LP and SDP relaxations for polynomial programming with applications in mathematical finance

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- LP and SDP-relaxations for polynomial programming

- Application in mathematical finance ; option pricing

Polynomial programming

$$\mathbf{P}: \quad f^* = \min\{\, f(x) \mid \quad x \in \mathbf{K}\},$$

with $\mathbf{K} \subset \mathbf{R}^n$ being the semi-algebraic set

$$\mathbf{K} \quad = \{x \in \mathbf{R}^n \mid \quad g_j(x) \geq 0,\ j = 1, \ldots, m\},$$

and $\{f, g_j\}_{j=1}^m \subset \mathbf{R}[x_1, \ldots, x_n]$.

- very general formulation which encompasses a lot of standard problems with many appplications. In particular, the set $\mathbf{K}$ can be nonconvex, non-connected, discrete.

Recent SDP-relaxations of $\mathbf{P}$ have been designed, extending earlier work by Shor, and later Nesterov. See e.g. De Klerk, Kojima, Lasserre, Laurent, Marshall, Parrilo, Schweighofer, ...

- Theory asserts asymptotic (and sometimes finite) convergence to f^*

- Practice seems to reveal fast (and finite) convergence.

Matlab based solvers : GLOPTIPOLY (Henrion and Lasserre), SOSTOOLS (Prajna, Papachristodoulou and Parrilo,) use the Sedumi SDP solver of J. Sturm.

LP-relaxations of Sherali-Adams, Lovász-Schrijver, Ceria-Balas-Cornuejols have finite convergence for $0-1$ programs.

LP-relaxations of the Sherali-Adams type for polynomial programming have been shown to also converge in the case $\mathbf{K}$ is a polytope (Lasserre, Parrilo and Sturmfels).

In fact, using an old representation result by Krivine, one may show that LP-relaxations also converge for general compact semi-algebraic sets under a relatively weak assumption (Lasserre).

Question: how do LP and SDP-relaxation compare?

BACKGROUND

Let $1, x_1, \ldots x_n, x_1^2, \ldots, x_n^r, \ldots$ be a basis for the vector space $\mathcal{P}$ of polynomials $\mathbf{R}^n \to \mathbf{R}$. A polynomial $f \in \mathcal{P}$ is written

$$x \mapsto f(x) = \sum_{\alpha \in \mathbf{N}^n} f_\alpha \, x^\alpha$$

with finitely many nonzero coefficients $\{f_\alpha\} \subset \mathbf{R}$.

Let $y = \{y_\alpha\}$, $\alpha \in \mathbf{N}^n$, be an infinite sequence indexed in this basis, and let $L_y : \mathcal{P} \to \mathbf{R}$ be the linear functional

$$f \mapsto L_y(f) = \sum_{\alpha \in \mathbf{N}^n} f_\alpha \, y_\alpha$$

Theorem [Schmüdgen, Putinar, Jacobi, Prestel]

Assume there is a polynomial $\mathbf{u} : \mathbf{R}^n \to \mathbf{R}$ *such that*

$$\mathbf{u} = \mathbf{u_0} + \sum_{\mathbf{k=1}}^{\mathbf{m}} \mathbf{g_k}\, \mathbf{u_k},$$

for some polynomials $\{\mathbf{u_k}\}_{k=0}^{m}$, *all sums of squares (s.o.s.), and such that the level set* $\{x \,|\, \mathbf{u}(x) \geq 0\}$ *is compact. Then:*

Every polynomial $p > 0$ *on* $\mathbf{K}$ *has the representation:*

$$(*) \qquad \mathbf{p} = \mathbf{q_0} + \sum_{\mathbf{k=1}}^{\mathbf{m}} \mathbf{g_k}\, \mathbf{q_j}$$

for some family of s.o.s. polynomials $\{\mathbf{q_j}\}_{j=0}^{m}$.

