



An Extension of Sums of Squares Relaxations to Polynomial Optimization Problems over Symmetric Cones

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Polynomial Optimization Problem (POP)

$$\begin{cases} \min & a(\mathbf{x}) \\ \text{s. t.} & b(\mathbf{x}) \geq 0 \end{cases}$$

⑥ $a : \mathbb{R}^n \rightarrow \mathbb{R}$: Polynomial

⑥ $b : \mathbb{R}^n \rightarrow \mathbb{R}^m$: Vector of Polynomials

$$b(\mathbf{x}) \geq 0 \Leftrightarrow b_j(\mathbf{x}) \geq 0 \ (j = 1, \dots, m)$$

POP over Various Cones

Polynomial SDP

$$\min x - 3y^3 \quad \text{s. t.} \quad \begin{pmatrix} x^2 + y & 2xy \\ 2xy & x - y^3 + 1 \end{pmatrix} \in \mathcal{S}_+^{2 \times 2}$$

$\mathcal{S}_+^{2 \times 2}$: 2×2 real symmetric PSD matrices.

POP over Various Cones

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Polynomial SOCP

$$\min x + 2x^3 - 3y^4 \quad \text{s. t.} \quad \begin{pmatrix} x + y^2 \\ 3x - x^2 - y^4 \\ -x + y + y^3 \end{pmatrix} \in SOC(3)$$

$SOC(k)$: the k dimensional second-order cone.

Symmetric Cones

⑥ A Generalization of Positive Orthant

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- ⑥ **A Unified Approach**

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- △ Positive Orthant \Rightarrow LP
- △ $\mathcal{S}^{n \times n}$: $n \times n$ symmetric positive semidefinite matrices \Rightarrow SDP
- △ Second-order cones

$$\left\{ \mathbf{x} \in \mathbb{R}^{1+n} \mid x_0 \geq \sqrt{\sum_{j=1}^n x_j^2} \right\} \Rightarrow \text{SOCP}$$

POP over Symmetric Cones

$$\langle POP \rangle \begin{cases} \min & a(\mathbf{x}) \\ \text{s. t.} & b(\mathbf{x}) \in \mathcal{E}_+ \end{cases}$$

- ⑥ $a : \mathbb{R}^n \rightarrow \mathbb{R}$, Polynomial
- ⑥ $b : \mathbb{R}^n \rightarrow \mathcal{E}$, \mathcal{E} -valued Polynomial
- ⑥ \mathcal{E} : Euclidean Jordan Algebra
- ⑥ \mathcal{E}_+ : Symmetric Cone associated with \mathcal{E} .

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$\langle POP \rangle$ includes the usual POP, polynomial SDP, and polynomial SOCP.

Contents of this talk

- ⑥ Euclidean Jordan algebra (EJA) and Symmetric Cones
- ⑥ Polynomials over EJAs
- ⑥ Overview of our Results
- ⑥ SOS relaxation and a generalization of Putinar's lemma
- ⑥ Relationship between SOS and SDP
- ⑥ Concluding Remarks

Euclidean Jordan Algebra (EJA) \mathcal{E}

Jordan Algebra : A finite dimensional real vector space having a **multiplication** $\circ : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$

(J1) $a \circ b = b \circ a$ for any $a, b \in \mathcal{E}$.

(J2) $(a^2 \circ b) \circ a = a^2 \circ (b \circ a)$ for any $a, b \in \mathcal{E}$.

(In general, $(a \circ b) \circ c \neq a \circ (b \circ c)$.)

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Euclidean Jordan Algebra : An **associative** inner product \bullet is defined on \mathcal{E} .

$$(a \circ b) \bullet c = a \bullet (b \circ c)$$

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Symmetric cone \mathcal{E}_+ associated with \mathcal{E}

$$\mathcal{E}_+ = \{ a^2 \mid a \in \mathcal{E} \}.$$

Examples of EJA

\mathbb{R}^n :

⑥ $(\mathbf{a} \circ \mathbf{b})_i = a_i b_i \ (i = 1, \dots, n).$

⑥ $\mathbf{a} \bullet \mathbf{b} = \mathbf{a}^T \mathbf{b}$

⑥ $\mathbb{R}_+ = \{ \mathbf{a}^2 \in \mathbb{R}^n \mid \mathbf{a} \in \mathbb{R}^n \} = \{ \mathbf{a} \mid a_i \geq 0 (i = 1, \dots, n) \}.$

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$\mathcal{S}^{N \times N}$, the set of $N \times N$ real symmetric matrices :

- ⑥ $X \circ Y = \frac{XY + YX}{2}$
- ⑥ $X \bullet Y = \text{trace}(XY)$
- ⑥ $\mathcal{S}_+^{N \times N}$: the set of $N \times N$ PSD matrices.

