Integer Points in Semi-Algebraic Cones

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Overview

- Integral bases and Integral Basis Method
- Existence of finite integral bases
- Motivating an extension
- Semi-algebraic cones and integral function bases
- Relaxed integral function bases

Primal Approach to Solve IP

$$\min\{c^\intercal x : Ax \leq b, x \in \mathbb{Z}^n_+\}$$

- Find a feasible solution x_0.
- Improve x_0 (and iterate) or assert optimality of x_0.

Integral Basis Method

$$\min\{c^\intercal x : Ax \leq b, x \in \mathbb{Z}_+^n\}$$

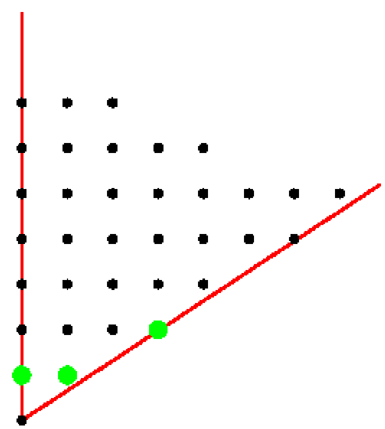
Idea:

- Reformulate simplex tableau by replacing one column with several columns using integral bases.
- Repeat until reduced cost shows optimality or an improving vector is found.

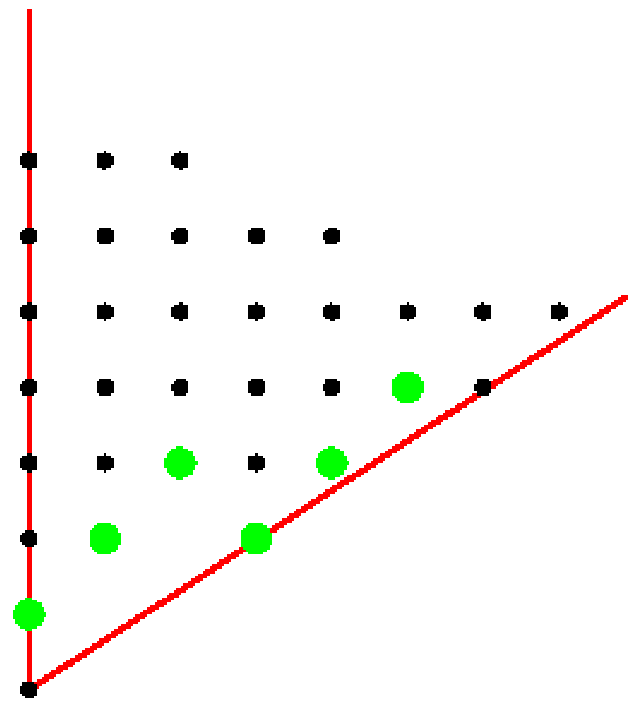
Integral Bases

$T \subseteq S \subseteq \mathbb{Z}^n$ is an **integral basis** of S, if for every $s \in S$ there exists a finite integer linear combination

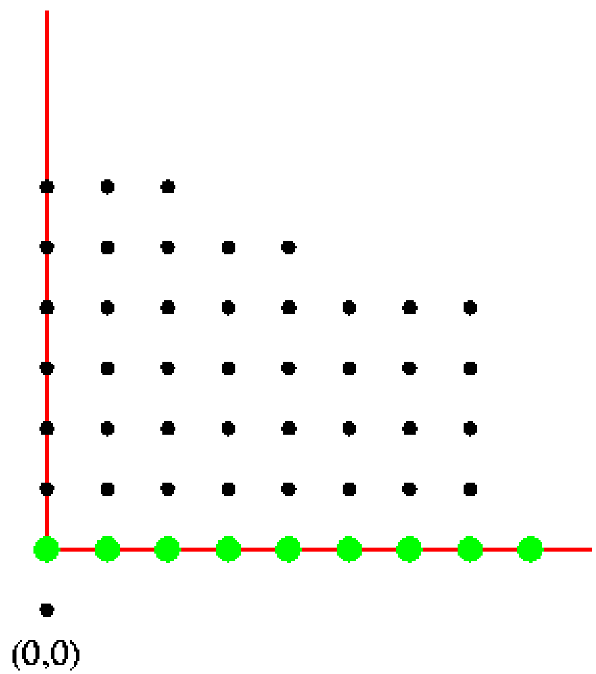
$$s = \sum \alpha_i t_i \quad \text{with} \quad t_i \in T \quad \text{and} \quad \alpha_i \in \mathbb{Z}_+.$$



