Semidefinite Programming in Polynomial Optimization

Monique LAURENT

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Polynomial Optimization Problem (P)

$$\boxed{p_{\min} := \inf_{x \in K} p(x)}$$

where $\boxed{K := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0 \ldots g_m(x) \geq 0\}}$

and $p, g_1, \ldots, g_m$ are polynomials in n variables

- **Unconstrained polynomial minimization:**

$$\boxed{p_{\min} := \ \inf_{x \in \mathbb{R}^n} p(x)}$$

Example: An $n \times n$ matrix M is *copositive* if $p_{\min} \geq 0$ for $p(x) := \sum_{i,j=1}^n x_i^2 x_j^2 M_{ij}$, i.e., $p(x)$ is nonnegative on $\mathbb{R}^n$

Example: A sequence $a_1, \ldots, a_n \in \mathbb{N}$ can be *partitioned* if $p_{\min} = 0$ for $p(x) = (\sum_{i=1}^n a_i x_i)^2 + \sum_{i=1}^n (x_i^2 - 1)^2$

- $0/1$ **linear programming:**

$$\boxed{\min \ c^T x \ \text{ s.t. } \ Ax \leq b, \ x_i^2 = x_i \ \forall i}$$

Example: The *stability number* $\alpha(G)$ of $G = (V, E)$ can be computed via any of the programs:

$$\alpha(G) = \max \sum_{i \in V} x_i \ \text{ s.t. } \ x_i + x_j \leq 1 \ (ij \in E), \ x_i^2 = x_i \ (i \in V)$$

$$\frac{1}{\alpha(G)} = \min \ x^T (I + A_G) x \ \text{ s.t. } \ \sum_{i \in V} x_i = 1, \ x_i \geq 0 \ (i \in V)$$

Hence: (P) is an NP-hard problem

Strategy:

Approximate (P) by a hierarchy of convex (semidefinite) relaxations

Shor (1987), Nesterov, Lasserre, Parrilo (2000-)

Such relaxations can be constructed using

representations of nonnegative polynomials as sums of squares of polynomials

and

the dual theory of moments

Underlying paradigm:

Testing whether a polynomial p is *nonnegative* is hard
but
one can test whether p is a *sum of squares of polynomials*
efficiently via *semidefinite programming*

Some notation about polynomials

- Multivariate polynomial:

$$p(x) = \sum_{\alpha \in \mathbb{Z}_+^n} p_\alpha \underbrace{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}_{x^\alpha} = \sum_{\alpha \in \mathbb{Z}_+^n} p_\alpha x^\alpha$$

- $\deg(x^\alpha) = |\alpha| := \sum_{i=1}^n \alpha_i$

- $\deg(p) = \max \deg(x^\alpha)$ s.t. $p_\alpha \neq 0$

- We identify a polynomial $p(x)$ with its sequence of coefficients $p = (p_\alpha)_{|\alpha| \leq d}$, if $\deg(p) = d$

- $p(x)$ is a *sum of squares of polynomials (SOS)* if

$$p(x) = \sum_{j=1}^m [u_j(x)]^2 \qquad \text{for some polynomials } u_1, \ldots, u_m$$

Then, $\deg(p)$ is even and $\deg(u_i) \leq \deg(p)/2$

Example: $x^2 + y^2 + 2xy + z^6 = (x+y)^2 + (z^3)^2$ is SOS

Recognizing sums of squares of polynomials via semidefinite programming [Powers-Wörmann 1998]

$$p(x) = \sum_{|\alpha| \leq 2d} p_\alpha x^\alpha \quad \text{is a sum of squares of polynomials}$$

$$\text{i.e., } p(x) = \sum_j [u_j(x)]^2, \quad \text{where } \deg(u_j) \leq d$$

$$\Updownarrow$$

$$p(x) = z^T \underbrace{(\sum_j u_j u_j^T)}_{X \succeq 0} z, \quad \text{setting } z := (x^\beta)_{|\beta| \leq d}$$

$$\Updownarrow$$

The following semidefinite program is feasible:

$$\begin{cases} X \succeq 0 \\ \sum_{\substack{|\beta|,|\gamma| \leq d \\ \beta+\gamma=\alpha}} X_{\beta,\gamma} = p_\alpha \quad (|\alpha| \leq 2d) \end{cases}$$

$\leadsto$ system in matrix variable X of order $\binom{n+d}{d} \times \binom{n+d}{d}$, with $\binom{n+2d}{2d}$ equations

$\leadsto$ polynomial size *fixing either* n *or* d

When does a polynomial have a decomposition as a sum of squares?