If one restricts the degree of the polynomials $\{\mathbf{q_j}\}$, then testing (*) translates into LMIs on the coefficients of the $\mathbf{q_j}$'s

Equivalent (dual) moment point of view

Let $\mathbf{y} = \{\mathbf{y}_\alpha\} \subset \mathbf{R}$ *be an infinite sequence. Then* $\mathbf{y}$ *has a representing measure* μ *with support contained in* $\mathbf{K}$*, that is,*

$$\exists\, \mu \quad \textit{s.t.} \quad \mathbf{y}_\alpha = \int_{\mathbf{K}} \mathbf{x}^\alpha\, \mathbf{d}\mu, \quad \forall\, \alpha \in \mathbf{N}^{\mathbf{n}}$$

if and only if

$$(**) \quad \mathbf{L_y}(\mathbf{f^2}) \geq \mathbf{0}; \quad \mathbf{L_y}(\mathbf{f^2}\,\mathbf{g_j}) \geq 0, \quad \forall j = 1, \ldots, m,$$

for all polynomials $\mathbf{f} \in \mathbf{R}[x_1, \ldots, x_n]$

If one restricts the degree of the polynomials $\mathbf{f}$, then $(**)$ translates into LMIs on $\mathbf{y}$

SDP-relaxations

Moment matrix $M_r(y)$. With $y = \{y_\alpha\}_{\alpha \in \mathbf{N}^n}$

$M_r(y)(i,1) = y_\alpha$ and $M_r(y)(1,j) = y_\beta \Rightarrow M_r(y)(i,j) = y_{\alpha+\beta}$.

$$M_2(y) = \left[\begin{array}{c|cc|ccc} 1 & y_{1,0} & y_{0,1} & y_{2,0} & y_{1,1} & y_{0,2} \\ \hline y_{1,0} & y_{2,0} & y_{1,1} & y_{3,0} & y_{2,1} & y_{1,2} \\ y_{0,1} & y_{1,1} & y_{0,2} & y_{2,1} & y_{1,2} & y_{0,3} \\ \hline y_{2,0} & y_{3,0} & y_{2,1} & y_{4,0} & y_{3,1} & y_{2,2} \\ y_{1,1} & y_{2,1} & y_{1,2} & y_{3,1} & y_{2,2} & y_{1,3} \\ y_{0,2} & y_{1,2} & y_{0,3} & y_{2,2} & y_{1,3} & y_{0,4} \end{array}\right]$$

$$M_r(y) \succeq 0 \quad \Leftrightarrow \quad L_y(f^2) \geq 0, \quad \forall f, \deg(f) \leq r$$

Localizing matrix.

Given a polynomial $x \mapsto \theta(x) = \sum_{\alpha \in \mathbf{N}^n} \theta_\alpha \, x^\alpha$, and $y = \{y_\alpha\}_{\alpha \in \mathbf{N}^n}$, let $M_r(\theta \, y)$ be the **localizing matrix** with respect to θ.

If $M_r(y)(i,j) = y_\beta$ then $M_r(\theta \, y)(i,j) = \sum_\alpha \theta_\alpha \, y_{\beta+\alpha}$.

For instance, with $x \mapsto \theta(x) = 1 - x_1^2 - x_2^2$,

$$M_1(\theta \, y) = \left[\begin{array}{ccc} 1 - y_{20} - y_{02}, & y_{10} - y_{30} - y_{12}, & y_{01} - y_{21} - y_{03} \\ y_{10} - y_{30} - y_{12}, & y_{20} - y_{40} - y_{22}, & y_{11} - y_{21} - y_{12} \\ y_{01} - y_{21} - y_{03}, & y_{11} - y_{21} - y_{12}, & y_{02} - y_{22} - y_{04} \end{array} \right].$$

$$M_r(\theta \, y) \succeq 0 \quad \Leftrightarrow \quad L_y(f^2 \, \theta) \geq 0, \quad \forall f, \deg(f) \leq r$$

With $f \in \mathbf{R}[x]$, introduce the family $\{\mathbf{Q}_r\}$ of SDP-relaxations

$$\mathbf{Q}_r \begin{cases} \min\limits_{y} \quad L_y(f) & \\ L_y(h^2) \quad \geq 0, & \forall h, \deg(h) \leq r \\ L_y(h^2\, g_j) \quad \geq 0 & \forall h, \deg(h^2\, g_j) \leq 2r \\ & j = 1, \ldots, m. \end{cases}$$

and the family $\{\mathbf{Q}_r^*\}$ of their dual

$$\mathbf{Q}_r^* \begin{cases} \max\limits_{\lambda, q_0, \ldots, q_m} \quad \lambda \\ f - \lambda = q_0 + \sum_{j=1}^m q_j\, g_j \\ q_j \quad s.o.s. \quad \deg(q_j\, g_j) \leq 2r, \quad \forall j = 0, \ldots, m \end{cases}$$