Example: Integral basis of $S \neq \{x : Ax \leq b\} \cap \mathbb{Z}^n$



Integral bases need NOT be finite!



$$S = \{x \geq 0, y \geq 1\} \cap \mathbb{Z}^2$$

Finiteness Criterion for $S = \{x : Ax \leq b\} \cap \mathbb{Z}^n$

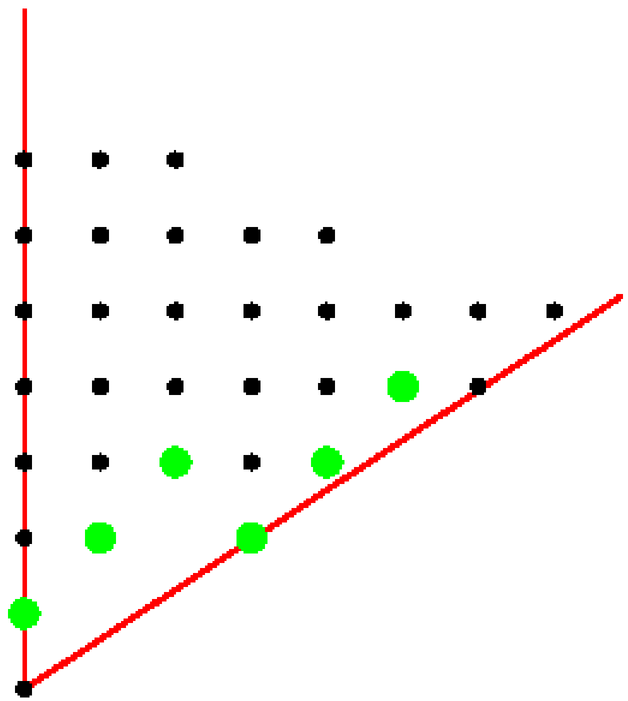
Theorem. [Bertsimas & Weismantel] *For $A \in \mathbb{Z}^{d\times n}$ and $b \in \mathbb{Z}^d$, let*

$$S = \{x : Ax \leq b, x \geq 0\} \cap \mathbb{Z}^n, \text{ and } C = \{x : Ax \leq 0, x \geq 0\}.$$

(a) *There exists a finite integral generating set of S if and only if $(C \cap \mathbb{Z}^n) \setminus S$ is finite.*

(b) *If a finite integral generating set of S exists, then there is a unique integral basis of S.*

What if there are “holes”?



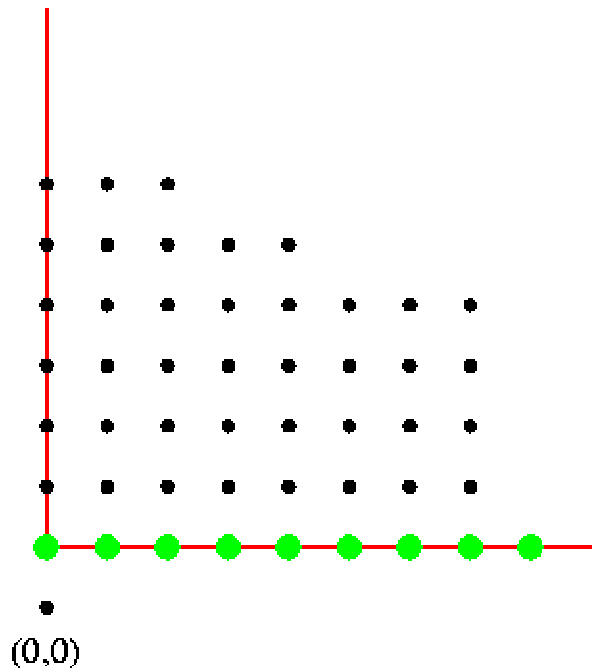
Finiteness Criterion for General S

Theorem. [H. & Weismantel] *Let $S \subseteq \mathbb{Z}^n$ be any set of lattice points in $\mathbb{Z}^n$.*

