# The Maximum Stable Set Problem: Basic Polyhedral Results and The Theta Function $\vartheta(G)$

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## The maximum stable set problem:

Given a graph $G = (V, E)$ with node weights $w \in \Re_+^V$,
find a stable set $S \subseteq V$ of maximum weight

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|---|
| $\alpha(G) :=$ maximum cardinality of a stable set<br>$\mathrm{STAB}(G) := \mathrm{conv}\{\chi^S \in \Re^V \mid S \subseteq V \text{ stable set}\}$ |

## Some valid inequalities for STAB($G$):

(i) trivial constraints: $0 \leq x_i \leq 1 \quad$ for $i \in V$

(ii) the edge inequalities: $x_i + x_j \leq 1 \quad$ for $ij \in E$

(iii) the odd circuits ineq: $\sum_{i \in C} x_i \leq \frac{|C|-1}{2} \quad$ for $C$ odd circuit

(iv) the clique inequalities: $\sum_{i \in Q} x_i \leq 1 \quad$ for $Q$ clique

## Some facts:

• FRAC($G$) := polytope determined by (i), (ii)
$\mathrm{FRAC}(G) = \mathrm{STAB}(G) \Longleftrightarrow G$ bipartite

• CSTAB($G$) := polytope determined by (i),(ii),(iii)
Call $G$ *$t$-perfect* if $\mathrm{CSTAB}(G) = \mathrm{STAB}(G)$

**Theorem:** One can optimize a linear objective function over CSTAB($G$) in polynomial time.
Hence, the maximum stable set problem can be solved in polynomial time in $t$-perfect graphs.

**Proof:** Indeed, one can solve the separation problem for the system (i),(ii),(iii) in polynomial time.
Let $x \in Q^V$ satisfying (i),(ii) and set

$$z_{ij} := 1 - x_i - x_j \text{ for all edges } ij \in E.$$

Then, $z \in Q_+^E$ and

$$\boxed{x \text{ satisfies (iii)} \iff z(C) \geq 1 \ \forall C \text{ odd circuit}}$$

**Fact:** One can find a shortest odd circuit in $G$ in polynomial time.

**Proof:** Let $H$ be the new graph obtained from $G$ by replacing each node $u$ by two copies $u_1$ and $u_2$, and each edge $uv$ by the edges $u_1v_2$ and $u_2v_1$.

$$\boxed{\text{odd circuit in } G \text{ through node } u \longrightarrow \text{ path in } H \text{ from } u_1 \text{ to } u_2}$$

Hence, one can find a shortest odd circuit in $G$ by $n$ shortest path computations in $H$

• QSTAB($G$) := polytope determined by (i), (iii)

Call $G$ *perfect* if QSTAB($G$) = STAB($G$).

**Theorem [Fulkerson 1970] [Chvátal 1975]:**

$G$ is perfect $\iff$ $\omega(G') = \chi(G')$
for all induced subgraphs $G'$ of $G$
[original definition of Berge (1962)]

$\chi(G)$ := minimum number of colours needed to colour the nodes in such a way that no two adjacent nodes receive the same colour

$\omega(G) := \alpha(\overline{G})$ = maximum cardinality of a clique

1) Odd circuits and their complements are not perfect

2) Bipartite graphs, and their complements are perfect

3) Line graphs of bipartite graphs, and their complements are perfect

**Perfect graph theorem [Lovász 72]:**

$G$ is perfect $\iff$ its complement $\overline{G}$ is perfect

**Strong perfect graph theorem [Chudnovsky, Robertson, Seymour, Thomas 2002]:**

$G$ is perfect $\iff$ $G$ contains no odd circuit or its complement as an induced subgraph

# How to solve the stable set problem in perfect graphs?

**Note:** Optimizing over QSTAB($G$) does not help! Testing membership in QSTAB($G$) is indeed an instance of the maximum weight clique problem ...