Theorem : *Assume there is a polynomial* $\mathbf{u} : \mathbf{R}^n \to \mathbf{R}$ *such that*

$$\mathbf{u} = \mathbf{u_0} + \sum_{\mathbf{k}=1}^{\mathbf{m}} \mathbf{g_k}\, \mathbf{u_k},$$

for some polynomials $\{\mathbf{u_k}\}_{k=0}^{m}$*, all sums of squares (s.o.s.), and such that the level set* $\{x \mid \mathbf{u}(x) \geq 0\}$ *is compact.*

Then $\min \mathbf{Q}_r \uparrow f^*$ as $r \to \infty$.

In practice the convergence is **fast** and even **finite**.

If $x^* \in \mathbf{R}^n$ is the unique global minimizer of $\mathbf{P}$, convergence of first-order moments $\{y_\alpha\}_{|\alpha|=1}$ to x^* occurs (Schweighofer)

finite convergence eventually occurs for 0-1 (nonlinear) programs (and discrete optimization)

LP relaxations : Background.

Assumption (generating): Let $g_0 \equiv 1$. The polynomials $\{0, g_0, \ldots, g_m\}$ **generate** the $\mathbf{R}$-algebra $\mathbf{R}[x_1, \ldots, x_n]$, that is, $\mathbf{R}[x_1, \ldots, x_n] = \mathbf{R}[g_1, \ldots, g_m]$.

Let $0 \leq \overline{g}_j := \max_{x \in \mathbf{K}} g_j(x)$ for all $j = 1, \ldots, m$, and let

$$\widehat{g}_j = \begin{cases} g_j / \overline{g}_j & \text{if } \overline{g}_j > 0 \\ g_j & \text{otherwise} \end{cases} \qquad j = 1, \ldots, m$$

so that $0 \leq g_j \leq 1$ on $\mathbf{K}$ for all $j = 0, 1, \ldots, m$.

- One may also take for $\overline{g}_j$ any upper bound of g_j on $\mathbf{K}$;

- One may also introduce *redundant constraints* $x_k \geq \underline{x}_k$ in the definition of $\mathbf{K}$ to enforce the generating assumption.

Upper bounds on g_j or lower bounds on x_k can be obtained by running the SDP relaxations $\mathbf{Q}_r$ with $f := -g_j$, or $f \equiv x_k$, respectively.

Theorem : [Krivine, Becker and Schwartz, Marshall, Vasilescu]
Let the generating assumption hold. If $f \in \mathbf{R}[x_1, \ldots, x_m]$ is positive on $\mathbf{K}$ then

$$(***) \quad f = \sum_{\alpha,\beta \in \mathbf{N}^m} c_{\alpha\beta}\, \widehat{g}^{\alpha}\, (1 - \widehat{g})^{\beta},$$

for finitely many positive coefficients $\{c_{\alpha\beta}\}$.

Testing (***) with $|\alpha + \beta| \leq r$ reduces to solving a LP.

Equivalent (dual) moment point of view.

Let $\mathbf{y} = \{\mathbf{y}_\alpha\} \subset \mathbf{R}$ *be an infinite sequence. Then* $\mathbf{y}$ *has a representing measure* μ *with support contained in* $\mathbf{K}$*, that is,*

$$\exists\,\mu \quad \textit{s.t.} \quad \mathbf{y}_\alpha = \int_{\mathbf{K}} \mathbf{x}^\alpha\,\mathbf{d}\mu, \quad \forall\,\alpha \in \mathbf{N}^{\mathbf{n}}$$

if and only if

$$(****) \quad \mathbf{L}_{\mathbf{y}}(\widehat{\mathbf{g}}^\alpha\,(\mathbf{1}-\widehat{\mathbf{g}})^\beta) \geq \mathbf{0} \quad \forall\,\alpha,\beta \in \mathbf{N}^{\mathbf{m}}.$$

$\longrightarrow$ Countably many linear inequalities on the vector $\mathbf{y}$.....