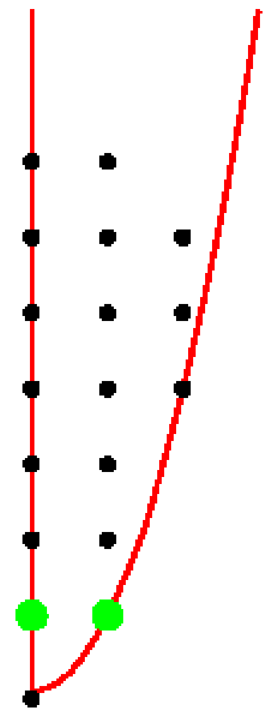
(a) *S has a finite integral basis if and only if $\mathrm{cone}(S)$ is a rational polyhedral cone.*

(b) *If the cone $\mathrm{cone}(S)$ is rational and pointed, there is a unique finite integral basis that is minimal with respect to set inclusion.*

$(C \cap \mathbb{Z}^n) \setminus S$ **not finite.** $\mathrm{cone}(S)$ **not rational.**



$\mathrm{cone}(S)$ **IS rational. Finite integral basis exists!**



Any hope if $\mathrm{cone}(S)$ is not rational?

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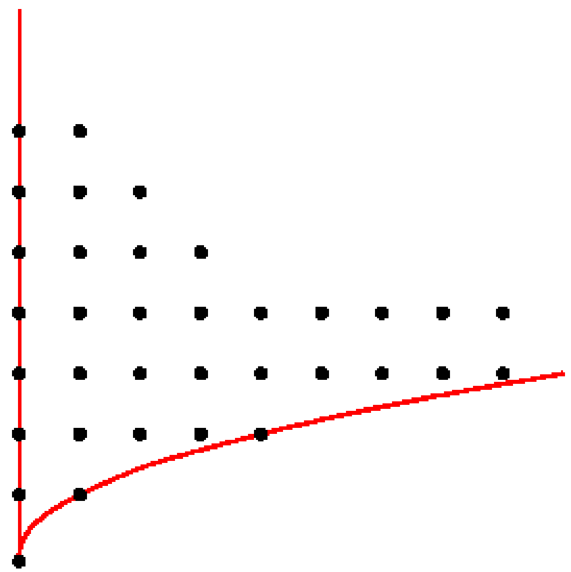
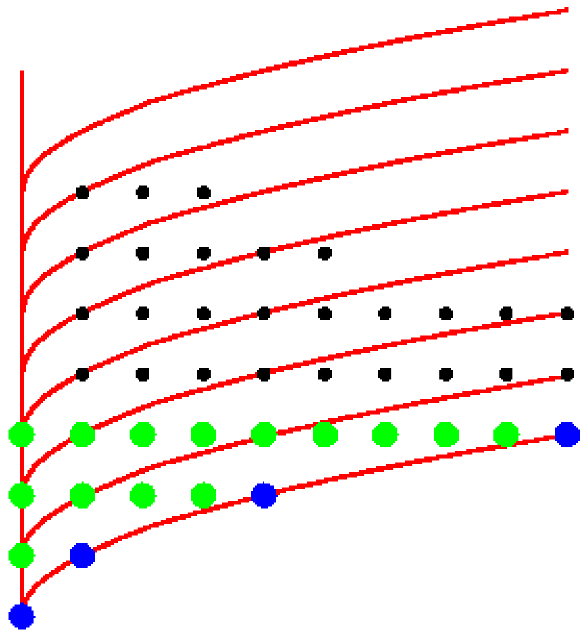
We think, YES.

