{A}$ has an even intersection with every $I \in \mathcal{I}(\overline{n})$ by (39) and (40), we deduce that A has an even intersection with $I\Delta I_0$ and thus $|A \cap I| \equiv |A \cap I_0|$ mod 2. □

Let $A \in \mathcal{A}$. Set $\epsilon_A := (-1)^{|A\cap I_0|}$; then, $(-1)^{|A\cap I|} = \epsilon_A$ for all $I \in \mathcal{I}(n)$ by Lemma 32. If $\epsilon_A = 1$, set $A' := A$ and if $\epsilon_A = -1$, set $A' := A \cup \{n\}$. Then,

$$(-1)^{|A'\cap I|} = 1 = y_I \quad \text{for all } I \in \mathcal{I}(n).$$

Define $z \in \mathbb{R}^{\mathcal{E}_{2t+2}(V)}$ by

$$z := \sum_{A\in\mathcal{A}} \lambda_A y^{A'}$$

and $Z := \widetilde{M}_{t+1}(z)$. To conclude the proof we show that $y = z$ or, equivalently, $Y = Z$. We already know that $z_I = y_I$ for all $I \in \mathcal{E}_{2t+2}(V \setminus \{n\})$ and $I \in \mathcal{I}(n)$. Hence the submatrices of Y and Z indexed by $\mathcal{U}_{t+1}(V \setminus \{n\})$ are identical and it suffices now to verify that $Y(H,K) = Z(H,K)$ for all $H, K \in \mathcal{U}_{t+1}(V)$ with $n \in H \setminus K$. Pick such H, K and let $K' \in \mathcal{U}_{t+1}(V \setminus \{n\})$ for which $H \sim K'$ (which exists by Lemma 30). Then, $Ye_H = Ye_{K'}$ and $Ze_H = Ze_{K'}$. Therefore, $Y(H,K) = Y(K,K') = Z(K,K') = Z(H,K)$. Thus we have shown that $Y = Z$. This concludes the proof of Theorem 20.

4.2 Proof of Theorem 23

In view of the switching symmetry it suffices to show Theorem 23 for four cut matrices $\widetilde{M}_2(S_i)$ $(i = 0,1,2,3)$ where $S_0 := \emptyset$. We proceed as follows. Set $A := \frac{1}{4}\sum_{i=0}^{3} \widetilde{M}_2(S_i)$; then, $\ker(A) = \bigcap_{i=0}^{3} \ker\ \widetilde{M}_2(S_i)$. Our goal is to show that $F(A)$, the smallest face of $\mathcal{F}_1(n)$ containing A, is equal to the convex hull of the cut matrices $\widetilde{M}_2(S_i)$; that is, any $Y \in F(A)$ can be written as a convex combination of the matrices $\widetilde{M}_2(S_i)$ $(i = 0,1,2,3)$. The following result will be used in the proof.

Lemma 33. *If all the cuts $x := \delta(S_i)$ $(i = 0,1,2,3)$ satisfy some linear equality: $\sum_{jk\in E_n} u_{jk}x_{jk} = u_0$, then any matrix $Y \in F(A)$ satisfies the equation:*

$$\sum_{jk\in E_n} u_{jk} Y e_{jk} = u_0 Y e_{\emptyset}; \tag{42}$$

that is, $\sum_{jk\in E_n} u_{jk}e_{jk} - u_0 e_{\emptyset} \in \ker Y$.

Proof. Note first that (42) holds for every $Y := \widetilde{M}_2(S_i)$; this follows from the fact that $Y_{jk,\emptyset} = \delta(S_i)_{jk}$ and $Y_{jk,rs} = \delta(S_i)_{jk} \cdot \delta(S_i)_{rs}$. Hence (42) holds for A and thus for $Y \in F(A)$, since the vector $\sum_{jk} u_{jk}e_{jk} - u_0 e_{\emptyset}$ belongs to $\ker A \subseteq \ker Y$. □

Theorem 23 holds for $n = 3$ since $\mathcal{F}_1(3)$ is a simplex. We now assume that $n \geq 4$. We first settle the case $n = 4$.

Lemma 34. *For $S_0 = \emptyset$, $S_i = \{1, i+1\}$ $(i = 1, 2, 3)$, the cut matrices $\widetilde{M}_2(S_i)$ $(i = 0, 1, 2, 3)$ form a face of $\mathcal{F}_1(4)$.*

Proof. Let $Y \in F(A)$; that is, $Y = \widetilde{M}_2(y) \succeq 0$ for some $y \in \mathbb{R}^{\mathcal{E}_4(V)}$ and $\ker Y \supseteq \ker A$. As each cut vector y^{S_i} satisfies the equation $y_{1234} = 1$, the same holds for y (since $e_{12} - e_{34} \in \ker A \subseteq \ker Y$, by Lemma 1 (ii)) and thus $Y_{12,34} = Y_{13,24} = Y_{14,23} = 1$. The principal submatrix X of Y indexed by the set $\{\emptyset, 12, 13, 23\}$ belongs to the simplex $\mathcal{F}_1(3)$ and thus can be written as $X = \alpha_1 \widetilde{M}_2(\emptyset) + \alpha_2 \widetilde{M}_2(12) + \alpha_3 \widetilde{M}_2(13) + \alpha_4 \widetilde{M}_2(23)$ with $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$. Then, $Y = \alpha_1 \widetilde{M}_2(\emptyset) + \sum_{i=2}^4 \alpha_i \widetilde{M}_2(1i)$, where the cut matrices are now matrices in $\mathcal{F}_1(4)$. (To see it, note that $\widetilde{M}_2(23) = \widetilde{M}_2(14)$ and use the fact that $y_{1234} = 1$.) $\square$