\mathcal{E} -valued Polynomials

\mathcal{E} -valued polynomial $f : \mathbb{R}^N \rightarrow \mathcal{E}$

$$f(\mathbf{x}) = \sum_{\alpha \in \mathcal{G}} f_{\alpha} \mathbf{x}^{\alpha}$$

- ⑥ \mathcal{G} , a finite subset of \mathbb{Z}^n
- ⑥ $f_{\alpha} \in \mathcal{E}$ for every $\alpha \in \mathcal{G}$,
- ⑥ $\mathbf{x}^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$.

$\mathcal{E}[\mathbf{x}]$: The set of \mathcal{E} -valued polynomials.

ε -valued Polynomials and EJA

⑥ EJA $\Leftrightarrow \circ$ and \bullet

\mathcal{E} -valued Polynomials and EJA

⑥ EJA $\Leftrightarrow \circ$ and \bullet

⑥ For $f, g \in \mathcal{E}[\mathbf{x}]$,

$$(f \circ g)(\mathbf{x}) = f(\mathbf{x}) \circ g(\mathbf{x})$$

$$(f \bullet g)(\mathbf{x}) = f(\mathbf{x}) \bullet g(\mathbf{x})$$

$$(f \circ g \in \mathcal{E}[\mathbf{x}], f \bullet g \in \mathbb{R}[\mathbf{x}])$$

\mathcal{E} -valued Polynomials and EJA

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⑥ For $f, g \in \mathcal{E}[\mathbf{x}]$,

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$$(f \bullet g)(\mathbf{x}) = f(\mathbf{x}) \bullet g(\mathbf{x})$$

⑥ $\mathcal{E}[\mathbf{x}]^2$: Sums of Squares of \mathcal{E} -valued polynomials:

$$\mathcal{E}[\mathbf{x}]^2 = \left\{ \sum_{i=1}^q f_i \circ f_i \mid q > 0, f_i \in \mathcal{E}[\mathbf{x}] \ (i = 1, \dots, q) \right\}$$

A Primal and Dual pair

$$\langle POP \rangle \begin{cases} \min & a(\mathbf{x}) \\ \text{s. t.} & \mathbf{x} \in K = \{ \mathbf{x} \in \mathbb{R}^n \mid b(\mathbf{x}) \in \mathcal{E}_+ \} . \end{cases}$$

- ⑥ $a \in \mathbb{R}[\mathbf{x}]$: \mathbb{R} -valued polynomial
- ⑥ $b \in \mathcal{E}[\mathbf{x}]$: \mathcal{E} -valued polynomial
- ⑥ \mathcal{E}_+ : the symmetric cone associated with \mathcal{E} .

A Primal and Dual pair

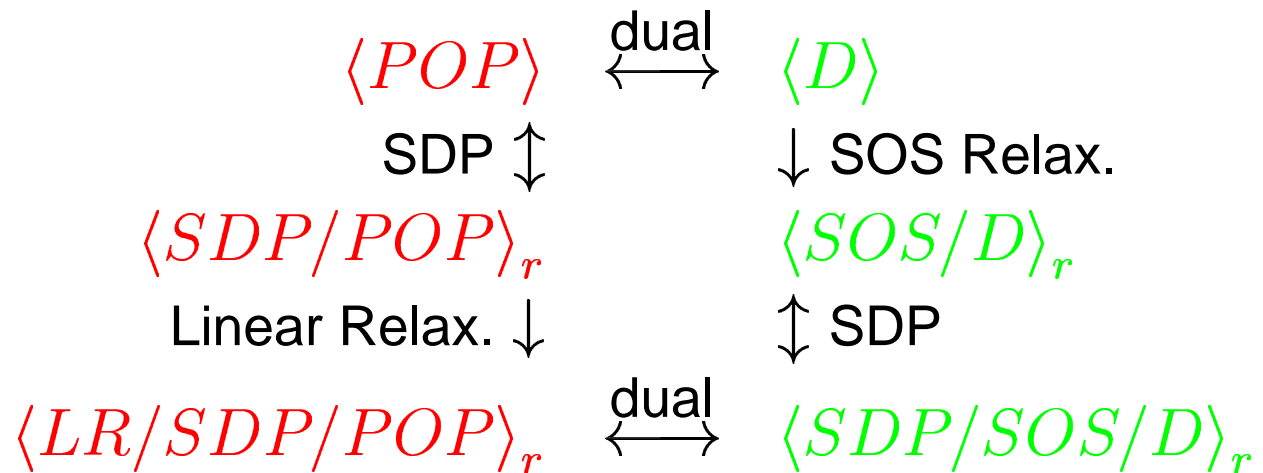
$$\langle POP \rangle \begin{cases} \min & a(\mathbf{x}) \\ \text{s. t.} & \mathbf{x} \in K = \{ \mathbf{x} \in \mathbb{R}^n \mid b(\mathbf{x}) \in \mathcal{E}_+ \} . \end{cases}$$

