Polynomial Optimization

and

Sums of Squares of Polynomials

Monique LAURENT

CWI, Amsterdam

Some definitions about polynomials

- **Multivariate polynomial:**

$$p(x) = \sum_{\alpha \in Z_+^n} p_\alpha \underbrace{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}_{x^\alpha} = \sum_{\alpha \in Z_+^n} p_\alpha x^\alpha$$

- The **degree** of the monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is

$$|\alpha| := \sum_{i=1}^{n} \alpha_i$$

- The **degree** of the polynomial $p(x)$ is the maximum degree of a monomial x^α for which $p_\alpha \neq 0$.

Example: $x^2y^3z - y^4 + 2xz - 1$ is a polynomial in three variables with degree 6

- $S_n^d := \{\alpha \in Z_+^n : |\alpha| \leq d\}$, $|S_n^d| = \binom{n+d}{d}$
- One can identify a polynomial $p(x)$ with degree $\leq d$ with its sequence of coefficients $p = (p_\alpha)_{\alpha \in S_n^d}$
- The set $\Re_d[x_1, \ldots, x_n]$ of polynomials with degree $\leq d$ is a vector space of dimension $\binom{n+d}{d}$, isomorphic to $\Re^{S_n^d}$.

- A polynomial $p(x)$ is a **sum of squares (SOS)** if

$$p(x) = \sum_{j=1}^{m} p_j(x)^2$$

for some polynomials p_j. Necessarily, $p(x)$ has an even degree $2d$ and each $p_j(x)$ has degree $\leq d$.

Example: $x^2 + y^2 + 2xy + z^6 = (x+y)^2 + (z^3)^2$ is SOS

• A **form** (homogeneous polynomial) is a polynomial whose monomials all have the same degree:

$$p(x) = \sum_{|\alpha|=d} p_\alpha x^\alpha$$

Lemma:
A form of degree $2d$ is a SOS if and only if it is a sum of squares of forms (of degree d).

• $p(x)$ polynomial of degree d in n variables
$\longrightarrow$ its **homogenization** $f(x,t)$, the form of degree d in $n+1$ variables:

$$f(x,t) = \sum_{|\alpha|\leq d} p_\alpha x^\alpha t^{d-|\alpha|}$$

Lemma:
(i) $p(x)$ is non-negative on $\Re^n \iff f(x,t)$ is non-negative on $\Re^{n+1}$
(ii) $p(x)$ is a sum of squares $\iff f(x,t)$ is a sum of squares [of forms]

Unconstrained polynomial programming problem:

$$\boxed{p_{\min} := \inf \; p(x) \;\; \text{subject to} \;\; x \in \Re^n}$$

where $p(x) \in \Re[x_1, \ldots, x_n]$ is a multivariate polynomial with even degree $2d$

Example: One can verify that $p_{\min} = 0$ for the (Motzkin) polynomial:

$$p(x, y, z) := x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$$

Equivalently:

$$\boxed{p_{\min} = \sup \;\; \lambda \;\; \text{s.t.} \;\; p(x) - \lambda \geq 0 \;\; \text{for all } x \in \Re^n}$$

Idea: Obtain a lower bound for $p_{\min}$ by replacing the non-negativity condition by a stronger condition:

$$\boxed{p_{\min} \geq p_{sos} := \sup \;\; \lambda \;\; \text{s.t.} \;\; p(x) - \lambda \text{ is a sum of squares}}$$

Motivation: While testing whether a polynomial $p(x)$ is non-negative is a hard problem, one can test whether $p(x)$ is a sum of squares efficiently via semidefinite programming !

Recognizing sums of squares of polynomials via semidefinite programming

$$p(x) = \sum_{\substack{\alpha \in Z_+^n \\ |\alpha| \le 2d}} p_\alpha x^\alpha \text{ polynomial of degree } 2d$$

Theorem:

(1) $p(x)$ is a sum of squares of polynomials

$$\Updownarrow$$

(2) $p(x) = z^T X z$ for some matrix $X = (X_{\beta,\gamma})_{|\beta|,|\gamma| \le d} \succeq 0$

where $z := (x^\beta)_{|\beta| \le d}$

$$\Updownarrow$$

(3) The following semidefinite program is feasible:

$$\begin{cases} X \succeq 0 \\ \sum_{\beta,\gamma \in S_n^d | \beta+\gamma=\alpha} X_{\beta,\gamma} = p_\alpha \quad (|\alpha| \le 2d) \end{cases}$$

$\rightsquigarrow$ system in matrix variable X of order $\binom{n+d}{d} \times \binom{n+d}{d}$, with $\binom{n+2d}{2d}$ equations

Proof:

(1) $\Longrightarrow$ (2) Assume $p(x)$ is a SOS. That is,

$$p(x) = \sum_{j=1}^{m} p_j(x)^2,$$

for some polynomials p_j of degree $\leq d$.