Hilbert [1888] classified the pairs (n, d) for which *every nonnegative polynomial of degree d in n variables is a sum of squares of polynomials*:

- $n = 1$
- $d = 2$
- $n = 2,\ d = 4$

E.g., the **Motzkin polynomial:** $x^4y^2 + x^2y^4 - 3x^2y^2 + 1$ is nonnegative but *not* a SOS

Artin [1927] solved **Hilbert's 17th problem** [1900]:

$$\boxed{p \geq 0 \text{ on } \mathbb{R}^n \Longrightarrow p = \sum_i \left(\frac{p_i}{q_i}\right)^2, \text{ where } p_i, q_i \in \mathbb{R}[x_1, \ldots, x_n]}$$

That is,

$$\boxed{p \cdot q^2 \text{ is SOS for some } q \in \mathbb{R}[x_1, \ldots, x_n]}$$

Sometimes, the common denominator is known:

Pólya [1928] + Reznick [1995]: For p *homogeneous*

$$\boxed{p > 0 \text{ on } \mathbb{R}^n \setminus \{0\} \Longrightarrow p \cdot \left(\sum_{i=1}^n x_i^2\right)^r \text{ SOS for some } r \in \mathbb{N}}$$

(Dual) SOS approximations for (P)

following [Lasserre 2001]

Recall: $K = \{x \in \mathbb{R}^n \mid g_\ell(x) \geq 0 \ (\ell = 1, \ldots, m)\}$

$$\boxed{p_{\min} := \inf_{x \in K} \ p(x) = \sup \ \lambda \ \text{ s.t. } \ p(x) - \lambda \geq 0 \ \forall x \in K}$$

Replacing the *nonnegativity* condition by a stronger *SOS* condition, one can formulate a hierarchy of tractable SDP relaxations:

(SOSt)

$$\boxed{\begin{array}{ll} \sigma_t := \sup \ \lambda \ \text{ s.t.} & p(x) - \lambda = s_0(x) + \sum_{\ell=1}^m s_\ell(x) g_\ell(x) \\ & s_0, \ s_\ell \text{ are SOS} \\ & \deg(s_0), \ \deg(s_\ell g_\ell) \leq 2t \end{array}}$$

for any $t \geq \lceil \deg(p)/2 \rceil, \ \lceil \deg(g_\ell)/2 \rceil$

$$\boxed{\sigma_t \leq \sigma_{t+1} \leq p_{\min}}$$

Example: $K = \mathbb{R}^n$ (unconstrained case), $\deg(p) = 2d$

$\sigma_t = \sigma_d \leq p_{\min}$, with equality iff $p(x) - p_{\min}$ is SOS

Equality in the univariate case (Shor [1987], Nesterov [2000])

(Primal) moment approximations for (P)

$$\boxed{p_{\min} = \inf_{x \in K} p(x) = \inf_{\mu \in \mathcal{P}_K} \int p(x)\mu(dx) = \inf_{y \in \mathcal{M}_K} p^T y}$$

$$\boxed{\int p(x)\mu(dx) = \sum_\alpha p_\alpha \int x^\alpha \mu(dx)}$$

$\mu \in \mathcal{P}_K$ if μ is a probability measure supported by K

$$y \in \mathcal{M}_K \text{ if } y_\alpha = \underbrace{\int x^\alpha \mu(dx)}_{\substack{\text{moment of order } \alpha \\ \text{of the measure } \mu}} \ \forall \alpha, \text{ for some } \mu \in \mathcal{P}_K$$

Lemma: If y is the sequence of moments of $\mu \in \mathcal{P}_K$ then

$$y_0 = 1$$

$$\boxed{M_t(y) := (y_{\alpha+\beta})_{|\alpha|,|\beta| \le t} \succeq 0} \quad \text{for } t \ge 0$$

moment matrix of order t

$$\boxed{M_t(g_\ell y) := (\textstyle\sum_\gamma (g_\ell)_\gamma y_{\alpha+\beta+\gamma})_{|\alpha|,|\beta| \le t} \succeq 0} \quad \text{for } t \ge 0$$

localizing matrix

Moreover: $\boxed{\operatorname{supp}(\mu) \subseteq \operatorname{Zeros}(p)} \quad \forall p \in \operatorname{Ker} M_t(y)$

