Progress on Turán's brick factory problem

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Outline

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- Quadratic optimization on the simplex (standard QP);
- Approximating standard QP's using semidefinite programming;
- Lower bounding the crossing numbers using standard QP: new results.

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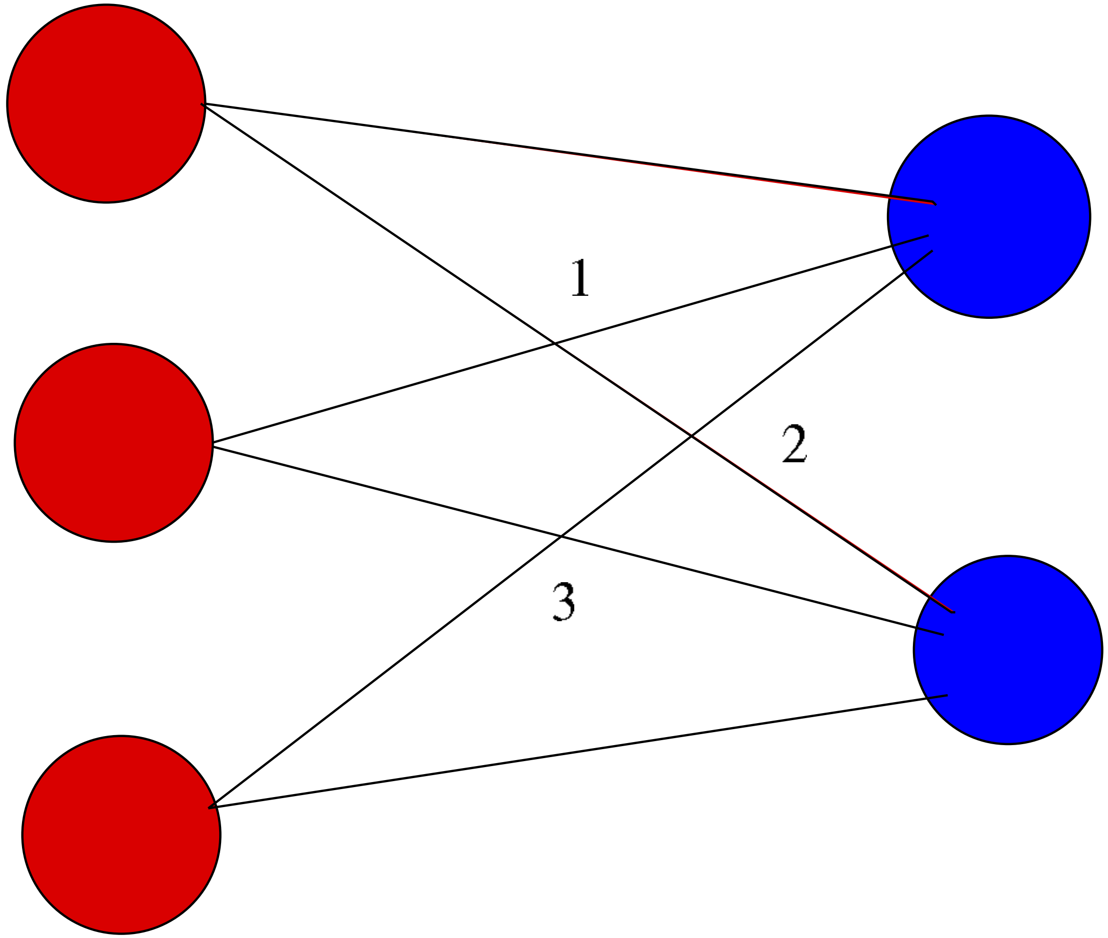
- We denote the *complete bipartite graph* with color classes of size m and n by $K_{n,m}$.
- In any drawing of $K_{n,m}$ we count the number of times that two edges cross.
- The minimum possible number of crossings is called the *crossing number* of $K_{n,m}$ denoted by $\mathrm{cr}(K_{n,m})$.