A Motivating Example



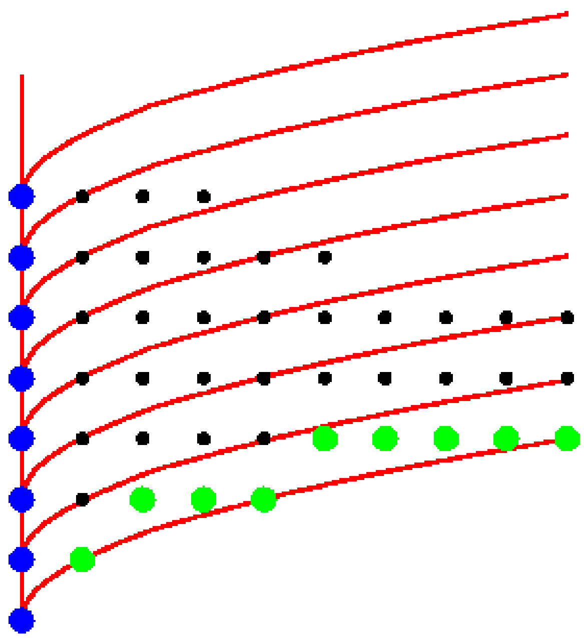
$$S = \{(x,y) : y^2 \geq x, x, y \geq 0\} \cap \mathbb{Z}^2$$

A Trivial 2-dim Representation



$$S = \{(x^2 - s, x) : s \leq x^2, x, s \in \mathbb{Z}_+\}$$

Our Goal: A Simpler 2-dim Representation



$$S = \{(0,t) : t \in \mathbb{Z}_+\} + \{(x^2 - s, x) : s \leq 2x - 2, x, s \in \mathbb{Z}_+\} \cup \{(0,0)\}$$

A bit more formal...

Definition: Semi-Algebraic Cones

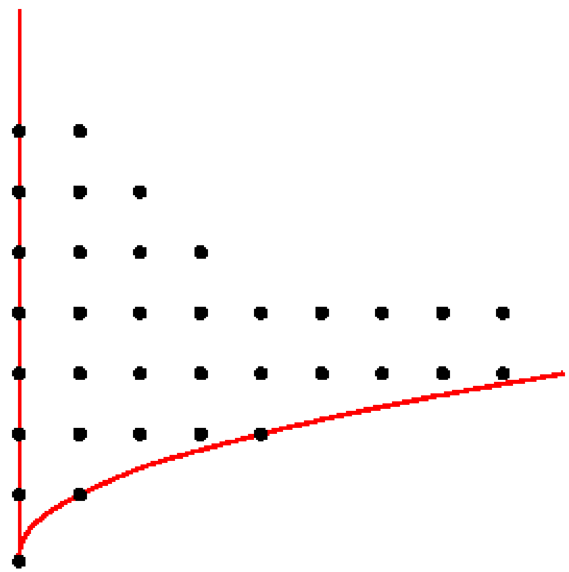
Let $g : \mathbb{R}^n \to \mathbb{R}^d$ and denote by $g_i : \mathbb{R}^n \to \mathbb{R}$ the d components of g.

Then we call the set

$$\mathcal{C} = \{y \in \mathbb{R}^d : y = g(\lambda), \lambda \geq 0\}$$

a semi-algebraic cone.

Example: Semi-Algebraic Cone



$$\mathcal{C} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda_1^2 \\ \lambda_1 + \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 \\ \lambda_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}, \lambda \geq 0 \right\}$$

Integral Function Bases

Given $S \subseteq \mathbb{Z}^n$, consider family

$$T_i := \{f_i(t_i) : t_i \in \mathbb{R}_+^{n_i} \cap S_i\} \subseteq S$$

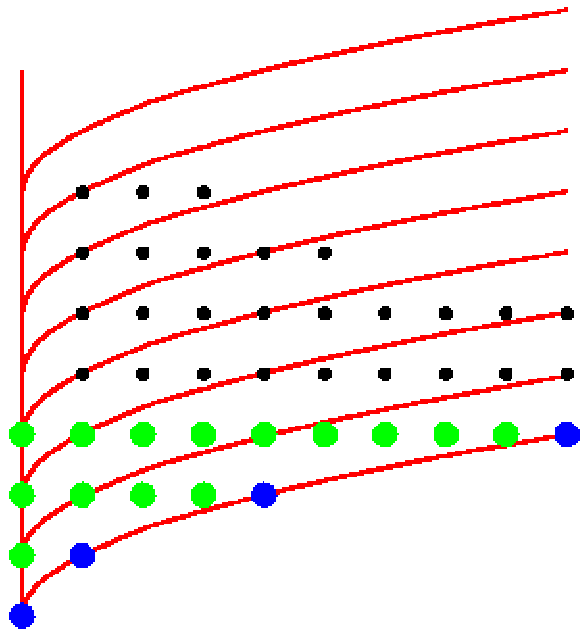
with polynomial functions

$$f_i : \mathbb{R}_+^{n_i} \rightarrow \mathbb{R}^n, \quad f_i(\mathbb{Z}_+^{n_i}) \subseteq \mathbb{Z}^n,$$

where S_i are semi-algebraic sets such that $T_i \subseteq S$.