Proposition 35. *Any four cut matrices form a face of $\mathcal{F}_1(4)$.*

Proof. Consider four cut matrices $\widetilde{M}_2(S_i)$ where $S_0 = \emptyset$. The case when all S_i are even sets has been settled in Lemma 34; hence we can assume without loss of generality that $S_1 = \{1\}$. Let $Y = \widetilde{M}_2(y) \in F(A)$; that is, $Y \succeq 0$ and $\ker Y \supseteq \ker A$. The vector y can be decomposed as $y = \begin{pmatrix} y_\emptyset = 1 \\ \widetilde{y} \\ y_{1234} \end{pmatrix}$ where $\widetilde{y}$ is indexed by the pairs of $V = \{1, \ldots, 4\}$. By Lemma 33, $\widetilde{y}$ satisfies any triangle equality that is satisfied at equality by all the cuts $\delta(S_i)$ $(i = 0, 1, 2, 3)$. Denote by $\mathcal{T}$ the set of common triangle equalities satisfied by all the cuts $\delta(S_i)$. As $Q_1(K_4) = \mathrm{CUT}(K_4)$, $\widetilde{y}$ can be written as a convex combination of cuts. Let us assume that the following conditions (43) and (44) hold:

$$\mathcal{T} \neq \emptyset. \tag{43}$$

$$\text{The only cuts satisfying all the equalities from } \mathcal{T} \text{ are } \delta(S_i) \ (i = 0, 1, 2, 3). \tag{44}$$

Then $\widetilde{y} = \sum_{i=0}^3 \alpha_i \delta(S_i)$ for some $\alpha_i \geq 0$ with $\sum_i \alpha_i = 1$. Therefore, the matrices Y and $Y' := \sum_{i=0}^3 \alpha_i \widetilde{M}_2(S_i)$ coincide everywhere except maybe at their $(12, 34)$, $(13, 24)$, $(14, 23)$-entries. By (43), there exists a triangle equality that is satisfied by all the cuts $\delta(S_i)$. Suppose, to fix ideas, that this triangle equality is $-y_{12} + y_{13} - y_{23} = -1$. By Lemma 33, we deduce that $Y(-e_{12} + e_{13} - e_{23} + e_\emptyset) = 0$. Computing entry 34, we find that $Y_{12,34} = y_{14} - y_{24} + y_{34}$ can be expressed in terms of entries of $\widetilde{y}$ only. As the matrix Y' satisfies the same identity $Y'(-e_{12} + e_{13} - e_{23} + e_\emptyset) = 0$, it follows that $Y_{12,34} = Y'_{12,34}$. This shows, therefore, that $Y = Y' = \sum_{i=0}^3 \alpha_i \widetilde{M}_2(S_i)$, which concludes the proof.

We now proceed to show the claims (43) and (44). Since $S_0 = \emptyset$ and $S_1 = \{1\}$, the only possible triangle equalities in $\mathcal{T}$ are of the form:

$$y_{rs} - y_{rt} - y_{st} = -1 \text{ for } t \in \{2, 3, 4\} \text{ and } r \neq s \in \{1, 2, 3, 4\} \setminus \{t\}. \tag{45}$$

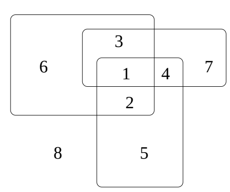
We discuss according to the number of singletons among the sets S_i. If $S_i = \{i\}$ for $i = 1, 2, 3$, then $\mathcal{T}$ consists of the three triangle equalities in (45) for $t = 4$. If

$S_i = \{i\}$ for $i = 1, 2$ and $S_3 = \{1, 2\}$, then $\mathcal{T}$ consists of the triangle equalities from (45) for ($t = 3$, $rs = 14, 24$) and for ($t = 4$, $rs = 13, 23$). If $S_i = \{i\}$ for $i = 1, 2$ and, say, $S_3 = \{1, 3\}$, then $\mathcal{T}$ consists of the triangle equalities from (45) for ($t = 3$, $rs = 12, 14$) and for ($t = 4$, $rs = 12, 23$). Finally, if $S_2 = \{1, 2\}$ and $S_3 = \{1, 3\}$, then $\mathcal{T}$ consists of the triangle equalities from (45) for ($t = 2$, $rs = 14, 34$), ($t = 3$, $rs = 14, 24$) and for ($t = 4$, $rs = 23$). In each case one can verify that (44) holds. $\square$