If one restricts to $|\alpha+\beta| \leq r$ then (****) translates into finitely many linear inequalities on $\mathbf{y}$

With $f \in \mathbf{R}[x]$, introduce the family $\{\mathbf{L}_r\}$ of LP-relaxations

$$\mathbf{L}_r \begin{cases} \min\limits_{y} \quad L_y(f) \\ \\ L_y(\widehat{g}^{\alpha}\,(1-\widehat{g})^{\beta}) \;\geq 0, \qquad |\alpha+\beta| \leq 2r \end{cases}$$

and the family $\{\mathbf{L}_r^*\}$ of their dual

$$\mathbf{L}_r^* \begin{cases} \max\limits_{\lambda,\{c_{\alpha\beta}\}} \quad \lambda \\ \\ f - \lambda = \sum\limits_{\alpha,\beta \in \mathbf{N}^m} c_{\alpha\beta}\,\widehat{g}^{\alpha}\,(1-\widehat{g})^{\beta} \\ \\ c_{\alpha\beta} \;\geq 0, \qquad \forall\, |\alpha+\beta| \leq 2r \end{cases}$$

Theorem : *Assume that* $\mathbf{K}$ *is compact and* $\mathbf{R}[x_1, \ldots, x_n] = \mathbf{R}[g_1, \ldots, g_m]$. *Then the LP-relaxations converge, that is,*

$$\max \mathbf{L}_r^* = \min \mathbf{L}_r \uparrow f^* \quad \text{as } r \rightarrow \infty.$$

Primal LP-relaxation L_r^*	Primal SDP-relaxation Q_r^*
$\min_{y} \quad L_y(f)$ $L_y(\widehat{g}^{\alpha}(1-\widehat{g})^{\beta}) \quad \geq 0$ $\forall\, \alpha, \beta, \quad \|\alpha+\beta\| \quad \leq 2r$	$\min_{y} \quad L_y(f)$ $L_y(h^2 g_j) \quad \geq 0$ $\forall h, \mathrm{deg}(hg_j) \leq 2r, \quad j \in \{0,\ldots,m\}$
Dual LP-relaxation L_r	**Dual SDP-relaxation Q_r**
$\max_{\lambda,\{c_{\alpha\beta}\}} \quad \lambda$ $f-\lambda \quad = \sum_{\alpha,\beta\in\mathbb{N}^m} c_{\alpha\beta}\widehat{g}^{\alpha}(1-\widehat{g})^{\beta}$ $\|\alpha+\beta\| \quad \leq 2r$ $c_{\alpha\beta} \quad \geq 0 \quad \forall \alpha,\beta$	$\max_{\lambda,\{q_j\}} \quad \lambda$ $f-\lambda \quad = \sum_{j=0}^{m} q_j\, g_j$ $\mathrm{deg}(q_j\, g_j) \quad \leq 2r, \; j \in \{0,\ldots,m\}$ $q_j \quad s.o.s., \; j \in \{0,\ldots,m\}$

Some remarks ....

1. Notice the presence of binomial coefficients in both primal and dual LP-relaxations ... which yields numericall ill-conditioning for relatively large r.

2. Let $x^* \in \mathbf{K}$ be a global minimizer, and for $x \in \mathbf{K}$, let $J(x)$ be the set of active constraints $\widehat{g}_j(x) = 0$ and ...$1 - \widehat{g}_k(x) = 0$.

Then FINITE convergence CANNOT occur

- (a) If there exists nonoptimal $x \in \mathbf{K}$ with $J(x) \supseteq J(x^*)$,

- (b) or if $\overset{\circ}{\mathbf{K}} = \{x \in \mathbf{R}^n |\, g_j(x) > 0,\ j = 1, \ldots, m\}$ and $x^* \in \overset{\circ}{\mathbf{K}}$ (whenever such $\overset{\circ}{\mathbf{K}}$ exists)

3. If $\mathbf{K}$ is a Polytope then FINITE convergence is possible only if every global minimizer is a vertex of $\mathbf{K}$.

Hence if f is convex .. the LP-relaxations cannot be exact !!