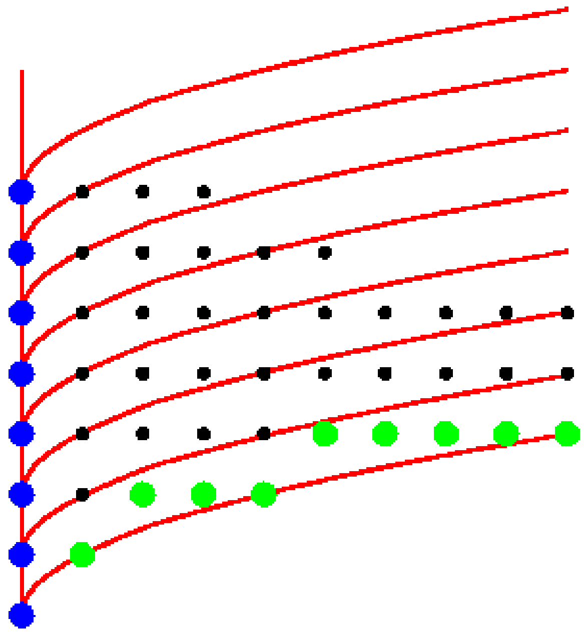
Such a family $\{T_i\}$ is an integral function basis of S, if for every $s \in S$ there exists a finite combination $s = \sum f_i(t_i)$ with $t_i \in \mathbb{Z}_+^{n_i}$ and $f_i(t_i) \in T_i$.

Trivial Integral Function Basis



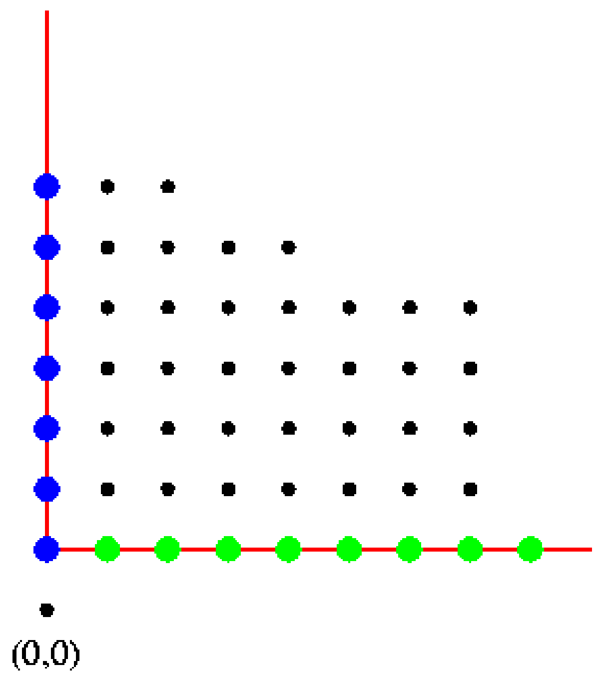
$$S = T_1 = \{(x^2 - s, x) : s \leq x^2, x, s \in \mathbb{Z}_+\}$$

Simpler Integral Function Basis



$$S = T_1 + T_2 = \{(0,t) : t \in \mathbb{Z}_+\} + \{(x^2 - s, x) : s \leq 2x - 2, x, s \in \mathbb{Z}_+\} \cup \{(0,0)\}$$

Problem repaired!



$$S = T_1 = \{(0,1) + (0,t) + (s,0) : s,t \in \mathbb{Z}_+\}$$

In fact...

...all rational polyhedra have a finite integral function basis.