We now consider the general case $n \geq 4$. As before, $S_0 = \emptyset$. If some cell in the Venn diagram of S_1, S_2, S_3 contains two distinct points, say 1 and n, then each cut matrix $\widetilde{M}_2(S_i)$ satisfies the equation $Y_{\emptyset,1n} = 1$ and thus the same holds for any $Y \in F(A)$. Hence each matrix in $F(A)$ is the 1-extension of some matrix in $\mathcal{F}_1(n-1)$. Using Lemma 13 and 14, one can verify that the cut matrices $\widetilde{M}_2(S_i)$ form a face of $\mathcal{F}_1(n)$ if and only if the cut matrices $\widetilde{M}_2(S_i \setminus \{n\})$ form a face of $\mathcal{F}_1(n-1)$. Repeating this argument, we arrive at the conclusion that one can assume that each cell in the Venn diagram contains at most one point and thus $n \leq 8$. We first settle the case when each cell contains exactly one point; that is, $n = 8$ and, say,

$$S_1 := \{1, 2, 3, 6\},\ S_2 := \{1, 2, 4, 5\},\ S_3 := \{1, 3, 4, 7\} \tag{46}$$

with Venn diagram:



Proposition 36. *Assume that $S_0 = \emptyset$ and the three sets S_1, S_2, S_3 are as in (46). Then the four cut matrices $\widetilde{M}_2(S_i)$ ($i = 0, 1, 2, 3$) form a face of $\mathcal{F}_1(8)$.*

Proof. We use the following result from [24]:

$$\text{The cut matrices } \widetilde{M}_1(S_i)\ (i = 0, 1, 2, 3) \text{ form a face of } \mathcal{F}_0(8). \tag{47}$$

Set $A_1 := \frac{1}{4}\sum_{i=0}^{3} \widetilde{M}_1(S_i)$. Observe that $\widetilde{M}_1(S_i)$ (resp. A_1) is equal to the principal submatrix of $\widetilde{M}_2(S_i)$ (resp. A) indexed by the set $\mathcal{I} := \{\emptyset, 12, \ldots, 18\}$. Let $Y \in F(A)$ and let X be the principal submatrix of Y indexed by $\mathcal{I}$. Say, $Y = \widetilde{M}_2(y)$ where $y \in \mathbb{R}^{\mathcal{E}_4(V)}$. Then $X = \widetilde{M}_1(y)$ and $\ker X \supseteq \ker A_1$ (by Lemma 1 (i)). Therefore, X belongs to $F(A_1)$, the smallest face of $\mathcal{F}_0(8)$ containing A_1 which, using (47), implies that $X = \sum_{i=0}^{3} \alpha_i \widetilde{M}_1(S_i)$ for some nonnegative α_i with $\sum_i \alpha_i = 1$. Set $Y' := \sum_{i=0}^{3} \alpha_i \widetilde{M}_2(S_i)$. Our task is now to show that $Y = Y'$. By construction, Y and Y' have the same (I, J)-entries for $I, J \in \mathcal{I}$. It suffices therefore to verify that $Y_{ab,cd} = Y'_{ab,cd}$ for any 4-tuple $abcd \subseteq V$.

Let $\mathcal{T}$ denote the set of 4-tuples $T := abcd$ for which $|T \cap S_i|$ is even for all i; then $\widetilde{M}_2(S_i)_{ab,cd} = 1$ for all i implying $A_{ab,cd} = 1$ and thus $Y_{ab,cd} = 1$ (by Lemma

1). One can verify that $\mathcal{T}$ consists of the following 4-tuples: 1567, 2348, 1245, 3678, 1347, 2568, 1236, 4578, 1358, 2467, 1278, 3456, 1468, and 2357. The relation

$$ab \sim cd \ \text{ if } abcd \in \mathcal{T} \tag{48}$$

is an equivalence relation on the set E_8 of pairs of points of V and induces the partition of E_8 into the following seven equivalence classes:

$$\begin{array}{c} \mathcal{L}_1 := \{12, 36, 45, 78\},\ \mathcal{L}_2 := \{13, 26, 47, 58\}, \\ \mathcal{L}_3 := \{14, 25, 37, 68\},\ \mathcal{L}_4 := \{15, 24, 38, 67\}, \\ \mathcal{L}_5 := \{16, 23, 48, 57\},\ \mathcal{L}_6 := \{17, 28, 34, 56\},\ \mathcal{L}_7 := \{18, 27, 35, 46\}. \end{array} \tag{49}$$

The following property of the set $\mathcal{T}$ holds:

$$\text{For every 4-tuple } T \notin \mathcal{T},\ \text{there exists } T_0 \in \mathcal{T} \text{ such that } |T \cap T_0| = 3. \tag{50}$$

Indeed, let $T := abcd$ and let $\mathcal{L}_i$ be the equivalence class containing the pair ab. Then, $\mathcal{L}_i$ contains the pair cc' for some $c' \in V \setminus \{a, b, c\}$ which implies that $T_0 := abcc'$ belongs to $\mathcal{T}$ and meets T in three elements.

Based on this, one can verify that $Y_{ab,cd} = Y'_{ab,cd}$ for any 4-tuple $T := abcd$. Indeed, if $T \in \mathcal{T}$, then $Y_{ab,cd} = Y'_{ab,cd} = 1$. Otherwise, let $T_0 \in \mathcal{T}$ such that $|T \cap T_0| = 3$ (which exists by (50)); say, $T_0 := abcd'$. Then, $Y_{ab,cd} = (Ye_{ab})_{cd} = (Ye_{cd'})_{cd} = Y_{\emptyset,dd'}$ and, similarly, $Y'_{ab,cd} = Y'_{\emptyset,dd'} = Y_{\emptyset,dd'}$. Thus $Y = Y'$, which concludes the proof. $\square$

We finally consider the case when some cell in the Venn diagram of $S_1, \ldots, S_3$ is empty. In other words, given $W \subseteq V = \{1, \ldots, 8\}$, we have to show that the cut matrices $\widetilde{M}_2(S_i \cap W)$ form a face of $\mathcal{F}_1(n)$, where $n := |W| \leq 8$.