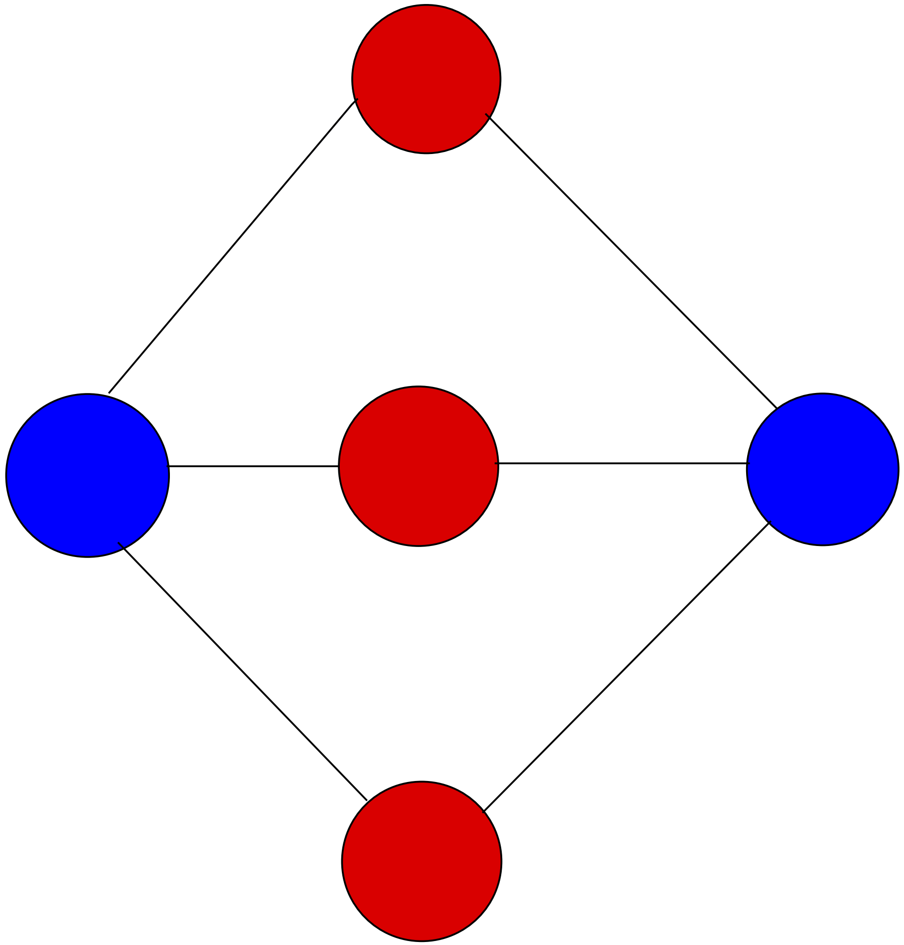
Example

This drawing of $K_{2,3}$ has 3 crossings.



Example (ctd.)

This drawing of $K_{2,3}$ has no crossings.



Zarankiewicz conjecture

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- Conjecture: $\mathbf{cr}\,(K_{m,n}) = Z(n)Z(m)$.
- Proven for the case $\min(m, n) \leq 6$. Case $m = 7$ still open.

This talk

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- Previous result (Kleitman 1970):

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  that follows from $\text{cr}\,(K_{6,n}) = Z(6)Z(n)$.
- Slight improvement to 0.8001 (Nahas 2003).