Proof: For $p = (p_\alpha)_{|\alpha| \le t}$, $p^T M_t(y) p = \int p(x)^2 \mu(dx) \ge 0$

$$p^T M_t(g_\ell y) p = \int g_\ell(x) p(x)^2 \mu(dx) \ge 0$$

If we relax the moment condition "$y \in \mathcal{M}_K$" by requiring positive semidefiniteness of its moment matrix, we obtain a hierarchy of tractable SDP relaxations:

(MOMt)

$$\boxed{\begin{array}{llll} p_t := \inf & p^T y & \text{s.t.} & y_0 = 1,\ M_t(y) \succeq 0 \\ & & & M_{t-d_\ell}(g_\ell y) \succeq 0 \ (\ell = 1, \ldots, m) \end{array}}$$

for any $t \geq \lceil \deg(p)/2 \rceil$, $d_\ell := \lceil \deg(g_\ell)/2 \rceil$

$$\boxed{p_t \leq p_{t+1} \leq p_{\min}}$$

Lemma:
The semidefinite programs **(SOSt)** and **(MOMt)** are dual of each other.

Weak duality: $\boxed{\sigma_t \leq p_t}$

No duality gap, e.g., if K has a nonempty interior

Properties of the bounds:

$$\boxed{\sigma_t \leq p_t \leq p_{\min}}$$

• For *fixed* t, one can compute σ_t, p_t in **polynomial time** (to any fixed precision)

• **Asymptotic (finite) convergence** to $p_{\min}$ via some representation theorems for positive polynomials from real algebraic geometry

• **Optimality certificate** via some theorems of Curto and Fialkow about moment matrices

• **Extracting global minimizers** via the eigenvalue method for solving systems of polynomial equations

Asymptotic convergence

Representation theorem: [Putinar 1993]
(more elementary proof by [Schweighofer 2003])

If $\quad \exists u_\ell$ SOS : $\{x \mid \sum_\ell u_\ell(x) g_\ell(x) \geq 0\}$ is compact $\quad (*)$

then $p > 0$ on $K \Longrightarrow p = s_0 + \sum_\ell s_\ell g_\ell$ with s_0, s_ℓ SOS

Convergence theorem: [Lasserre 2001]

$$\text{If } (*) \text{ then } \boxed{\lim_{t\to\infty} \sigma_t = \lim_{t\to\infty} p_t = p_{\min}}$$

Note: $(*)$ holds, e.g.,
- if $\{x \mid g_\ell(x) \geq 0\}$ is compact for some ℓ
- if the equations $x_i^2 = x_i$ are present in the description of K **(0/1 case)**
- if the radius R of a ball containing K is known, then add the (redundant) constraint $R^2 - \sum_i x_i^2 \geq 0$ to the description of K

Note: A representation result valid for "$p \geq 0$ on K" would give a *finite* convergence result

Finite convergence: The finite variety case

Assume the equations: $h_1(x) = 0 \ldots h_k(x) = 0$ are present in the description of K and the set

$V := \{x \in \mathbb{C}^n \mid h_1(x) = 0 \ldots h_k(x) = 0\}$ is *finite*

Then: $\boxed{p_t = p_{\min} \quad \text{for } t \text{ large enough}}$ [Laurent 2002]

Moreover: $\boxed{\sigma_t = p_t = p_{\min} \quad \text{for } t \text{ large enough}}$

if $h_1, \ldots, h_k$ form a *Groebner basis* [Laurent 2002]
or if they generate a *radical ideal* [Parrilo 2002]

Representation result in radical case: [Parrilo 2002]
$p \geq 0$ on $K \Longrightarrow p = s_0 + \sum_\ell s_\ell g_\ell$ (s_0, s_ℓ SOS)

In the 0/1 case: $\boxed{p_t = p_{\min} \quad \text{for } t \geq n + d}$

where $d := \max_\ell \lceil \deg(g_\ell)/2 \rceil$ [Lasserre 2001]

The hierarchy of bounds p_t refines other combinatorial hierarchies, e.g., by Lovász-Schrijver [1991] (cf. [Laurent 2003])