As $z = (x^\beta)_{|\beta| \leq d}$ is the vector containing all monomials of degree $\leq d$, we have:

$$p_j(x) = \sum_{|\beta| \leq d} (p_j)_\beta x^\beta = z^T p_j$$

Therefore,

$$p(x) = \sum_j (z^T p_j)^2 = \sum_\ell z^T p_j p_j^T z = z^T \left(\sum_j p_j p_j^T \right) z$$

This shows that

$$p(x) = z^T X z \quad \text{for some matrix } X = (X_{\beta,\gamma})_{\beta,\gamma \in S_n^d} \succeq 0$$

(2) $\Longrightarrow$ (1) Reverse the above argument

(2) $\Longleftrightarrow$ (3) Equate the coefficients of the two polynomials $p(x)$ and $z^T X z$. Indeed,

$$\begin{aligned} z^T X z &= \sum_{\beta,\gamma \in S_n^d} x^\beta x^\gamma X_{\beta,\gamma} = \sum_{\beta,\gamma \in S_n^d} x^{\beta+\gamma} X_{\beta,\gamma} \\ &= \sum_{\alpha \in S_n^{2d}} x^\alpha \left(\sum_{\beta,\gamma | \beta+\gamma=\alpha} X_{\beta,\gamma} \right) \end{aligned}$$

An example [Parrilo 2000]

Is the polynomial: $p(x,y) := 2x^4 + 2x^3y - x^2y^2 + 5y^4$ a sum of squares ?

Solution: Decide whether there exists a matrix $X := \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \succeq 0$, indexed by the monomials of degree 2 [because p is a form of degree 4], for which:

$$p(x,y) \equiv (x^2\ y^2\ xy) \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \\ xy \end{pmatrix}$$

Compute the coefficients of the monomials of degree 4:

$$\begin{array}{ll} x^4 = x^2 \cdot x^2 & 2 = a \\ x^3y = x^2 \cdot xy & 2 = 2c \\ x^2y^2 = (xy)\cdot(xy) = x^2 \cdot y^2 & -1 = 2b + f \\ y^3 = y^2 \cdot xy & 0 = 2e \\ y^4 = y^2 \cdot y^2 & 5 = d \end{array}$$

Hence, $X = \begin{pmatrix} 2 & b & 1 \\ b & 5 & 0 \\ 1 & 0 & -1-2b \end{pmatrix} \succeq 0$ if $b = -3$, with

$$X = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ -3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

This gives the following SOS decomposition:

$$p(x) = \frac{1}{2}(2x^2 - 3y^2 + xy)^2 + \frac{1}{2}(y^2 + 3xy)^2$$

Does every nonnegative polynomial of degree d in n variables have a decomposition as a sum of squares ?

Hilbert's theorem: This holds precisely in three cases:
(i) $n = 1$ (monovariate polynomial)
(ii) $d = 2$ (polynomial of degree 2)
(iii) $n = 2$ and $d = 4$ (polynomial of degree 4 in 2 variables)

Equivalent formulation for forms: Every nonnegative form of degree d in n variables is a SOS precisely in the following three cases:
(i) $n = 2$ (binary form)
(ii) $d = 2$ (quadratic form)
(iii) $n = 3$ and $d = 4$ (ternary quartic form)

Example: The Motzkin form: $x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$ is non-negative but not a SOS

Hilbert's 17th problem (1900): Does every nonnegative polynomial $p(x)$ have a decomposition as a sum of squares of rational functions? I.e., does there exist a polynomial $q(x)$ for which

$$q(x)^2 \cdot p(x) \text{ is a SOS}$$

Answer: Yes [Artin 1927]