Cyclic orders

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- The *antidistance* $\bar{d}(a, b) := d(a, \bar{b})$, where $\bar{b}$ is obtained by inverting b, e.g:

$$\bar{d}(123, 123) = d(123, \overline{123}) = d(123, 321) = 1.$$

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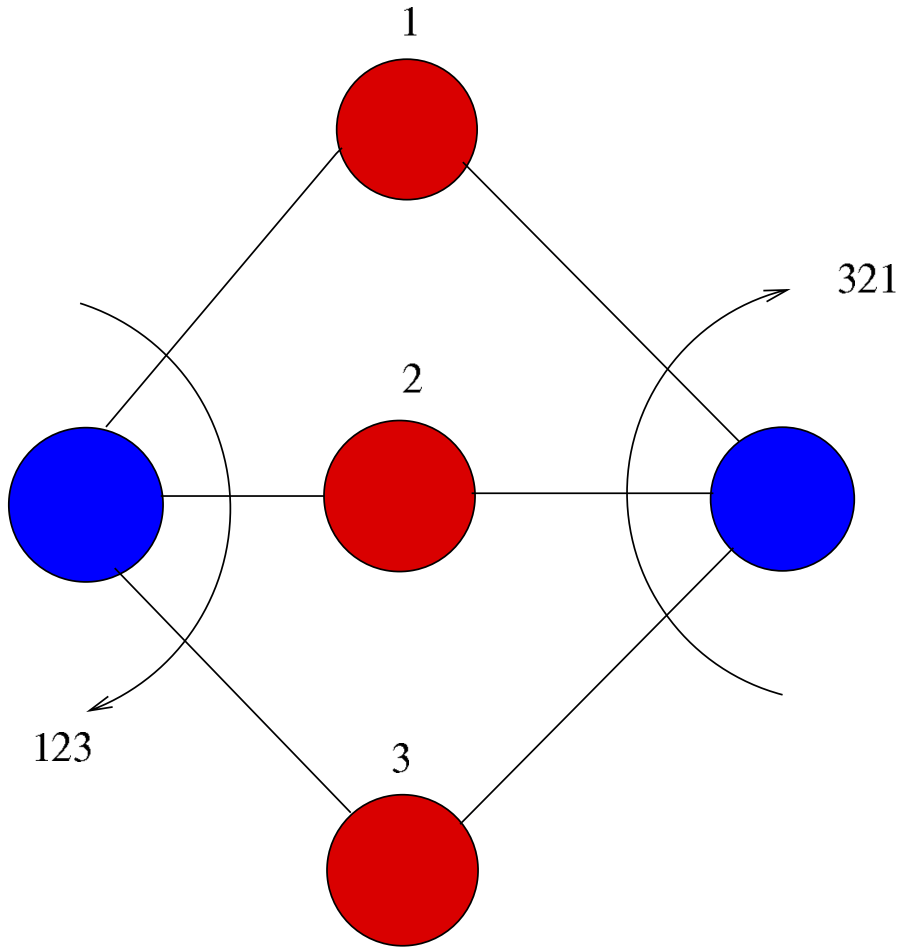
- Let $a \in C_m$ (resp. $b \in C_m$) denote the *cyclic, clockwise order* in which the edges leave α (resp. β).
- **Theorem** (Kleitman, 1970): One has