Ex: $\min\{x(x-1) \,|\, 0 \leq x \leq 1\} \Rightarrow x^* = 0.5$ and $f^* = -0.25$.

$$f(x) + 0.25 = x^2 - x + 0.25 = (x - 0.5)^2, \qquad x \in \mathbf{R},$$

and the SDP-relaxation $\mathbf{Q}_1$ is exact whereas one CANNOT write

$$f(x) - f^* = f(x) + 0.25 = \sum_{i,j \in \mathbf{N}} c_{ij}\, x^i (1-x)^j,$$

because

$$0 = f(x^*) + 0.25 = \sum_{i,j \in \mathbf{N}} c_{ij}\, 2^{-i-j} > 0.$$

In addition, the convergence $\min \mathbf{L}_r \uparrow -0.25$ is very slow...

$$\lambda_2 = \lambda_4 = -1/3; \quad \lambda_6 = -0.3; \quad \lambda_{10} = -0.27, \quad \ldots$$

Consider now the concave minimization problem:

$\min \{x(1-x) \,|\, 0 \leq x \leq 1\} \Rightarrow$, with $f^* = 0$ and $x^* = 0$ or $x^* = 1$, both vertices of $\mathbf{K}$.

$$f(x) - f^* = x\,(1-x), \qquad x \in \mathbf{R},$$

so that the LP-relaxation $\mathbf{L}_1$ is exact (the SDP-relaxation $\mathbf{Q}_2$ is also exact).

Hence we have the paradox that the LP-relaxations behave much better for the concave minmization problem than for the convex one!!

Mathematical finance applications

Joint work with

- T. Prieto-Rumeau (Computinense, Madrid, Spain)

- M. Zervos (King's College, London)

Consider a multi-dimensional SDE:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x_0.$$

Generator of the process: $\mathcal{A} : \mathcal{D} \to \mathcal{D} \subset C_0(\mathbf{R}^n)$.

$$\mathcal{A}\, f(x) = \sum_i b_i(x) \frac{\partial f}{\partial x_i}(x) + \sum_{i,j} (\sigma\sigma')_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

τ is a stopping time. In some examples $\tau = T$.

We want to evaluate

$$J = \mathbf{E}\left[\int_0^\tau J_0(X_t)dt + J_1(X_\tau)\right],$$

where J_0 and J_1 are (piecewise) polynomials.

Define the expected occupation measure (before τ) as:

$$\mu_0(B) = \mathbf{E}\left[\int_0^\tau 1_{\{X_s \in B\}} ds\right], \quad \text{for measurable sets } B.$$

and the exit location probability measure (at τ) as:

$$\mu_1(B) = P(X_\tau \in B), \quad \text{for measurable sets } B.$$

Let $\{y_k\}$ and $\{z_k\}$ be the moments of μ_0 and μ_1:

$$\int x^k \mu_0(dx) = y_k, \quad \text{where } x^k = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}.$$

For every $f \in \mathcal{D}(\mathcal{A})$:

$$f(X_t) - f(x_0) - \int_0^t (\mathcal{A}f)(X_s)ds, \quad t \geq 0$$

is a martingale. If $\mathbf{E}[\tau] < +\infty$ then:

$$\mathbf{E}[f(X_\tau)] - f(x_0) - \mathbf{E}\left[\int_0^\tau (\mathcal{A}f)(X_s)ds\right] = 0.$$

The moment approach

The martingale property yields the basic adjoint equation:

$$\dagger \qquad \int f\, d\mu_1 - f(x_0) - \int (\mathcal{A}f)\, d\mu_0 = 0.$$

If $f(x) \equiv x^k$ then $(\mathcal{A}f)(x) = \sum c_i(k)\, x^i$ and so, $\dagger$ yields

$$\dagger\dagger \qquad y_k - x_0^k - \sum_j c_j(k)\, z_j = 0, \qquad k \in \mathbf{N}.$$

The basic adjoint equation is relaxed to:

(i) the martingale moment conditions $\dagger\dagger$

(ii) moment conditions on $\{y_k\}$ and $\{z_k\}$ to be moments of some measures μ_0, μ_1,

for moments up to some order r

If $J_0, J_1 \in \mathbf{R}[x]$ write $J_0 = \sum_k A_k x^k$ and $J_1 = \sum_k D_k x^k$, to obtain

$$J = \mathbf{E}\left[\int_0^\tau J_0(X_t)dt + J_1(X_\tau)\right],$$

or, equivalently,

$$J = \sum_k A_k y_k + D_k z_k.$$

To obtain upper and lower bounds on J we solve the problems:

$$\inf_{y,z} \; and \; \sup_{y,z} \; \sum_k (A_k y_k + D_k z_k),$$

subject to: (i) martingale and (ii) moment conditions, for r moments.

Numerical comparison of LP and SDP-relaxations.