Note: One can use the equations $h_1(x) = 0 \ldots h_k(x) = 0$ to *reduce the number of variables* in the moment relaxations (MOMt)

Optimality certificate

Theorem: [Henrion-Lasserre 2005]
If y is an optimum solution to the program **(MOMt)** satisfying:

(RC) $\boxed{\mathrm{rank} M_t(y) = \mathrm{rank} M_{t-d}(y),}$

where $d = \max_\ell \lceil \deg(g_\ell)/2 \rceil$

then: $\boxed{p_t = p_{\min}}$

Proof:
By a theorem of [Curto-Fialkow 1996] (cf. [Laurent 2004])

y is the sequence of moments of
an r-atomic nonnegative measure μ, i.e.,

$$\mu = \sum_{i=1}^{r} \lambda_i \delta_{x_i}, \text{ where } \lambda_i > 0, \ \sum_i \lambda_i = 1, \ x_i \in K$$

$$\Longrightarrow \quad y = \sum_{i=1}^{r} \lambda_i (x_i^\alpha)_{|\alpha| \le 2t}$$

$$\Longrightarrow \quad p_{\min} \ge p_t = p^T y = \sum_{i=1}^{r} \lambda_i p(x_i) \ge p_{\min}$$

Moreover: $\boxed{x_1, \ldots, x_r \text{ are global minimizers over } K}$

Extracting global minimizers

Same setting: y is an optimum solution to (MOMt) satisfying (RC)

Then, $\text{supp}(\mu) = \{x_1, \ldots, x_r\} \subseteq$ {global minimizers}
Moreover,

$$\text{supp}(\mu) = \bigcap_{p \in \text{Ker} M_t(y)} \text{Zeros}(p)$$

Therefore: one can find the global minimizers $x_1, \ldots, x_r$ by *computing the common zeros to a system of polynomial equations*

$\rightsquigarrow$ Use the **'eigenvalue method'**, based on computing the eigenvalues of the $r \times r$ multiplication matrices in the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]/I$, where I is the ideal generated by the kernel of $M_t(y)$

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Implementations of the SOS/moment relaxation method:

GloptiPoly by Henrion, Lasserre

SOSTOOLS by Prajna, Papachristodoulou, Parrilo

Example 1:

$$\begin{array}{ll} \min & p = -25(x_1-2)^2 - (x_2-2)^2 - (x_3-1)^2 \\ & \quad -(x_4-4)^2 - (x_5-1)^2 - (x_6-4)^2 \\ \text{s.t.} & (x_3-3)^2 + x_4 \geq 4, \quad (x_5-3)^2 + x_6 \geq 4 \\ & x_1 - 3x_2 \leq 2, \quad -x_1 + x_2 \leq 2, \ x_1 + x_2 \leq 6, \\ & x_1 + x_2 \geq 2, \ 1 \leq x_3 \leq 5, \ 0 \leq x_4 \leq 6, \\ & 1 \leq x_5 \leq 5, \ 0 \leq x_6 \leq 10, \ x_1, x_2 \geq 0 \end{array}$$

order t	bound p_t	solution extracted
1	unbounded	none
2	-310	(5,1,5,0,5,10)

The global minimum is found at the relaxation of order $t = 2$

Example 2:

$$\begin{array}{ll} \min & p = -x_1 - x_2 \\ \text{s.t.} & x_2 \leq 2x_1^4 - 8x_1^3 + 8x_1^2 + 2 \\ & x_2 \leq 4x_1^4 - 32x_1^3 + 88x_1^2 - 96x_1 + 36 \\ & 0 \leq x_1 \leq 3, \ 0 \leq x_2 \leq 4 \end{array}$$

order t	bound p_t	solution extracted
2	-7	none
3	-6.6667	none
4	-5.5080	(2.3295,3.1785)

The global minimum is found at the relaxation of order $t = 4$

Unconstrained polynomial minimization

$$\boxed{p_{\min} = \inf_{x \in \mathbb{R}^n} p(x)} \text{ where } \deg(p) = 2d$$

As $K = \mathbb{R}^n$, the relaxation scheme just gives *one* bound: $\sigma_d = p_d \leq p_{\min}$, with equality iff $p(x) - p_{\min}$ is SOS

Idea:

Get better bounds by transforming the *unconstrained* problem into a *constrained* problem