$$\operatorname{cr}(D) \geq \bar{d}(a, b).$$

Example 1

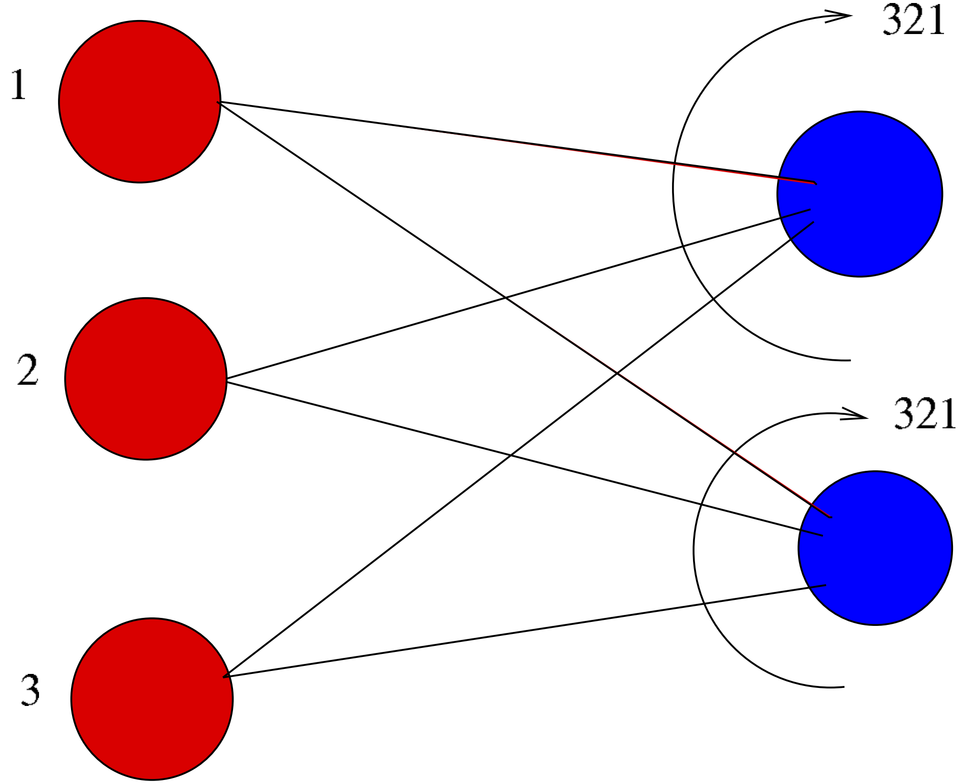
This drawing D has $\mathrm{cr}(D) = 0$ crossings.



$\bar{d}(123, 321) = d(123, 123) = 0 = \mathrm{cr}(D).$

Example 2

This drawing D has $\mathrm{cr}(D) = 3$ crossings.



$$\bar{d}(321, 321) = d(321, 123) = 1 < \mathrm{cr}(D).$$

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- Denote by x_i the number of the n vertices that have edge orientation a_i;
- Now

$$\mathrm{cr}(D) \geq \frac{1}{2} \sum_{i,j} \bar{d}(a_i, a_j) x_i x_j - \frac{9}{2} n.$$

Standard quadratic opt.

We get a *lower bound* on $\mathrm{cr}(K_{7,n})$ via

$$\mathrm{cr}(K_{7,n}) \geq \min_x \frac{1}{2} \sum_{i,j} \bar{d}(a_i, a_j) x_i x_j - \frac{9}{2} n$$

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Scale the variables to get

$$\mathrm{cr}(K_{7,n}) \geq \min_x \frac{1}{2} n^2 \sum_{i,j} \bar{d}(a_i, a_j) x_i x_j - \frac{9}{2} n$$

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- **Problem:** the relaxations are too large to solve in the naive formulation.
- **Solution:** reduce the problem size by exploiting the invariance properties of the quadratic function, a'la Schrijver, Gatermann-Parrilo.

The basic relaxation

Our problem is of the form

$$p_{\min} := \min_{x} \left\{ x^T Q x \; : \; e^T x = 1, \; x \geq 0 \right\},$$

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Basic relaxation we used:

$$\min_{X} \left\{ \mathrm{tr}(QX) \ : \ \mathrm{tr}\left(ee^T X\right) = 1, \ X \succeq 0, \ X \geq 0 \right\},$$

where $X \succeq 0$ means X positive semidefinite.

Exploiting group symmetry

Our matrix Q is invariant under the action of a group G of permutation matrices $P \in G$, in the sense that

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Idea: If X is optimal for the SDP relaxation then so is $\frac{1}{k} \sum_{P \in G} P^T X P$. We can work in the low dimensional subspace $\mathcal{A}$.

Results

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- Further implication:

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- Previous best:

$$\operatorname{cr}(K_{m,n}) \geq 0.8001Z(m)Z(n).$$

Further reading

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- Preprint with our results available at

  `www.math.uwaterloo.ca/~edeklerk/publications/`