Pólya's theorem: If $p(x)$ is a positive even form [i.e., $p(x) > 0 \ \forall x \neq 0$, p has only even monomials $x^{2\alpha}$], then

$$(\sum_{i=1}^{n} x_i^2)^r \cdot p(x) \text{ is SOS} \quad \text{for some } r \geq 0$$

Back to the optimization problem:

$$\boxed{p_{\min} := \inf_{x \in \Re^n} p(x) = \sup \ \lambda \text{ s.t. } p(x) - \lambda \geq 0 \ \forall x \in \Re^n}$$

and its relaxation:

$$\boxed{p_{sos} := \sup \ \lambda \text{ s.t. } p(x) - \lambda \text{ is SOS}}$$

$$\boxed{p_{\min} \geq p_{sos}}$$

Equality holds for monovariate polynomials
$\leadsto$ approach of Nesterov (1997)

How to strengthen the lower bound p_{sos} ?

Idea of Lasserre: Suppose that we know a priori a ball $B(0, R) := \{x \mid R^2 - \sum_{i=1}^{n} x_i^2 \geq 0\}$ containing a global optimizer of $p(x)$. Then,

$$\boxed{\begin{aligned} p_{\min} = \min_{x \in B(0,R)} p(x) \ = \ \max \ \lambda \text{ s.t. } \ & p(x) - \lambda \geq 0 \\ & \text{for all } x \in B(0, R) \end{aligned}}$$

Again, replace the non-negativity condition by a stronger condition:

$$p(x) - \lambda = p_0(x) + p_1(x) \left(R^2 - \sum_{i=1}^{n} x_i^2 \right) \text{ where } p_0, p_1 \text{ are SOS}$$

No upper bound is known a priori on the degrees of the polynomials p_0, p_1

Define a hierarchy of lower bounds p_t for $p_{\min}$:

$$\boxed{p_t := \max\ \lambda \text{ s.t. } \ p(x) - \lambda = p_0(x) + p_1(x)(R^2 - \sum_i x_i^2) \\ \text{where } p_0, p_1 \text{ are SOS,} \\ \deg(p_0) \leq 2t,\ \deg(p_1) \leq 2t-2}$$

$$\boxed{p_t \leq p_{t+1} \leq p_{\min}}$$

Theorem [Lasserre 2000]: $\boxed{\lim_{t\to\infty} p_t = p_{\min}}$

The proof is based on the following result.

Theorem [Putinar 1993] (Special case):
If $p(x)$ is positive on the set

$$B(0,R) = \{x \mid R^2 - \sum_i x_i^2 \geq 0\}$$

then

$$(P) \qquad p(x) = p_0(x) + p_1(x)(R^2 - \sum_i x_i^2)$$

for some polynomials p_0, p_1 that are SOS.

Proof of convergence:
Let $\epsilon > 0$. Then, the polynomial $p(x) - p_{\min} + \epsilon$ is positive on $B(0,R)$. Therefore, by Putinar's theorem, it has a decomposition (P). Hence, $\lambda := p_{\min} - \epsilon$ is feasible for the t-th relaxation, for some t large enough. This implies that

$$p_{\min} - \epsilon \leq p_t$$

Extension to constrained polynomial programming:

$$\boxed{\begin{array}{rcl} p_{\min} & := & \min \;\; p(x) \\ & & \text{s.t. } \; g_\ell(x) \geq 0 \;\; (\ell = 1, \ldots, m) \\ & & \\ & = & \sup \;\; \lambda \;\; \text{s.t. } \; p(x) - \lambda \geq 0 \text{ for all } x \in F \end{array}}$$

setting

$$\boxed{F := \{x \in \Re^n \mid g_\ell(x) \geq 0 \;\; (\ell = 1, \ldots, m)\}}$$

where $p, g_1, \ldots, g_m$ are polynomials in $x_1, \ldots, x_n$

Again, one can replace the non-negativity condition by a stronger condition in terms of sums of squares, and formulate a hierarchy of lower bounds p_t for $p_{\min}$:

$$\boxed{\begin{array}{ll} p_t := \sup \;\; \lambda \;\; \text{s.t.} & p(x) - \lambda = p_0(x) + \sum_{\ell=1}^{m} p_\ell(x) g_\ell(x) \\ & \qquad \text{where } p_0, \; p_1, \ldots, p_m \text{ are SOS} \\ & \qquad\quad \deg(p_0), \; \deg(p_\ell g_\ell) \leq 2t \end{array}}$$

for all t s.t. $2t \geq \deg(p)$, $\deg(g_\ell)$

Theorem [Lasserre 2000]:

If F is compact and ‘good’, then $\boxed{\lim_{t \to \infty} p_t = p_{\min}}$

Proof: Apply Putinar’s theorem.