Accuracies of the LP and SDP-relaxations for the Cox-Ingersoll-Ross interest rate model.

r	5	10	15	20	25
LP	58%	7.38%	1.55%	0.59%	0.17%
SDP	31%	0.14%	0.0052%	0.0045%	0.0026%

Financial Models: Stochastic differential equation:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x_0.$$

Price dynamics

1. Geometric Brownian motion (Black and Scholes model):

$$dX_t = \mu X_t dt + \sigma X_t \, dW_t,$$

2. Ornstein-Uhlenbeck process:

$$dX_t = k(\theta - X_t)dt + \sigma \, dW_t,$$

3. Fleming process (Cox-Ingersoll-Ross interest rate model):

$$dX_t = k(\theta - X_t)dt + \sigma \sqrt{X_t} \, dW_t,$$

European (call) options: $e^{-rT}\mathbf{E}[(X_T - K)^+]$.

Barrier options (down-and-out): $e^{-rT}\mathbf{E}[(X_T - K)^+ \mathbf{1}_{\{\tau \geq T\}}]$.

Asian options: $e^{-rT}\mathbf{E}\left[\left(\frac{1}{T}\int_0^T X_t dt - K\right)^+\right]$.

Parameters:

Option's maturity: T.
Option's strike price: K.
Discount factor: r.
Option's knockout barrier: $H < \min\{K, x_0\}$.
Stopping time: $\tau = \inf\{t \geq 0 \mid X_t \leq H\}$.

Handling piecewise polynomials, e.g. in $J = \int (x-K)^+ \, d\mu$?