Denote by E_W the set of pairs ij $(1 \leq i < j \leq 8)$ that are contained in W. If $|W| \leq 4$, then we are done by Proposition 35. We now assume that $|W| \geq 5$. This implies that E_W contains at least one edge e_i from each class $\mathcal{L}_i$ in (49). Set $E_0 := \{e_i \mid i = 1, \ldots, 7\}$. Let A^W denote the principal submatrix of A indexed by $E_W \cup \{\emptyset\}$. By the definition of E_0, the principal submatrix of A^W indexed by $E_0 \cup \{\emptyset\}$ is equal to A_1 (defined in the proof of Proposition 36 as the principal submatrix of A indexed by $\{\emptyset, 12, \ldots, 18\}$).

Let $X \in \mathcal{F}_1(n)$ with $\ker X \supseteq \ker A^W$; we have to show that X is a convex combination of the cut matrices $\widetilde{M}_2(S_i \cap W)$. Say, $X = \widetilde{M}_2(x)$, where $x \in \mathbb{R}^{\mathcal{E}_4(W)}$. We extend x to a vector $y \in \mathbb{R}^{\mathcal{E}(V)}$ in the following way: For $ij \in E_8 \setminus E_W$, let $ab \in E_0$ such that $ij \sim ab$ and set $y_{ij} := x_{ab}$. Let T be a 4-tuple of elements of V. If $T \in \mathcal{T}$, then let $y_T := 1$. Otherwise, let $T_0 \in \mathcal{T}$ such that $|T \cap T_0| = 3$ (recall (50)); then $|T \Delta T_0| = 2$ and set $y_T := y_{T\Delta T_0}$. Finally set $Y := \widetilde{M}_2(y)$.

Then, $Y \succeq 0$ (since Y is an extension of X). Moreover, $\ker Y \supseteq \ker A$. This follows from the fact that $\ker A$ is spanned by the vectors $\begin{pmatrix} u \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ for $u \in \ker A^W$, and the vectors $e_{ab} - e_{ij}$ for $ab \sim ij$ with $ab \in E_0$, $ij \in E_8$. We know from Proposition

36 that $Y = \sum_{i=0}^{3} \alpha_i \widetilde{M_2}(S_i)$ for some nonnegative scalars α_i with $\sum_i \alpha_i = 1$. Restricting to the entries in $E_W \cup \{\emptyset\}$, we deduce that $X = \sum_i \alpha_i \widetilde{M_2}(S_i \cap W)$. This concludes the proof of Theorem 23.

5 Numerical Comparison of the Various Relaxations for Small n

In this section we examine in detail how the Lasserre relaxations $Q_t(K_n)$ approximate the cut polytope $\mathrm{CUT}(K_n)$ for small n and t. In particular, we compare them with the Anjos-Wolkowicz relaxation F_n (defined in (23)) and with the Lovász-Schrijver relaxation $N_+^t(K_n)$. Some of our results have been obtained using the software package SeDuMi[2] for solving semidefinite programs. Recall the inclusions:

$$\mathrm{CUT}(K_n) \subseteq Q_t(K_n) \subseteq N_+(K_n) \subseteq F_n \subseteq \mathrm{MET}(K_n) \cap Q_0(K_n)$$

for $t \geq 1$. For $n = 3, 4$, one has:

$$\mathrm{CUT}(K_n) = \mathrm{MET}(K_n) = Q_1(K_n) \subset Q_0(K_n).$$

Indeed, the matrix $X := \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$ belongs to $\mathcal{F}_0(3)$, the matrix $\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}$ belongs to $\mathcal{F}_0(4)$ while the vector $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})^T$ does not belong to $\mathrm{CUT}(K_3)$.

For $n = 5$, one has:

$$\mathrm{CUT}(K_5) = N_+(K_5) = Q_2(K_5) \subset Q_1(K_5) \subset F_5.$$

The equality $\mathrm{CUT}(K_5) = N_+(K_5)$ is shown in [21]. The strict inclusion $\mathrm{CUT}(K_5) \subset Q_1(K_5)$ follows from the fact that the minimum of the linear objective function $\sum_{ij \in E_5} y_{ij}$ over $\mathrm{CUT}(K_5)$ is equal to -2 while its minimum over $Q_1(K_5)$ is equal to -2.5 attained at the matrix $Y = \widetilde{M_2}(y) \in \mathcal{F}_1(5)$ where $y_{ij} := -\frac{1}{4}$ $(ij \in E_5)$ and $y_{ijhk} := \frac{3}{8}$ $(1 \leq i < j < h < k \leq 5)$; note that the minimum over the relaxation F_5 is also equal to -2.5 [1]. We verified the strict inclusion $Q_1(K_5) \subset F_5$ using computer. For instance, the minimum of the linear objective function $14y_{12} + 13y_{13} + 14y_{14} + 12y_{15} + 13y_{23} + 15y_{24} + 17y_{25} + 13y_{34} + 11y_{35} + 14y_{45}$ is equal to -34.833887 over F_5 and to -34.3402792 over $Q_1(K_5)$. Note, however, that for many random linear objective functions, one finds the same optimum over F_5 and $Q_1(K_5)$.