Assumption on the semi-algebraic set F:
F is said to be **good** if there exists a polynomial $u(x)$ of the form

$$u(x) = u_0(x) + \sum_{\ell=1}^{m} u_\ell(x) g_\ell(x)$$

where $u_0, u_1, \ldots, u_m$ are SOS, and for which the set

$$\{x \mid u(x) \geq 0\}$$

is compact.

Examples:
• F is good if the set $\{x \mid g_\ell(x) \geq 0\}$ is compact for at least one of the constraints in its description
• F is good if the constraints $x_i^2 - x_i = 0$ $(i = 1, \ldots, n)$ are present in its description
$\rightsquigarrow$ 0/1 case

Theorem [Putinar 1993]:
Assume that the semi-algebraic set F is compact and good. Then, every polynomial $p(x)$ positive on F has a decomposition:

$$p(x) = p_0(x) + \sum_{\ell=1}^{m} p_\ell(x) g_\ell(x)$$

where $p_0, p_1, \ldots, p_m$ are SOS.

Dual formulation for $p_{\min} = \inf_{x \in \Re^n} \; p(x)$

$\mathcal{P}_{2d}$:= set of polynomials of degree $\leq 2d$ that are non-negative on $\Re^n$; $\mathcal{P}_{2d}$ is a cone in the space $\Re^{S_n^{2d}}$

What is the dual cone of $\mathcal{P}_{2d}$?

Which sequences $y = (y_\alpha) \in \Re^{S_n^{2d}}$ belong to $(\mathcal{P}_{2d})^*$, i.e., satisfy:

$$y^T p = \sum_{\alpha \in S_n^{2d}} y_\alpha p_\alpha \geq 0 \;\; \forall p \in \mathcal{P}_{2d} \,?$$

Definition: $\mathcal{M}_{2d}$:= set of $y \in \Re^{S_n^{2d}}$ that are the **sequences of moments** of some non-negative measure μ on $\Re^n$, i.e.,

$$y_\alpha = \int x^\alpha d\mu(x) \;\; (\alpha \in S_n^{2d})$$

Lemma: $\boxed{\mathcal{P}_{2d} = (\mathcal{M}_{2d})^*. \text{ Hence, } (\mathcal{P}_{2d})^* = cl(\mathcal{M}_{2d}).}$

Therefore:

$$\boxed{\begin{aligned} p_{\min} = \inf_{x \in \Re^n} \; p(x) &= \sup \;\; \lambda \text{ s.t. } \; p(x) - \lambda \in \mathcal{P}_{2d} \\ &= \inf \;\; p^T y \text{ s.t. } \; y \in \mathcal{M}_{2d}, \;\; y_0 = 1 \end{aligned}}$$

Proof:

$$\begin{aligned} p(x) - \lambda \in \mathcal{P}_{2d} &\iff (p(x) - \lambda)^T y \geq 0 \quad \forall y \in \mathcal{M}_{2d} \\ &\iff p^T y - \lambda y_0 \geq 0 \quad \forall y \in \mathcal{M}_{2d} \\ &\iff p^T y - \lambda \geq 0 \quad \forall y \in \mathcal{M}_{2d} \text{ with } y_0 = 1 \end{aligned}$$

Proof of the lemma: $\mathcal{P}_{2d} = (\mathcal{M}_{2d})^*$

$$\boxed{\mathcal{P}_{2d} \stackrel{?}{\subseteq} (\mathcal{M}_{2d})^*.}$$

Indeed, if $p \in \mathcal{P}_{2d}$, $y \in \mathcal{M}_{2d}$, then:

$$p^T y = \sum_{\alpha} p_\alpha y_\alpha = \sum_{\alpha} p_\alpha \int x^\alpha d\mu(x)$$

$$= \int (\sum_{\alpha} p_\alpha x^\alpha) d\mu = \int p(x) d\mu(x) \geq 0$$

$$\boxed{\mathcal{P}_{2d} \stackrel{?}{\supseteq} (\mathcal{M}_{2d})^*}$$

Let $p \notin \mathcal{P}_{2d}$. Then, $p(x_0) < 0$ for some $x_0 \in \Re^n$.
Let μ be the Dirac measure at x_0 [with mass 1 at x_0 and 0 elsewhere] and let y be its sequence of moments.
Then, $y \in \mathcal{M}_{2d}$, $y = (x_0^\alpha)_\alpha$, which implies that

$$p^T y = p(x_0) < 0$$

Therefore, $p \notin (\mathcal{M}_{2d})^*$.