\Updownarrow dual

$$\langle D \rangle \begin{cases} \max & \zeta \\ \text{s. t.} & a(\mathbf{x}) - \zeta \geq 0 \ (\forall \mathbf{x} \in K) . \end{cases}$$

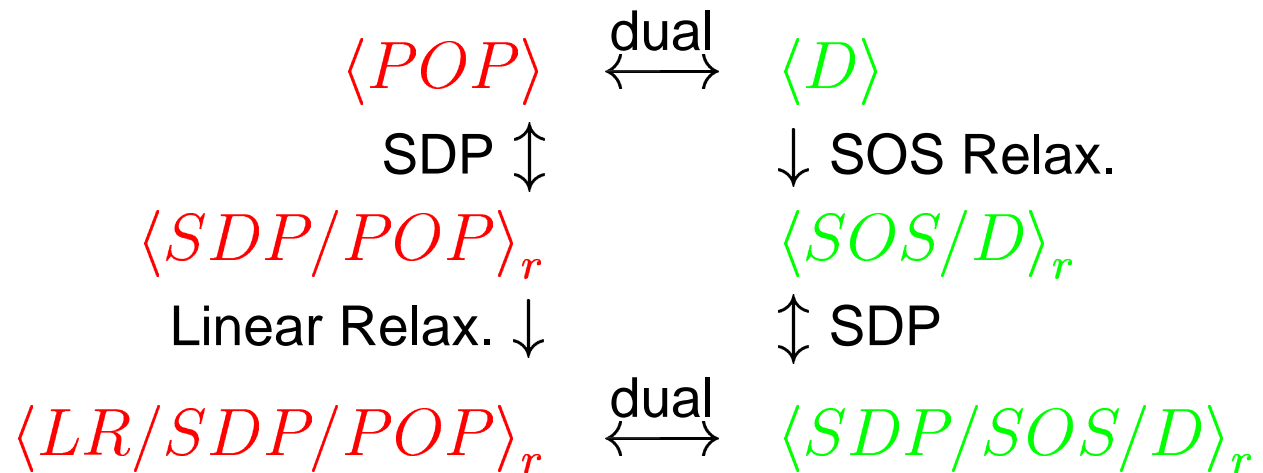
$$\text{optval} \langle POP \rangle = \text{optval} \langle D \rangle$$

Overview of Our Results



- ⑥ $\text{optval} \langle SOS/D \rangle_r \rightarrow \text{optval} \langle D \rangle$ as $r \rightarrow \infty$.
- ⑥ $\text{optval} \langle LR/SDP/POP \rangle_r \rightarrow \text{optval} \langle POP \rangle$ as $r \rightarrow \infty$.

Overview of Our Results



$\langle POP \rangle$ and $\langle D \rangle$ are POP over Symmetric Cones!

An SOS relaxation of $\langle D \rangle$

$$\langle D \rangle \max \quad \zeta \quad \text{s. t.} \quad a(\mathbf{x}) - \zeta \geq 0 \quad (\forall \mathbf{x} \in K)$$

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↓ SOS Relax.

$$\langle SOS/D \rangle_r \max \zeta \text{ s. t. } a - \zeta \in \mathbb{R}[\mathbf{x}]^2 + b \bullet \mathcal{E}[\mathbf{x}]_r^2$$

- ⑥ $\mathcal{E}[\mathbf{x}]_r$: \mathcal{E} -valued Polynomials, degree $\leq r$.
- ⑥ $\mathcal{E}[\mathbf{x}]_r^2$: Sums of Squares of $\mathcal{E}[\mathbf{x}]_r$.
- ⑥ $\mathbb{R}[\mathbf{x}]^2$: SOS of \mathbb{R} -valued Polynomials.

An SOS relaxation of $\langle D \rangle$

$$\langle D \rangle \max \zeta \text{ s. t. } a(\mathbf{x}) - \zeta \geq 0 \ (\forall \mathbf{x} \in K)$$

↓ SOS Relax.

$$\langle SOS/D \rangle_r \max \zeta \text{ s. t. } a - \zeta \in \mathbb{R}[\mathbf{x}]^2 + b \bullet \mathcal{E}[\mathbf{x}]_r^2$$

Theorem 1: $\text{optval} \langle SOS/D \rangle_r \nearrow \text{optval} \langle D \rangle$ as $r \rightarrow \infty$.