For $n = 6$, one has:

$$\mathrm{CUT}(K_6) = Q_2(K_6) \subset N_+(K_6).$$

Indeed, the minimum of the linear objective function $2\sum_{i=2}^{6} y_{1i} + \sum_{2 \leq i < j \leq 6} y_{ij}$ is equal to -4 over both $\mathrm{CUT}(K_6)$ and $Q_2(K_6)$, while it is equal to $-\frac{49}{12} < -4$ over

[2] This optimization software for semidefinite programming has been developed by J. Sturm and is accessible from his homepage *http://fewcal.kub.nl/sturm*.

$N_+(K_6)$ (cf. [21]). The equality $\mathrm{CUT}(K_6) = Q_2(K_6)$ now follows from the fact that the cut polytope $\mathrm{CUT}(K_6)$ is determined by the triangle inequalities (5), the *pentagonal inequalities*:

$$\sum_{1\leq i<j\leq 5} y_{ij} \geq -2, \tag{51}$$

the *hexagonal inequalities*:

$$2\sum_{i=2}^{6} y_{1i} + \sum_{2\leq i<j\leq 6} y_{ij} \geq -4 \tag{52}$$

and the inequalities obtained from (51) and (52) by permutation of the nodes and switching by cuts.

We now treat the case $n = 7$. Grishukhin [17] has computed that all the facets of $\mathrm{CUT}(K_7)$ are, up to permutation and switching, induced by one of the following eleven inequalities:

(i) the triangle inequality (5);

(ii) the pentagonal inequality (51);

(iii) the hexagonal inequality (52);

(iv) the inequality: $\sum_{ij\in E_7} y_{ij} \geq -3$;

(v) the inequality: $4y_{12} + \sum_{j=3}^{7} 2y_{1j} + 2y_{2j} + \sum_{3\leq i<j\leq 7} y_{ij} \geq -6$;

(vi) the inequality: $3\sum_{j=2}^{7} y_{1j} + \sum_{2\leq i<j\leq 7} y_{ij} \geq -7$;

(vii) the (bicycle odd wheel) inequality: $y_{12}+\sum_{j=3}^{7} y_{1j}+y_{2j}+\sum_{j=3}^{6} y_{j,j+1}+y_{37} \geq -4$;

(viii) the inequality: $4y_{12}+\sum_{j=3}^{7} 2y_{1j}+2y_{2j}+\sum_{3\leq i<j\leq 7} y_{ij}-y_{12}-y_{23}-y_{34}-y_{14} \geq -8$;

(ix) the inequality: $5y_{12} + 5y_{13} + 3y_{23} + \sum_{j=4}^{7} 3y_{1j} + 2y_{2j} + 2y_{3j} \geq -9$;

(x) the (parachute) inequality: $\sum_{j=1}^{5} y_{j,j+1} - \sum_{j=4,5,6} y_{1j} - \sum_{j=2,3} y_{j6} - \sum_{j=2}^{5} y_{j7} \geq -5$;

(xi) the (Grishukhin) inequality: $\sum_{1\leq i<j\leq 4} y_{ij} + y_{56} + y_{57} - 2\sum_{j=1}^{4} y_{j5} - y_{16} - y_{36} - y_{27} - y_{47} - y_{67} \geq -5$.

(Inequalities (i)-(vi) belong to the class of hypermetric inequalities and (vii)-(ix) to the class of clique-web inequalities; cf. Section 30.5 in [9] for details).

It is shown in [21] that the inequalities (i),(ii),(vii),(x) are valid for the Lovász-Schrijver relaxation $N_+(K_7)$ and thus for $Q_2(K_7)$ too (by Corollary 12). Using the computer program SeDuMi, we have computed the minimum of the linear objective

Inequality	Min. over $\mathrm{CUT}(K_7)$	Min. over $Q_2(K_7)$	Min. over $Q_1(K_7)$	Min. over F_7	Min. over $Q_0(K_7)$	Min. over $N_+(K_7)$
triangle (i)	-1	-1	-1	-1	-1.5	-1
pentagonal (ii)	-2	-2	-2.5	-2.5	-2.5	-2
hexagonal (iii)	-4	-4	-4.5	-4.5	-4.5	-49/12 ~ -4.0833
(iv)	-3	-3.5	-3.5	-3.5	-3.5	?
(v)	-6	-6.051882	-6.5	-6.5	-6.5	?
(vi)	-7	-7	-7.5	-7.5	-7.5	?
bicycle (vii)	-4	-4	-5	-5.0045	-5.8090	-4
(viii)	-6	-6	-6.5817	-6.6522	-7.9661	?
(ix)	-9	-9	-9.6433	-9.7036	-11.0166	?
parachute (x)	-4	-4	-4.7439	-4.8099	-5.9220	-4
Grishukhin (xi)	-5	-5	-5.6152	-5.7075	-6.9518	?