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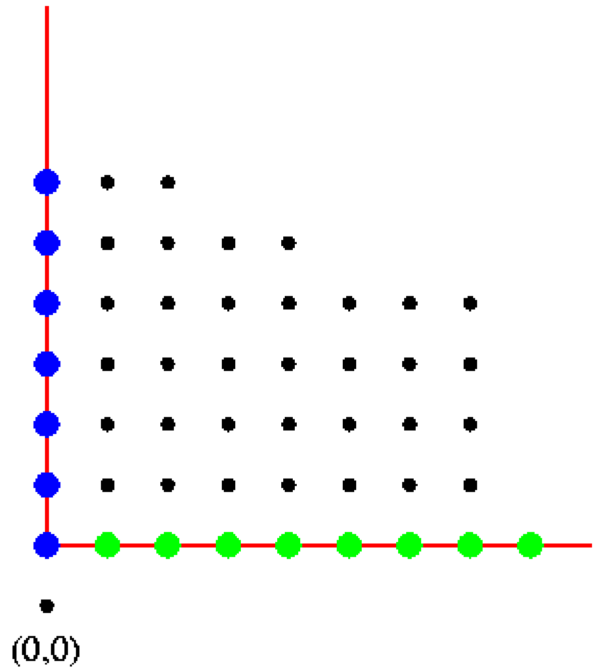
More precisely, every lattice point in P is a sum of a lattice point in the recession cone plus one of finitely many other points.

Look at the picture again



$$S = T_1 = \{(0,1) + (0,t) + (s,0) : s,t \in \mathbb{Z}_+\}$$

Goal: Finite representation results

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Applications:

- Generalize Integral Basis Method to nonlinear integer optimization problems.
- Primal approach for nonlinear integer optimization problems.

Some results...

Special Case

Lemma 1. *Let*

$$\mathcal{C} = \{y \in \mathbb{R}^d : y = g_1(\lambda_1) + \ldots + g_k(\lambda_k), \lambda_1, \ldots, \lambda_k \geq 0\}$$

where all but one g_i are linear functions.

Then $\mathcal{C}$ has a finite integral function basis.

Special Case

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$$\mathcal{C} = \{y \in \mathbb{R}^d : y = g_1(\lambda_1) + \ldots + g_k(\lambda_k), \lambda_1, \ldots, \lambda_k \geq 0\}$$

where all but one g_i are linear functions.

Then $\mathcal{C}$ has a finite integral function basis.

Taking union of such special cones, this result can be generalized.

Relaxed Integral Function Bases

A relaxed integral function basis for $S := \{y \in \mathbb{R}^n : y = g(\lambda)\} \cap \mathbb{Z}^d$ is a set of functions

$$g_l, g_u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

with

$$\mathrm{maxdeg}(g_l), \mathrm{maxdeg}(g_u) < \mathrm{maxdeg}(g)$$

and such that for every point $x \in S$ there exist $\lambda \in \mathbb{Z}_+^d$ and $v_x \in \mathbb{Z}^n$ with

$$x = g(\lambda) + v_x$$

and with

$$g_l(\lambda) \leq v_x \leq g_u(\lambda).$$

Note that v_x is not required to lie in S.

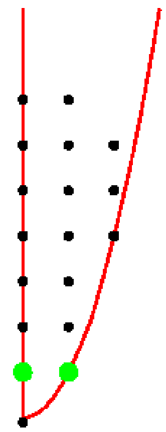
Fact

Lemma 2. *Every semi-algebraic cone*

$$\mathcal{C} := \{y \in \mathbb{R}^n : y = g(\lambda)\}$$

has a relaxed integral function basis.

Example



$$\mathcal{C} = \left\{ y \in \mathbb{R}^d : y = g(y) = \begin{pmatrix} y_1 \\ y_1^k + y_2 \end{pmatrix} \right\}$$

Each point $v \in \mathcal{C} \cap \mathbb{Z}^d$ can be written as $v = g(\lambda)$ for some $\lambda \in \mathbb{Z}_+^2$, showing that the correction term v_x is 0 in this case.

Some Open Questions

- When is $v_x \in S$?
- When is $v_x = 0$?

relaxed integral function basis $\Rightarrow$ integral function basis

- Which sets have "nice" and finite integral function bases?
- What is a "nice" integral function basis?

THE END

Thank you!!!