If p has a minimum:

$$\boxed{p_{\min} = p_{\text{grad}} := \inf_{x \in V_{grad}^{\mathbb{R}}} p(x)}$$

where $$\boxed{V_{\text{grad}}^{\mathbb{R}} := \{x \in \mathbb{R}^n \mid \frac{\partial p}{\partial x_i} = 0 \ (i = 1, \ldots, n)\}}$$

If, moreover, a bound R is known on the norm of a global minimizer:

$$\boxed{p_{\min} = p_{\text{ball}} := \inf_{R^2 - \sum_i x_i^2 \geq 0} p(x)}$$

When p attains its minimum:

The 'ball approach':

• p_{ball} can be approximated via Lasserre's relaxation scheme (asymptotic convergence, as Putinar's assumption holds!)

• seems to work well only if the radius R of the ball is not too large ...

The 'gradient variety' approach:

Representation result: [Demmel, Nie, Sturmfels 2004]

$$\boxed{p > 0 \text{ on } V_{\mathrm{grad}}^{\mathbb{R}} \Longrightarrow p = s_0 + \sum_{i=1}^{n} u_i \frac{\partial p}{\partial x_i}}$$

where s_0 SOS, $u_i \in \mathbb{R}[x_1, \ldots, x_n]$

Convergence result: [Demmel, Nie, Sturmfels 2004] There is asymptotic convergence of the SOS and moment bounds to p_{grad}

Moreover: When $\frac{\partial p}{\partial x_1}, \ldots, \frac{\partial p}{\partial x_n}$ generate a radical ideal, the representation result holds for $p \geq 0$, implying the *finite* convergence

What if p is not known to have a minimum?

Strategy: Perturb the polynomial p

[Hanzon-Jibetean 2003] [Jibetean-Laurent 2004]

$$\boxed{p_\epsilon(x) := p(x) + \epsilon(\sum_{i=1}^{n} x_i^{2d+2})} \quad \text{for small } \epsilon > 0$$

Then:

- p_ϵ has a minimum and $\boxed{\lim_{\epsilon \to 0} (p_\epsilon)_{\min} = p_{\min}}$

- the gradient variety of p_ϵ is *finite*
$\rightsquigarrow$ one can compute $(p_\epsilon)_{\min}$ via the 'gradient variety' approach
$\rightsquigarrow$ *finite convergence* of the moment bounds to $(p_\epsilon)_{\min}$ (in $2nd$ steps)

- one can use the gradient equations:

$$\frac{\partial p_\epsilon}{\partial x_i} = (2d+2)x_i^{2d+1} + \frac{\partial p}{\partial x_i} = 0 \quad \forall i$$

to reduce # variables ($\rightsquigarrow$ only variables y_α with $\alpha_i \leq 2d$ $\forall i$) and to express $(p_\epsilon)_{\min}$ via a SDP with a *single* LMI

- the limits of global minimizers of p_ϵ (as $\epsilon \to 0$) are global minimizers of p (if any)

Examples

$\rightsquigarrow$ compute the bounds $(p_\epsilon)_t$ via the 'gradient' approach

Motzkin polynomial: $p = x^2y^2(x^2+y^2-3)+1$

Then, $p_{\min} = 0$, attained at $(\pm 1, \pm 1)$, and p is *not* SOS

ϵ	order t	(r_t, r_{t-1}, r_{t-2})	$(p_\epsilon)_t$	extracted solutions
	3	(10,6,3)	$-3.5070\ 10^4$	none
10^{-1}	4	(4,4,4)	0.0316	$(\pm 0.9453, \pm 0.9453)$
10^{-2}	4	(4,4,4)	$5.0177\ 10^{-4}$	$(\pm 0.9935, \pm 0.9935)$
10^{-3}	5	(4,4,4)	$5.2951\ 10^{-6}$	$(\pm 0.9993, \pm 0.9993)$
10^{-4}	5	(4,4,4)	$5.3170\ 10^{-8}$	$(\pm 0.9999, \pm 0.9999)$