Write $\mu = \varphi + \psi$ with

$$\varphi([K, +\infty)) = 0; \qquad \psi((-\infty, K)) = 0,$$

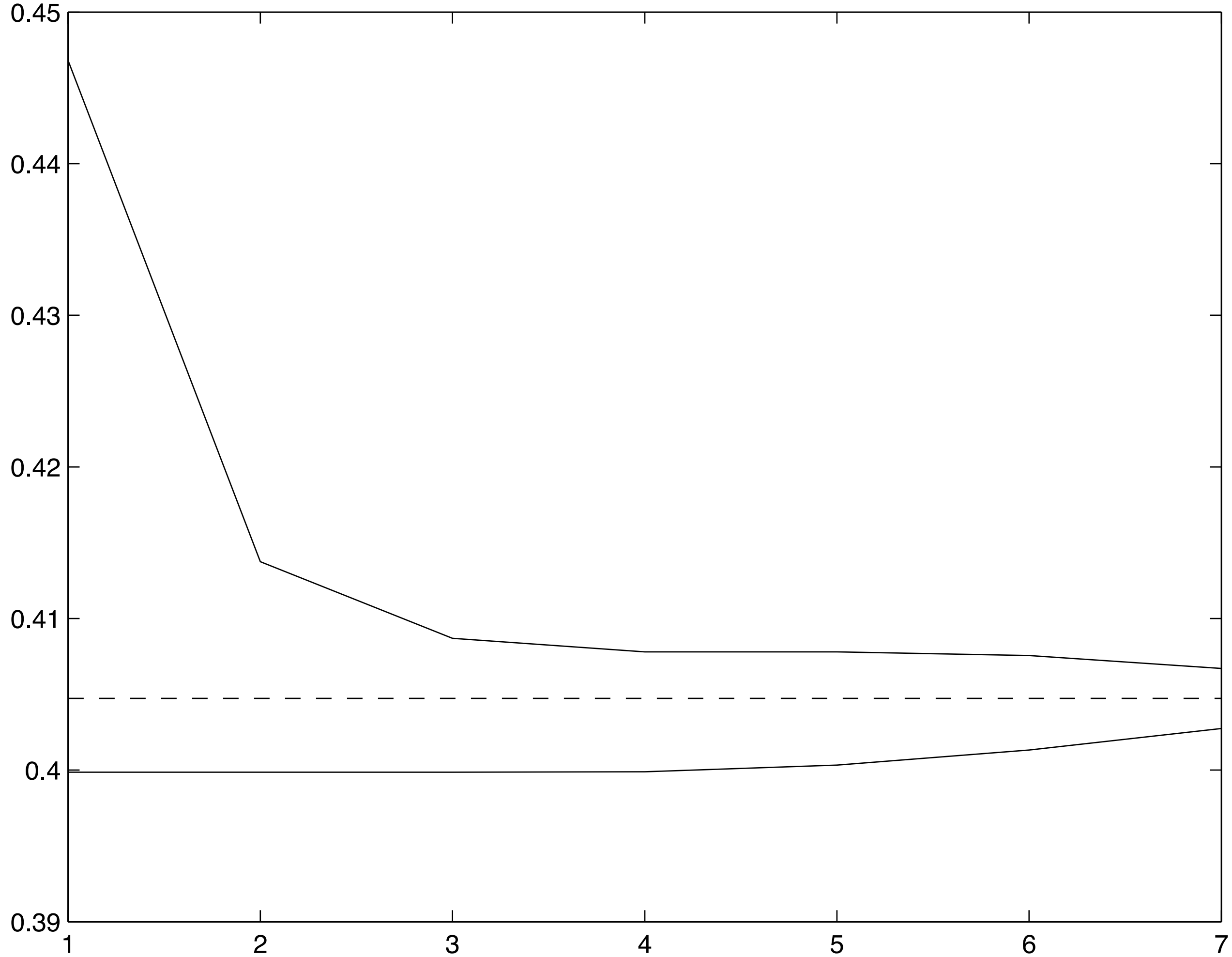
and

$$y_k = \int x^k \, d\mu = \int x^k \, d\varphi + \int x^k \, d\psi \; = \; u_k + v_k, \quad k = 0, 1, \ldots$$

so that $J = \int (x-K) \, d\psi \; = \; v_1 - K v_0$.

So it suffices to introduce the moment conditions

$$L_u(f^2(K-x)) \; \geq \; 0, \qquad L_v(f^2(x-K)) \; \geq \; 0, \quad \forall f \in \mathbf{R}[x],$$



European options. Black and Scholes model

Irreducible gap between upper and lower bounds ... because the log-normal distribution of X_T is not moment determinate.

Relative error := (UB- LB) / ((UB+LB)/2)

	$\sigma = 0.10$	$\sigma = 0.15$	$\sigma = 0.20$	$\sigma = 0.25$
$M = 4$	0.87%	3.42%	8.31%	12.42%
$M = 6$	0.50%	2.77%	4.60%	6.40%
$M = 8$	0.46%	1.92%	4.18%	6.38%
$M = 10$	0.34%	1.91%	3.36%	4.42%

European options. Ornstein-Uhlenbeck process

Drift $\mu = 0.14$

	$\sigma = 0.08$	$\sigma = 0.10$	$\sigma = 0.12$
Curran lower bound	0.16605	0.16658	0.16778
SDP lower bound	0.16642	0.16715	0.16796
SDP upper bound	0.16656	0.16772	0.16965
Relative error	0.08%	0.34%	1.01%

Drift $\mu = 0.16$

	$\sigma = 0.08$	$\sigma = 0.10$	$\sigma = 0.12$
Curran lower bound	0.18497	0.18518	0.18578
SDP lower bound	0.18534	0.18565	0.18704
SDP upper bound	0.18562	0.18652	0.18788
Relative error	0.15%	0.47%	0.45%

Asian options. Geom. Brownian motion. $M = 10$

	$\sigma = 0.05$	$\sigma = 0.10$	$\sigma = 0.15$	$\sigma = 0.20$	$\sigma = 0.25$
$M = 2$	0.88%	3.40%	7.22%	11.94%	17.20%
$M = 4$	0.03%	0.42%	1.98%	5.31%	10.21%
$M = 6$	0.03%	0.20%	1.58%	4.69%	7.48%
$M = 8$	0.02%	0.19%	1.52%	3.95%	5.54%

Asian options. Ornstein-Uhlenbeck process

Barrier options

	$\sigma = 0.10$	$\sigma = 0.15$	$\sigma = 0.20$	$\sigma = 0.25$
$M = 16$	2.63%	3.91%	0.52%	(1.07%)

Geometric Brownian motion

	$\sigma = 0.10$	$\sigma = 0.15$	$\sigma = 0.20$	$\sigma = 0.25$
$M = 18$	1.97%	2.19%	1.36%	2.8%

Ornstein-Uhlenbeck process

	$\sigma = 0.10$	$\sigma = 0.15$	$\sigma = 0.20$	$\sigma = 0.25$
$M = 18$	6.3%	2.85%	1.47%	0.83%

Fleming process

Conclusion

SDP might be preferable to LP-relaxations both for theoretical and practical (numerical) reasons ... However the status of SDP software packages is far from being comparable to that of LP packages ....