Dual formulation for the bound p_{sos} Derivation via LP-cone duality

$$p_{sos} = \sup\ \lambda \text{ s.t. } p(x) - \lambda \text{ is a SOS}$$

What is the dual cone of the SOS cone Σ_{2d} ?

$$\boxed{\Sigma_{2d} := \text{ set of polynomials of degree } \leq 2d \text{ that are SOS}}$$

Definition: Given $y \in \Re^{S_n^{2d}}$, its **moment matrix** is the matrix:

$$M_d(y) := (y_{\beta+\gamma})_{\beta,\gamma \in S_n^d}$$

Define the cone:

$$\boxed{\mathcal{M}_{2d}^+ := \{y \in \Re^{S_n^{2d}} \mid M_d(y) \succeq 0\}}$$

Lemma:
(i) $\Sigma_{2d} \subseteq \mathcal{P}_{2d}$
(ii) $\mathcal{M}_{2d} \subseteq \mathcal{M}_{2d}^+$
(iii) Σ_{2d}, $\mathcal{M}_{2d}^+$ are closed cones
(iv) $\Sigma_{2d} = (\mathcal{M}_{2d}^+)^*$

Therefore:

$$\boxed{\begin{aligned} p_{sos} &= \sup\ \lambda \text{ s.t. } p(x) - \lambda \in \Sigma_{2d} \\ &= \inf\ p^T y \text{ s.t. } y \in \mathcal{M}_{2d}^+,\ y_0 = 1 \end{aligned}}$$

Proof of the lemma:

(ii) Let $y \in \mathcal{M}_{2d}$, i.e., $y_\alpha = \int x^\alpha d\mu(x)$, for some non-negative measure μ. Then, for all $u \in \Re^{S_n^d}$,

$$u^T M_d(y) u = \sum_{\beta,\gamma \in S_n^d} u_\beta u_\gamma y_{\beta+\gamma} = \sum_{\beta,\gamma \in S_n^d} u_\beta u_\gamma \int x^{\beta+\gamma} d\mu(x)$$
$$= \int u(x)^2 d\mu(x) \geq 0$$

Therefore, $y \in \mathcal{M}_{2d}^+$.

(iv) $\boxed{\Sigma_{2d} \overset{?}{\subseteq} (\mathcal{M}_{2d}^+)^*}$

Let $p \in \Sigma_{2d}$, i.e., $p = \sum_j p_j^2$, and $y \in \mathcal{M}_{2d}^+$, i.e., $M_d(y) \succeq 0$.
Define $X := \sum_j u_j u_j^T$. We claim that

$$p^T y \overset{?}{=} \langle X, M_d(y) \rangle$$

Indeed,

$$\langle X, M_d(y) \rangle = \sum_{\beta,\gamma \in S_n^d} X_{\beta,\gamma} y_{\beta+\gamma} = \sum_{\alpha \in S_n^{2d}} y_\alpha \left(\sum_{\beta,\gamma | \beta+\gamma=\alpha} X_{\beta,\gamma} \right)$$
$$= \sum_{\alpha \in S_n^{2d}} y_\alpha p_\alpha$$

Therefore, $p^T y = \langle X, M_d(y) \rangle \geq 0$.

$$\boxed{(\mathcal{M}_{2d}^+)^* \overset{?}{\subseteq} \Sigma_{2d}, \text{ i.e., } (\Sigma_{2d})^* \overset{?}{\subseteq} \mathcal{M}_{2d}^+}$$

Let $y \notin \mathcal{M}_{2d}^+$, i.e., $M_d(y) \not\succeq 0$.
Hence, $u^T M_d(y) u < 0$ for some $u \in \Re^{S_n^d}$.
Then, the polynomial $p(x) := u(x)^2$ belongs to Σ_{2d} and $p^T y = \langle uu^T, M_d(y) \rangle = u^T M_d(y) u < 0$.
Therefore, $y \notin (\Sigma_{2d})^*$.