Example: $p = (xy-1)^2 + y^2$

Then, $p_{\min} = 0$ is *not* attained

ϵ	order t	(r_t, r_{t-1}, r_{t-2})	$(p_\epsilon)_t$	extracted solutions
	2	(4,2,1)	$6.8791\ 10^{-5}$	none
10^{-2}	3	(2,2,2)	0.3385	$\pm(1.3981, 0.4729)$
10^{-3}	3	(2,2,2)	0.2082	$\pm(1.9499, 0.4060)$
10^{-4}	3	(2,2,2)	0.1232	$\pm(2.6674, 0.3287)$
10^{-5}	3	(2,2,2)	0.0713	$\pm(3.6085, 0.2574)$
10^{-6}	3	(2,2,2)	0.0408	$\pm(4.8511, 0.1977)$
10^{-7}	3	(3,2,2)	0.0231	$\pm(6.4986, 0.1503)$
10^{-8}	3	(3,2,2)	0.0131	$\pm(8.6882, 0.1136)$
10^{-9}	3	(8,4,2)	0.0074	none
10^{-10}	3	(7,4,2)	0.0041	none

Testing copositivity of a matrix M

M copositive iff $p_{\min} = 0$ for $p(x) := \sum_{i,j=1}^{n} x_i^2 x_j^2 M_{ij}$

Example 1:

$$M = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix}$$

M is copositive, e.g., since $(\sum_i x_i^2)p$ is SOS [Parrilo 2000]

ϵ	order t	(r_t, r_{t-1}, r_{t-2})	$(p_\epsilon)_t$	extracted solutions
	2	(5,15)	$-1.4955\ 10^6$	none
10^{-2}	3	(18,6,3)	-1.5407	none
10^{-2}	4	(4,4,4)	$1.3854\ 10^{-5}$	$(\pm 0.7058, 0, 0, \pm 0.7058)$
10^{-4}	4	(9,7,5)	$1.5544\ 10^{-7}$	none

Example 2:

$$M = \begin{pmatrix} 1 & -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix}$$

ϵ	order t	(r_t, r_{t-1}, r_{t-2})	$(p_\epsilon)_t$	extracted solutions
1	3	(8,7,4)	-1.3333	none
1	4	(8,8,7)	-1.3333	$\pm(0.8165, 0.8165, 0.8165, 0, 0)$
10^{-1}	3	(8,7,4)	-133.3333	none
10^{-1}	4	(8,8,7)	-133.3333	$\pm(2.5820, 2.5820, 2.5820, 0, 0)$
10^{-2}	3	(8,7,4)	$-1.3333\ 10^4$	none

$\leadsto$ certificate that M is *not* copositive

Approximating the stability number via matrix copositivity and Pólya's representation result

[de Klerk-Pasechnik 2002] [Parrilo 2000]

M *strictly copositive*, i.e., $p_M := \sum_{i,j=1}^{n} x_i^2 x_j^2 M_{ij} > 0$ on $\mathbb{R}^n \setminus \{0\}$

$$\stackrel{\text{[Pólya]}}{\Longrightarrow} (\sum_i x_i^2)^r p_M \text{ is SOS for some } r \in \mathbb{N}$$

$$\Longrightarrow M \text{ copositive}$$

[Motzkin-Straus 1965]

$$\boxed{\alpha(G) = \min\ t \text{ s.t. } t(I + A_G) - J \text{ is copositive}}$$

Define: $\boxed{\vartheta^{(r)}(G) := \min\ t \text{ s.t. } t(I + A_G) - J \in \mathcal{K}^{(r)}}$

where $\boxed{\mathcal{K}^{(r)} := \{M \mid (\sum_i x_i^2)^r p_M \text{ is SOS}\}}$

Theorem: [dKP] $\lfloor \vartheta^{(r)}(G) \rfloor = \alpha(G)$ if $r \geq \alpha(G)^2$

Conjecture: [dKP] $\vartheta^{(r)}(G) = \alpha(G)$ if $r \geq \alpha(G) - 1$

Motivation: this rate of convergence holds for other hierarchies, e.g., by Lovász-Schrijver, Lasserre

Theorem: [Gvozdenović-L 2004] True for $\alpha(G) \leq 8$

Lasserre's hierarchy refines the hierarchy $\vartheta^{(r)}(G)$

Other results by [Peña-Vera-Zuluaga 2005]

Yet another representation result

Theorem: [Lasserre 2004]
$p \geq 0$ on $\mathbb{R}^n \Longrightarrow \forall \epsilon > 0\ \exists r \in \mathbb{N}$ for which

$$p_{r,\epsilon} := p + \epsilon \left(\sum_{k=0}^{r} \sum_{i=1}^{n} \frac{x_i^{2k}}{k!} \right) \text{ is SOS}$$