Figure 1. *Comparing the facet defining inequalities for* $\mathrm{CUT}(K_7)$

function $c^T y$ over $Q_t(K_7)$ for $t \leq 2$, when $c^T y$ is the left hand side of the remaining inequalities within (i)-(xi). Figure 1 summarizes our results.

The set F_7 improves the relaxation $Q_0(K_7)$ for the inequalities (i) and (vii)-(xi) that altogether comprise more than 96 percent of the total number of facets of $\mathrm{CUT}(K_7)$. On the other hand, the improvement of $Q_1(K_7)$ over F_7 does not seem to be very significant.

We know that $Q_3(K_7) = \mathrm{CUT}(K_7)$. Note that $Q_2(K_7)$ already approximates very well $\mathrm{CUT}(K_7)$; indeed, the minimum over $Q_2(K_7)$ is strictly less than the minimum over $\mathrm{CUT}(K_7)$ only for the inequalities (iv) and (v) which represent less than 1.3 percent of the total amount of facets of $\mathrm{CUT}(K_7)$.

Given $c \in \mathbb{Q}^{E_n}$ and $t \geq 0$, it is of interest to evaluate the *integrality ratio*:

$$\rho_t := \frac{\sum_{ij} c_{ij} - \min(c^T y \mid y \in \mathrm{CUT}(K_n))}{\sum_{ij} c_{ij} - \min(c^T y \mid y \in Q_t(K_n))};$$

that is, the ratio of the maximum weight of a cut with respect to the weights c by the maximum obtained by optimizing over the relaxation $Q_t(K_n)$. Goemans and Williamson [16] showed that

$$\rho_0 \geq 0.878$$

when $c \geq 0$ and Feige and Schechtman [14] have constructed graphs for which the integrality ratio ρ_0 attains the worst case value 0.878. It is however known that, in practice, the integrality ratio ρ_0 is larger than the worst case value. As

Inequality	$\sum_{ij} c_{ij}$	ρ_2	ρ_1	ρ_F	ρ_0
triangle (i)	3	1	1	1	$\frac{8}{9} \sim$ **0.888**
pentagonal (ii)	10	1	0.96	0.96	0.96
hexagonal (iii)	20	1	0.979	0.979	0.979
(iv)	21	**0.979**	0.979	0.979	0.979
(v)	34	0.998	0.987	0.987	0.987
(vi)	33	1	0.987	0.987	0.987
bicycle (vii)	16	1	**0.952**	**0.952**	0.917
(viii)	30	1	0.984	0.982	0.948
(ix)	47	1	0.988	0.987	0.965

Figure 2. *The integrality ratios for the facets of* K_7

Inequality	Min. over CUT(K_9)	Min. over $Q_2(K_9)$	Min. over $Q_1(K_9)$	Min. over F_9	Min. over $Q_0(K_9)$
AW_9^2	-6	-6	-6.8282634	-6.9937	-9
integ. ratio		1	0.966	0.960	$\frac{8}{9} \sim 0.888$

Figure 3. *The antiweb graph* AW_9^2

an indication we have computed the integrality ratio for the facets of K_7 and the relaxations $Q_t(K_7)$ and F_7 (in which case the ratio is denoted by ρ_F). Figure 2 gives the results. Observe that the worst case value for ρ_0 is $\frac{8}{9} \sim 0.888$ (attained at the triangle inequality) while the worst case values for ρ_F, ρ_1, ρ_2 are, respectively, 0.952, 0.952, 0.979.

Another example demonstrating the strict inclusion $Q_1(n) \subset F_n$ is as follows. Consider the circulant graph $G = AW_n^2$ on n nodes whose edgeset E consists of the pairs $(i, i+1)$ and $(i, i+2)$ $(i = 1, \ldots, n)$ (indices being taken modulo n). Figure 3 gives the values of the minimum of $\sum_{ij \in E} y_{ij}$ over CUT(K_9), $Q_1(K_9)$, $Q_2(K_9)$ and F_9. The last row shows the corresponding integrality ratios. In fact, CUT$(AW_n^2) = Q_2(AW_n^2)$ for any odd n, since contracting edge 12 in AW_n^2 produces then a planar graph (use Theorem 17).

6 Concluding Remarks

Application to the boolean quadric polytope. In this paper we have considered in detail the hierarchy of semidefinite relaxations $Q_t(G)$ $(t \geq 0)$ of the cut polytope CUT(G). All the results we have presented have counterparts for the analogous problem in 0/1 variables; namely, for the *unconstrained* 0/1 *quadratic programming problem*:

$$\max\ x^T Q x \ \text{ subject to } x \in \{0,1\}^n$$

and the associated *boolean quadric polytope*:

$$\mathrm{QP}_n := \mathrm{conv}(\{(x_i x_j)_{1 \leq i \leq j \leq n} \mid x \in \{0,1\}^n\})$$

studied in detail by Padberg [28]. Indeed, the mapping

$$x \in \{0,1\}^n \mapsto y := (1, 1-2x_1, \ldots, 1-2x_n)^T \in \{\pm 1\}^{n+1}$$

yields the correspondance $xx^T \mapsto yy^T$ between the vertices of QP_n and the vertices of $\mathrm{CUT}(K_{n+1})$. Therefore, as is well known, QP_n and $\mathrm{CUT}(K_{n+1})$ are in affine bijection.