Dual formulation for the bound p_{sos}
Direct derivation via SDP duality

Let p be a polynomial of degree $2d$

$$
\begin{aligned}
p_{sos} &= \sup \lambda \text{ s.t. } \quad p(x) - \lambda \text{ is a SOS} \\
&= \sup \lambda \text{ s.t. } \quad X \succeq 0 \\
& \qquad\qquad\qquad \langle X, A_\alpha \rangle [= \sum_{\beta,\gamma | \beta+\gamma=\alpha} X_{\beta,\gamma}] = p_\alpha \quad (\alpha \neq 0) \\
& \qquad\qquad\qquad \langle X, A_0 \rangle [= X_{00}] = p_0 - \lambda
\end{aligned}
$$

That is,

$$p_{sos} = p_0 + \ \sup \ -X_{00} \ \text{ s.t. } X \succeq 0, \ \langle X, A_\alpha \rangle = p_\alpha \ (\alpha \neq 0)$$

Dual semidefinite program:

$$p_{sos} \leq \ p_0 + \inf \ \sum_{\alpha \neq 0} p_\alpha y_\alpha \ \text{ s.t. } \ \sum_{\alpha \neq 0} y_\alpha A_\alpha + A_0 \succeq 0$$

$$= \inf \ p^T y \ \text{ s.t. } M_d(y) \succeq 0, \ y_0 = 1$$

No duality gap:

Let y be the sequence of moments of a probability measure with strictly positive density $f(x)$. Then, y is dual strictly feasible. Indeed,

$$u^T M_d(y) u = \int u(x)^2 f(x) dx > 0 \ \ \forall u \in \Re^{S_n^d}$$

Constrained case
Dual formulation for the bound:

$$p_t = \max\ \lambda \text{ s.t. } \quad p(x) - \lambda = p_0(x) + \sum_{\ell=1}^{m} p_\ell(x) g_\ell(x)$$
$$\text{where } p_0, p_\ell \text{ are SOS and}$$
$$\deg(p_0),\ \deg(p_\ell g_\ell) \leq 2t$$

Set $\deg(g_\ell) = 2v_\ell$, or $2v_\ell - 1$,
and define the cone:

$$\Sigma_{2t}(F) := \{p_0 + \sum_\ell p_\ell g_\ell \mid\ p_0, p_\ell \text{ are SOS},\ \deg(p_0) \leq 2t,\ \deg(p_\ell) \leq 2(t - v_\ell)\}$$

What is the dual cone of $\Sigma_{2t}(F)$?

Necessarily, $M_t(y) \succeq 0$ for $y \in (\Sigma_{2t}(F))^*$.
How to express the conditions $g_\ell(x) \geq 0$ defining the semi-algebraic set F ?

Definition: Given two sequences g and y, define the new sequence

$$g * y := (\sum_\beta g_\beta y_{\alpha+\beta})_\alpha$$

$$\mathcal{M}^+_{2t}(F) := \{y \in \Re^{S_n^{2t}} \mid M_t(y) \succeq 0,\ M_{t-v_\ell}(g_\ell * y) \succeq 0\ \ \forall \ell\}$$

Lemma:
$$\mathcal{M}^+_{2t}(F) = (\Sigma_{2t}(F))^*$$
implying: $(\mathcal{M}^+_{2t}(F))^* = cl(\Sigma_{2t}(F))$

Corollary:

$$p_t \leq \mu_t := \min\ p^T y \text{ s.t. } y \in \mathcal{M}^+_{2t}(F), y_0 = 1\ \leq p_{\min}$$