Note: $\|p_{r,\epsilon} - p\|_1 \to 0$ as $\epsilon \searrow 0$

Although there are *much more nonnegative polynomials than SOS polynomials* (at fixed degree, when $n \to \infty$) [Blekherman 2003]

$\rightsquigarrow$ new relaxation scheme for $p_{\min} = \inf_{x \in \mathbb{R}^n} p(x)$

$$\boxed{\inf\ y^T p_{r,\epsilon}\ \text{ s.t. } M_r(y) \succeq 0,\ y_0 = 1}$$

$\rightsquigarrow$ extension to the constrained case

Many other interesting results unfortunately not covered here ....

- *SDP versus LP relaxations* [Sherali-Adams] [Lasserre]

  LP relaxations arise from representations:

  $$p(x) = \sum_\alpha c_\alpha g_1^{\alpha_1} \cdots g_m^{\alpha_m} \quad \text{where } c_\alpha \in \mathbb{R}_+$$

  $\rightsquigarrow$ Asymptotic convergence when K is a polytope [Handelman 1988]

- *Exploiting the symmetry to reduce the size of the SDP*

  [Parrilo-Gaterman] [Schrijver] [Dukanovic-Rendl] [de Klerk-Pasechnik-Schrijver] [Laurent] ...

$\rightsquigarrow$ invariant theory, block-diagonalization of C^*-algebras

$\rightsquigarrow$ application, e.g., to coding theory

- *Exploiting the sparsity of the polynomials to reduce the size of the SDP*

  [Reznick 1978] [Kim-Kojima-Muramatsu-Waki] ...

- *Extension to matrix polynomial optimization*

  [Kojima] [Hol-Scherer] [Henrion-Lasserre] ...

Infeasibility certificates via SDP

Theorem: [Stengle 1974] (Positivstellensatz)
The following system (in real variable $x \in \Re^n$) is infeasible:

$$f_j(x) \geq 0 \ (j = 1, \ldots, s)$$
$$g_k(x) \neq 0 \ (k = 1, \ldots, t)$$
$$h_\ell(x) = 0 \ (\ell = 1, \ldots, u)$$

$$\Updownarrow$$

$$\exists \ f = \sum_{S \subseteq [1,s]} p_S \left(\prod_{j \in S} f_j \right) \quad \text{where } p_S \text{ are SOS}$$
$$\exists \ g = \prod_{k \in K} g_k \quad \text{where } K \subseteq [1,t]$$
$$\exists \ h = \sum_{\ell=1}^{u} q_\ell h_\ell \quad \text{where } q_\ell \text{ are polynomials}$$

such that

$$f + g^2 + h = 0$$

$\leadsto$ use SDP for searching for bounded degree infeasibility certificates

An example [Parrilo 2000]

$$\boxed{p(x,y) := 2x^4 + 2x^3y - x^2y^2 + 5y^4 \text{ is SOS ?}}$$

Can one find a decomposition:

$$p(x,y) = (x^2\ y^2\ xy)\underbrace{\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}}_{X \succeq 0}\begin{pmatrix} x^2 \\ y^2 \\ xy \end{pmatrix} \ ?$$

Compute the coefficients of the monomials of degree 4:

$$\begin{array}{lll}
x^4 = x^2 \cdot x^2 & 2 & = a \\
x^3y = x^2 \cdot xy & 2 & = 2c \\
x^2y^2 = (xy)\cdot(xy) = x^2 \cdot y^2 & -1 & = 2b + f \\
y^3 = y^2 \cdot xy & 0 & = 2e \\
y^4 = y^2 \cdot y^2 & 5 & = d
\end{array}$$

$$\text{Hence, } X = \begin{pmatrix} 2 & b & 1 \\ b & 5 & 0 \\ 1 & 0 & -1-2b \end{pmatrix}$$

$$\text{For } b = -3,\ X \succeq 0,\ X = \tfrac{1}{2}\begin{pmatrix} 2 & 0 \\ -3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

giving the SOS decomposition:

$$\boxed{p(x) = \tfrac{1}{2}(2x^2 - 3y^2 + xy)^2 + \tfrac{1}{2}(y^2 + 3xy)^2}$$