The Lasserre construction can be applied for constructing semidefinite relaxations of QP_n. Namely, for $t \geq 0$, let $\mathcal{Q}_t(n)$ denote the set of positive semidefinite matrices of the form

$$M_{t+1}(y) := (y(I \cup J))_{\substack{I,J \subseteq V \\ |I|,|J| \leq t+1}}$$

where $y \in \mathbb{R}^{\mathcal{P}_{2t+2}(V)}$ with $y_\emptyset = 1$ (comparing with (8), note that the symmetric difference is now replaced by the union). Then, the projection of $\mathcal{Q}_t(n)$ on the subspace indexed by the pairs ij with $1 \leq i \leq j \leq n$ is a semidefinite relaxation of QP_n. The set $\mathcal{Q}_0(n)$ is the basic semidefinite relaxation for QP_n, consisting of the symmetric positive semidefinite matrices Y of order $n+1$ having their main diagonal equal to their first row and $Y_{0,0} = 1$.

Given a graph $G = (V, E)$ with $V = \{1, \ldots, n\}$, Padberg [28] observed that the stable set polytope $\mathrm{ST}(G)$ of G arises as the projection of a face of the boolean quadric polytope QP_n; namely, $d \in \mathbb{R}^V$ belongs to $\mathrm{ST}(G)$ if and only if $(d, y) \in \mathrm{QP}_n$ for some $y \in \mathbb{R}^{E_n}$ satisfying $y_{ij} = 0$ for all edges $ij \in E$. Therefore, each relaxation $\mathcal{Q}_t(n)$ for QP_n yields a semidefinite relaxation for $\mathrm{ST}(G)$. More precisely, the projection on $\mathbb{R}^V$ of the set

$$\{M_{t+1}(y) \in \mathcal{Q}_t(n) \mid y_{ij} = 0 \ (ij \in E)\}$$

is a semidefinite relaxation of $\mathrm{ST}(G)$ which, in the case $t = 0$, coincides with the basic semidefinite relaxation $\mathrm{TH}(G)$; moreover, this semidefinite relaxation coincides with the set $Q_t(\mathrm{FR}(G))$ obtained by applying the Lasserre construction to the fractional stable set polytope $\mathrm{FR}(G)$ (defined in (14)). See [22] for more details.

Lower bounds for the rank of the Lasserre procedure. It would be interesting to find lower bounds for the *Lasserre rank* of a graph G, which is defined as the smallest integer t for which $\mathrm{CUT}(G) = Q_t(G)$; the *LS rank* of G is defined analogously as the smallest t for which $\mathrm{CUT}(G) = N_+^t(G)$. As $Q_t(G) \subseteq N_+^{t-1}(G)$, the Lasserre rank is less than or equal to the LS rank plus one. In the case of the stable set problem, it has been shown in [27] that the smallest t for which equality $N_+^t(\mathrm{FR}(G)) = \mathrm{ST}(G)$ holds satisfies $t \leq \alpha(G)$, with equality when G is the line graph of K_{2n+1} [30]. In the case of max-cut, the LS rank of K_n is conjectured to be equal to $n-4$; equality has been shown for $n = 4, 5, 6, 7$ [21]. We saw above that the Lasserre rank of K_n is equal to 1,2,2,3 for $n = 4, 5, 6, 7$, respectively.

It is shown in [21] that, for n odd, the inequality

$$\sum_{1 \leq i < j \leq n} y_{ij} \geq \frac{1-n}{2} \tag{53}$$

is valid for $N_+^{\frac{n-3}{2}}(K_n)$ and thus for $Q_{\frac{n-1}{2}}(K_n)$. We conjecture that the inequality (53) is *not* valid for $Q_{\frac{n-3}{2}}(K_n)$ for n odd. If true, this would imply that the Lasserre rank of K_n is at least $\frac{n-1}{2}$ and thus that the LS-rank is at least $\frac{n-3}{2}$ for n odd.

In order to show the above conjecture we have to find a positive semidefinite moment matrix $\widetilde{M}_{\frac{n-1}{2}}(y)$ with $\sum_{ij\in E_n} y_{ij} < \frac{1-n}{2}$. Set $a_0 := 1$ and, for $1 \le r \le \frac{n-1}{2}$,

$$a_{2r} := (-1)^r \prod_{i=0}^{r-1} \frac{2i+1}{n-2i-1}$$

and define $y \in \mathbb{R}^{\mathcal{E}(V)}$ by letting $y_I := a_{|I|}$ for all $I \in \mathcal{E}(V)$. Then, $\sum_{ij\in E_n} y_{ij} = \binom{n}{2}a_2 = -\binom{n}{2}\frac{1}{n-1} < \frac{1-n}{2}$. We conjecture that $\widetilde{M}_{\frac{n-1}{2}}(y) \succeq 0$ for all n odd. We verified that this fact is true for small $n = 3, 5, 7$.

Note added in proof: This conjecture has now been proved to hold for any odd $n \ge 3$ (Laurent, manuscript in preparation).

Bibliography

[1] M.F. Anjos and H. Wolkowicz, A strenghtened SDP relaxation via a second lifting for the Max-Cut problem. To appear in *Discrete Applied Mathematics.* Available at http://orion.math.uwaterloo.ca:80/˜hwolkowi/.