**Answer: Use semidefinite programming and the Lovász theta number $\vartheta(G)$**

$$\boxed{\begin{array}{rll} \vartheta(G) := & \max & \langle J, X\rangle \\ & \text{s.t.} & \langle I, X\rangle = 1 \qquad\qquad \text{(SDP)} \\ & & X_{ij} = 0 \ (ij \in E) \\ & & X \succeq 0 \end{array}}$$

$$\boxed{\alpha(G) \overset{(1)}{\leq} \vartheta(G) \overset{(2)}{\leq} \chi(\overline{G})}$$

**Proof:** (1) $S$ stable set $\rightsquigarrow X := \frac{\chi^S(\chi^S)^T}{|S|}$ feasible for (SDP)

(2) Let $Q_1, \ldots, Q_k$ be a partition of $V$ into $k := \chi(\overline{G})$ cliques and let $X$ be feasible for (SDP). Then,

$$0 \leq \sum_{h=1}^{k} (k\chi^{Q_h} - e)^T X (k\chi^{Q_h} - e)$$

$$= k^2 \langle \sum_{h=1}^{k} \chi^{Q_h}(\chi^{Q_h})^T, X\rangle - 2k(\sum_{h=1}^{k} \chi^{Q_h})^T Xe + k\langle J, X\rangle$$

$$= k^2\langle X, I\rangle - k\langle J, X\rangle = k^2 - k\langle J, X\rangle\rangle$$

**Corollary:** $\boxed{\vartheta(G) = \alpha(G) \text{ for } G \text{ perfect}}$

## Dual formulation for $\vartheta(G)$:

$$\boxed{\vartheta(G) = \min\ \lambda \text{ s.t. } \lambda I + \sum_{ij\in E} \lambda_{ij} E_{ij} - J \succeq 0}$$

No duality gap as $X := I/n$ is strictly feasible for (SDP)

## Reformulating $\vartheta(G)$ using the semidefinite relaxation $\mathrm{TH}(G)$ of $\mathrm{STAB}(G)$

$$\begin{aligned}\mathrm{TH}(G) := \{x \in \Re^V \mid \binom{1}{x} = Ye_0 \quad & \text{for some sym. matrix } Y \\ & \text{indexed by } \{0\}\cup V \text{ s.t.} \\ & \text{(i)} \quad Y \succeq 0 \\ & \text{(ii)} \quad Y_{ii} = Y_{0i} \quad (i \in V) \\ & \text{(iii)} \quad Y_{ij} = 0 \quad (ij \in E)\}\end{aligned}$$

**Lemma:** $\boxed{\mathrm{STAB}(G) \overset{(1)}{\subseteq} \mathrm{TH}(G) \overset{(2)}{\subseteq} \mathrm{QSTAB}(G)}$

**Proof:** (1) $S$ stable set $\leadsto x := \chi^S \in \mathrm{TH}(G)$ as the matrix $Y := \binom{1}{x}\binom{1}{x}^T$ satisfies (i), (ii), (iii)
(2) Let $x \in \mathrm{TH}(G)$, let $Y$ satisfying (i), (ii), (iii) s.t. $x = Ye_0$, and let $Q$ be a clique. As the principal submatrix of $Y$ indexed by $Q\cup\{0\}$ is psd, this implies that $\sum_{i\in Q} x_i \leq 1$.

**Theorem:** $\boxed{\vartheta(G) = \max_{x\in \mathrm{TH}(G)} e^T x}$

## Proof:

$$\boxed{\vartheta(G) = \max_{X \succeq 0,\ \langle I, X\rangle = 1,\ X_{ij}=0\ (ij \in E)} \langle J, X\rangle \overset{?}{\leq} \mu_G := \max_{x \in \mathrm{TH}(G)} e^T x}$$

Let $X$ be an optimum solution for the program defining $\vartheta(G)$, and let $v_1, \ldots, v_n$ be vectors s.t. $X_{ij} = v_i^T v_j$ for all $i, j \in V$. Thus,

$$\vartheta(G) = \langle J, X\rangle = \sum_{i,j=1}^{n} X_{ij} = \sum_{i,j=1}^{n} v_i^T v_j = \| \sum_{i=1}^{n} v_i \|^2$$

$$1 = \langle X, I\rangle = \sum_{i=1}^{n} X_{ii} = \sum_{i=1}^{n} \|v_i\|^2$$

$$0 = X_{ij} = v_i^T v_j \quad \text{for all edges } ij \in E$$

## Set:

$$P := \{i \in V \mid v_i \neq 0\},\ u_0 := \frac{\sum_{i=1}^{n} v_i}{\| \sum_{i=1}^{n} v_i \|},\ u_i := \frac{v_i}{\|v_i\|}\ (i \in P)$$

and let $\{u_i \mid i \in V \setminus P\}$ be an orthonormal basis of the orthogonal complement of the space $\Re(u_i \mid i \in P)$.