Proof of the corollary:

$$\boxed{p_t \overset{?}{\leq} \mu_t}$$

By definition, p_t is the maximum λ for which the polynomial $p(x) - \lambda$ belongs to the cone $\Sigma_{2t}(F)$. But:

$$\begin{aligned} p(x) - \lambda \in \Sigma_{2t}(F) &\Longrightarrow p(x) - \lambda \in cl(\Sigma_{2t}(F)) = (\mathcal{M}_{2t}^{+}(F))^* \\ &\Longleftrightarrow (p(x) - \lambda)^T y \geq 0 \quad \forall y \in \mathcal{M}_{2t}^{+}(F) \\ &\Longleftrightarrow p^T y \geq \lambda y_0 \quad \forall y \in \mathcal{M}_{2t}^{+}(F) \\ &\Longleftrightarrow p^T y \geq \lambda \quad \forall y \in \mathcal{M}_{2t}^{+}(F) \text{ with } y_0 = 1 \end{aligned}$$

$$\boxed{\mu_t \overset{?}{\leq} p_{\min}}$$

Given $x \in F$, consider the sequence $y := (x^\alpha)_{\alpha \in S_n^{2t}}$.
Thus, $y_0 = 1$ and $p(x) = p^T y$.
Hence, it suffices to verify that $y \in \mathcal{M}_{2t}^{+}(F)$, i.e.,
$M_t(y) \succeq 0$ and $M_{t-v_\ell}(g_\ell * y) \succeq 0$ for all ℓ.

$$\begin{aligned} u^T M_t(y) u &= \sum_{\beta,\gamma} u_\beta u_\gamma y_{\beta+\gamma} = \sum_{\beta,\gamma} u_\beta u_\gamma x^{\beta+\gamma} \\ &= \sum_{\beta,\gamma} u_\beta u_\gamma x^\beta x^\gamma = u(x)^2 \geq 0 \end{aligned}$$

Similarly,

$$u^T M_{t-v_\ell}(g_\ell * y) u = \sum_{\beta,\gamma} u_\beta u_\gamma (g_\ell * y)_{\beta+\gamma}$$

$$= \sum_{\beta,\gamma} u_\beta u_\gamma \left(\sum_\delta (g_\ell)_\delta y_{\beta+\gamma+\delta}\right) = \sum_{\beta,\gamma} u_\beta u_\gamma \left(\sum_\delta (g_\ell)_\delta x^{\beta+\gamma+\delta}\right)$$

$$= \sum_{\beta,\gamma} u_\beta u_\gamma \left(\sum_\delta (g_\ell)_\delta x^\beta x^\gamma x^\delta\right) = \left(\sum_\delta g_\delta x^\delta\right) \cdot \left(\sum_{\beta,\gamma} u_\beta u_\gamma x^\beta x^\gamma\right)$$

$$= g_\ell(x) \cdot u(x)^2 \geq 0$$

Proof of the lemma:

Polynomial $0/1$ programming

$$\boxed{\begin{array}{ll} F = \{x \in \Re^n \mid & g_\ell(x) \geq 0 \ \forall \ell = 1, \ldots, m \\ & h_i(x) = x_i - x_i^2 = 0 \ \forall i = 1, \ldots, n\} \end{array}}$$

Lasserre [2001] shows finite convergence of the hierarchy of bounds p_t. Namely,

$$\boxed{p_t = p_{\min} \ \text{ for } t \geq n + v - 1, \quad \text{where } v := \max_\ell \left\lceil \frac{\deg(g_\ell)}{2} \right\rceil}$$

Lasserre's proof uses results of Curto-Fialkow about moment sequences

When F is the set of $0/1$ points in a polytope, we find again the convergence result in n steps, obtained earlier via lift-and-project techniques

Elementary proof: Reduction to (combinatorial) moment matrices indexed by $\mathcal{P}(V)$

$$\alpha \in Z_+^n \rightsquigarrow \overline{\alpha} \in \{0,1\}^n \text{ with } \overline{\alpha}_i := \min(\alpha_i, 1)$$

$$M(h_i * y) = 0 \rightsquigarrow y_{\overline{\alpha}} = y_\alpha \ \ \forall \alpha \in Z_+^n$$

For $t \geq n$,

$$M_t(y) = (y_{\alpha+\beta})_{\substack{\alpha,\beta \in Z_+^n \\ |\alpha|,|\beta| \leq t}} \succeq 0$$

$$\Updownarrow$$

$$M_n(y) = (y_{\alpha+\beta})_{\substack{\alpha,\beta \in Z_+^n \\ |\alpha|,|\beta| \leq n}} \succeq 0$$

$$\Updownarrow$$

$$M_V(y) = (y_{\alpha+\beta})_{\alpha,\beta \in \{0,1\}^n} \succeq 0$$