[2] M.F. Anjos and H. Wolkowicz, Geometry of semidefinite Max-Cut relaxations via ranks. Report CORR 2001-39, Waterloo University, Canada, 2001. Available at http://orion.math.uwaterloo.ca:80/˜hwolkowi/.

[3] E. Balas, S. Ceria, and G. Cornuéjols, A lift-and-project cutting plane algorithm for mixed 0-1 programs. *Mathematical Programming*, 58:295–324, 1993.

[4] F. Barahona, On cuts and matchings in planar graphs. *Mathematical Programming*, 60:53–68, 1993.

[5] F. Barahona, On the computational complexity of Ising spin glass models. *Journal of Physics A, Mathematical and General*, 15:3241–3253, 1982.

[6] F. Barahona and A.R. Mahjoub, On the cut polytope. *Mathematical Programming*, 36:157–173, 1986.

[7] V. Chvátal, Edmonds polytopes and a hierarchy of combinatorial problems. *Discrete Mathematics*, 4:305–337, 1973.

[8] W. Cook and S. Dash, On the matrix-cut rank of polyhedra. *Mathematics of Operations Research*, 26:19–30, 2001.

[9] M. Deza and M. Laurent, *Geometry of Cuts and Metrics.* Springer Verlag, Berlin, 1997.

[10] M. Deza, M. Laurent, and S. Poljak, The cut cone III: On the role of triangle facets. *Graphs and Combinatorics*, 8:125–142, 1992.

[11] R.D. Hill and S.R. Waters, On the cone of positive semidefinite matrices. *Linear Algebra and its Applications*, 90:81–88, 1987.

[12] F. Eisenbrand, On the membership problem for the elementary closure of a polyhedron. *Combinatorica*, 19:299-300, 1999.

[13] F. Eisenbrand and A.S. Schulz, Bounds on the Chvátal rank of polytopes in the 0/1 cube. In G. Cornuéjols, R.E. Burkard, and G.J. Woeginger, eds., IPCO 1999, *Lecture Notes in Computer Science*, 1610:137–150, 1999.

[14] U. Feige and G. Schechtman, On the optimality of the random hyperplane rounding technique for MAX CUT. Preprint, 2000. Available at http://www.wisdom.weizmann.ac.il/˜feige/.

[15] M.X. Goemans and L. Tunçel, When does the positive semidefiniteness constraint help in lifting procedures. To appear in *Mathematics of Operations Research*. Available at http://www-math.mit.edu/˜goemans/.

[16] M.X. Goemans and D.P. Williamson, Improved approximation algorithms for maximum cuts and satisfiability problems using semidefinite programming. *Journal of the Association for Computing Machinery*, 42:1115–1145, 1995.

[17] V.P. Grishukhin, All facets of the cut cone C_n for $n = 7$ are known. *European Journal of Combinatorics*, 11:115–117, 1990.

[18] J.B. Lasserre, Optimality conditions and LMI relaxations for $0-1$ programs. Technical Report N. 00099, LAAS, Toulouse, 2000.

[19] J.B. Lasserre, Global optimization with polynomials and the problem of moments. *SIAM Journal on Optimization*, 11:796–817, 2001.

[20] J.B. Lasserre, An explicit exact SDP relaxation for nonlinear $0-1$ programs. In K. Aardal and A.M.H. Gerards, eds., *Lecture Notes in Computer Science*, 2081:293–303, 2001.

[21] M. Laurent, Tighter linear and semidefinite relaxations for max-cut based on the Lovász-Schrijver lift-and-project procedure. *SIAM Journal on Optimization*, 12:345–375, 2001.

[22] M. Laurent, A comparison of the Sherali-Adams, Lovász-Schrijver and Lasserre relaxations for $0-1$ programming. Report PNA-R0108, CWI, Amsterdam, 2001. Available at http://www.cwi.nl/˜monique/.

[23] M. Laurent and S. Poljak, On a positive semidefinite relaxation of the cut polytope. *Linear Algebra and its Applications*, 223/224: 439–461, 1995.

[24] M. Laurent and S. Poljak, On the facial structure of the set of correlation matrices. *SIAM Journal on Matrix Analysis and its Applications*, 17:530–547, 1996.

[25] M. Laurent and F. Rendl, Semidefinite programming and integer programming. Chapter in *Handbook on Discrete Optimization*, K. Aardal, G. Nemhauser and R. Weismantel, eds., to appear.

[26] L. Lovász, On the Shannon capacity of a graph. *IEEE Transactions on Information Theory*, IT-25:1–7, 1979.

[27] L. Lovász and A. Schrijver, Cones of matrices and set-functions and $0-1$ optimization. *SIAM Journal on Optimization*, 1:166–190, 1991.

[28] M. Padberg, The boolean quadric polytope: Some characteristics, facets and relatives. *Mathematical Programming*, 45: 139–172, 1989.

[29] H.D. Sherali and W.P. Adams, A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems. *SIAM Journal on Discrete Mathematics*, 3:411–430, 1990.

[30] T. Stephen and L. Tunçel, On a representation of the matching polytope via semidefinite liftings. *Mathematics of Operations Research*, 24:1–7, 1999.