## Define:

$$Z := (u_i^T u_j)_{i,j=0}^n$$

$$D := \operatorname{diag}(u_0^T u_0, u_0^T u_1, \ldots, u_0^T u_n)$$

$$Y := DZD = ((u_i^T u_j)(u_0^T u_i)(u_0^T u_j))_{i,j=0}^n$$

## Then:

- $Y$ satisfies (i), (ii), (iii)

For (i): $Y_{0i} = (u_0^T u_i)^2 = Y_{ii}$ for $i = 1, \ldots, n$

- Therefore, $(Y_{0i})_{i=1}^n \in \mathrm{TH}(G)$, implying $\sum_{i=1}^n Y_{0i} \leq \mu_G$
- $\vartheta(G) \overset{?}{\leq} \sum_{i=1}^n Y_{0i}$

Indeed, $u_0 = \frac{\sum_i v_i}{\|\sum_i v_i\|} = \frac{\sum_i v_i}{\sqrt{\vartheta(G)}}$. Taking the inner product with $u_0$ implies:

$$\vartheta(G) = \left(\sum_{i=1}^n u_0^T v_i\right)^2 = \left(\sum_{i=1}^n u_0^T u_i \|v_i\|\right)^2$$

$$\leq \left(\sum_{i=1}^n (u_0^T u_i)^2\right) \cdot \left(\sum_{i=1}^n \|v_i\|^2\right) = \sum_{i=1}^n Y_{0i}$$

**Proof:**

$$\boxed{\vartheta(G) = \max_{X\succeq 0,\ \langle I,X\rangle=1,\ X_{ij}=0\ (ij\in E)} \langle J, X\rangle \overset{?}{\geq} \mu_G := \max_{x\in \mathrm{TH}(G)} e^T x}$$

Let $x \in \mathrm{TH}(G)$ be optimum for $\mu_G$,
let $Y$ be a matrix satisfying (i), (ii), (iii) s.t. $x = Ye_0$,
and let $v_0, v_1, \ldots, v_n$ be vectors s.t.
$Y_{ij} = v_i^T v_j$ for $i, j = 0, \ldots, n$.

**Hence:**

$$\mu_G = e^T x = \sum_{i=1}^n Y_{0i} = \sum_{i=1}^n Y_{ii} = \sum_{i=1}^n \|v_i\|^2$$

$$0 = Y_{ij} = v_i^T v_j \text{ for all } ij \in E$$

**The matrix:** $X := \frac{1}{\mu_G}(v_i^T v_j)_{i,j=1}^n$

**satisfies:**

$$X \succeq 0,\ X_{ij} = 0\ (ij \in E)$$

$$\langle X, I\rangle = \sum_{i=1}^n X_{ii} = \frac{1}{\mu_G}\sum_{i=1}^n \|v_i\|^2 = 1$$

**We have:**

$$\vartheta(G) \geq \langle J, X\rangle = \sum_{i,j=1}^n X_{ij} = \frac{1}{\mu_G}\sum_{i,j=1}^n v_i^T v_j = \frac{1}{\mu_G}\|\sum_{i=1}^n v_i\|^2$$

$$\mu_G = \sum_{i=1}^n Y_{0i} = \sum_{i=1}^n v_0^T v_i = v_0^T(\sum_{i=1}^n v_i) \leq \|v_0\|\cdot\|\sum_{i=1}^n v_i\| = \|\sum_{i=1}^n v_i\|$$

**Implying:**

$$\vartheta(G) \geq \mu_G$$

**Corollary [Grötschel, Lovász, Schrijver 1988]:**
One can compute $\alpha(G)$ in polynomial time for perfect graphs.
One can also find in polynomial time a maximum weight stable set in a perfect graph.

**Proof for the cardinality case:**
Let $G$ be perfect and let $v_1, \ldots, v_n$ be an ordering of its nodes. One can construct a maximum stable set in the following way.

• Compute $\alpha(G \backslash v_1)$.
(i) If $\alpha(G \backslash v_1) = \alpha(G)$, set $G_1 := G \backslash v_1$
(ii) else, set $G_1 := G$. [$v_1$ must be in a maximum stable set]
Hence, $\alpha(G_1) = \alpha(G)$.

• Iterate with $G_1$, $v_2$ in place of $G$, $v_1$, etc.

After $n$ steps, the last found graph $G_n$ is a stable set of size $\alpha(G)$.

As $G$ is perfect, the stability numbers of all intermediary graphs $G_i$ can be computed in polynomial time, since $\alpha(G_i) = \vartheta(G_i)$ for all $i$.

## How good is the approximation of $\alpha(G)$ by $\vartheta(G)$?

The situation is less nice than for the max-cut problem ...

No constant approximation algorithm can exist ...

**Theorem [Arora et al. 1992]:**
Exists $\epsilon > 0$ for which the following holds:
If P $\neq$ NP, there is no polynomial time algorithm permitting to find in a graph $G$ a stable set $S$ s.t.

$$\frac{|S|}{\alpha(G)} \geq \frac{1}{n^\epsilon}$$

The gap between $\vartheta(G)$ and $\alpha(G)$ can be made arbitrarily large ...

**Theorem [Feige 1995]:**
$\forall \epsilon > 0$, there is a family of graphs for which

$$\frac{\vartheta(G)}{\alpha(G)} \geq n^{1-\epsilon}$$

## Semidefinite programming does not seem to help for the vertex cover problem

$T \subseteq V$ is *vertex cover* if every edge is adjacent to a node in $T$; i.e., if $V \setminus T$ is a stable set.

$vc(G) :=$ minimum size of a vertex cover $= n - \alpha(G)$

$$= \min e^T x \;\text{ s.t. }\; x \in \{0,1\}^n,\; x_i + x_j \leq 1\; (ij \in E)$$

$lp(G) := \min e^T x \;\text{ s.t. }\; x \in [0,1]^n,\; x_i + x_j \leq 1\; (ij \in E)$

$$\boxed{lp(G) \overset{(1)}{\leq} vc(G) \overset{(2)}{\leq} 2\, lp(G)}$$

(1) obvious
(2) $x$ optimum solution of the LP $\Longrightarrow T := \{i \mid x_i \geq 1/2\}$ is a vertex cover and $|T| \leq 2\, lp(G)$

**Corollary:** Exists 2-approximation algorithm for the minimum vertex cover problem.

**Up to now:** No $(2-\epsilon)$-approx. alg. is known for any $\epsilon > 0$. [It is known that no 1.36-approximation alg. exists, unless P=NP]

**The SDP bound $n - \vartheta(G)$ satisfies:** $\boxed{n - \vartheta(G) \leq lp(G)}$

**But,** there exist graphs for which $\frac{vc(G)}{n - \vartheta(G)} \simeq 2$.
Hence, in general, the basic SDP bound is no better than the basic LP bound!

## How to define stronger semidefinite relaxations of the stable set polytope ?

- use the Lovász-Schrijver $N$ and $N_+$ operators [see later]
- use 'lifting' to higher dimensional space

**Note:** $x \in \mathrm{STAB}(G) \cap \{0,1\}^V \rightsquigarrow Y = \binom{1}{x}\binom{1}{x}^T$ satisfying:

$$\boxed{\begin{array}{l} Y \succeq 0 \\ Y_{0i} = Y_{ii} \quad (i \in V) \\ Y_{ij} = 0 \quad (ij \in E) \end{array}}$$

$$\rightsquigarrow \mathrm{TH}(G)$$

**Idea: Lift** $x$ **to** $y \in \Re^N$ **(**$N > n$**) and consider** $Y = yy^T$

Namely, $x \rightsquigarrow y = \left(\prod_{i \in I} x_i\right)_{\substack{I \subseteq V \\ |I| \le t}} \rightsquigarrow Y := yy^T$ satisfying:

$$\boxed{\begin{array}{l} Y \succeq 0 \\ Y_{IJ} \text{ depends only on the union } I \cup J \\ Y_{IJ} = 0 \quad \text{if } I \cup J \text{ is not stable} \end{array}}$$

$$\rightsquigarrow Q_t(G) \text{ [Lasserre relaxations]}$$

$$\boxed{Q_1(G) = \mathrm{TH}(G) \subseteq \ldots \subseteq Q_t(G) \subseteq \ldots \subseteq Q_{\alpha(G)}(G) = \mathrm{STAB}(G)}$$