Idea of the Proof

ζ^* : $\text{optval}\langle D \rangle$, $\epsilon > 0$: fixed.

$$a(\mathbf{x}) - (\zeta^* - \epsilon) > 0 \quad (\forall \mathbf{x} \in K)$$

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\Downarrow

A Generalized
Putinar's lemma

$$\exists r > 0, a - (\zeta^* - \epsilon) \in \mathbb{R}[\mathbf{x}]^2 + b \bullet \mathcal{E}[\mathbf{x}]_r^2$$

Idea of the Proof

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A Generalized
Putinar's lemma

$$\exists r > 0, a - (\zeta^* - \epsilon) \in \mathbb{R}[\mathbf{x}]^2 + b \bullet \mathcal{E}[\mathbf{x}]_r^2$$



$\langle SOS/D \rangle_r$ has a feasible solution with $\zeta = \zeta^* - \epsilon$

A Generalization of Putinar's Lemma

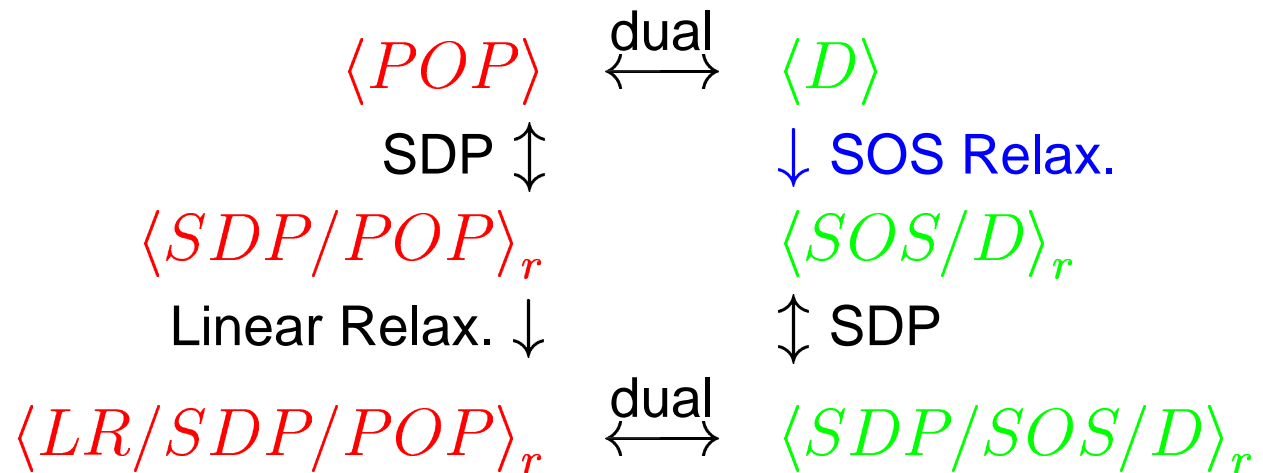
$$\begin{aligned} K &= \{ \mathbf{x} \in \mathbb{R}^n \mid b(\mathbf{x}) \in \mathcal{E}_+ \} \\ \tilde{\mathcal{C}} &= \mathbb{R}[\mathbf{x}]^2 + b \bullet \mathcal{E}[\mathbf{x}]^2 \end{aligned}$$

If K is compact, then

any positive polynomials on K belongs to $\tilde{\mathcal{C}}$
 \Updownarrow
 $\exists p \in \tilde{\mathcal{C}}$ such that $\{ \mathbf{x} \mid p(\mathbf{x}) \in \mathcal{E}_+ \}$ is compact.

- ⑥ Putinar's original Lemma : $\mathcal{E} = \mathbb{R}^N$.
- ⑥ We use this lemma assuming the latter condition.

Overview of Our Results



- ⑥ $\text{optval} \langle SOS/D \rangle_r \rightarrow \text{optval} \langle D \rangle$ as $r \rightarrow \infty$.
- ⑥ $\text{optval} \langle LR/SDP/POP \rangle_r \rightarrow \text{optval} \langle POP \rangle$ as $r \rightarrow \infty$.

An Equivalent SDP of SOS problem

- ⑥ \mathcal{D} : a finite dimensional linear subspace of $\mathcal{E}[x]$.
(e.g. $\mathcal{D} = \mathcal{E}[x]_r$)
- ⑥ \mathcal{A} : a basis of \mathcal{D} .
- ⑥ \mathcal{D}^2 : Sums of Squares of polynomials in \mathcal{D} .

An Equivalent SDP of SOS problem

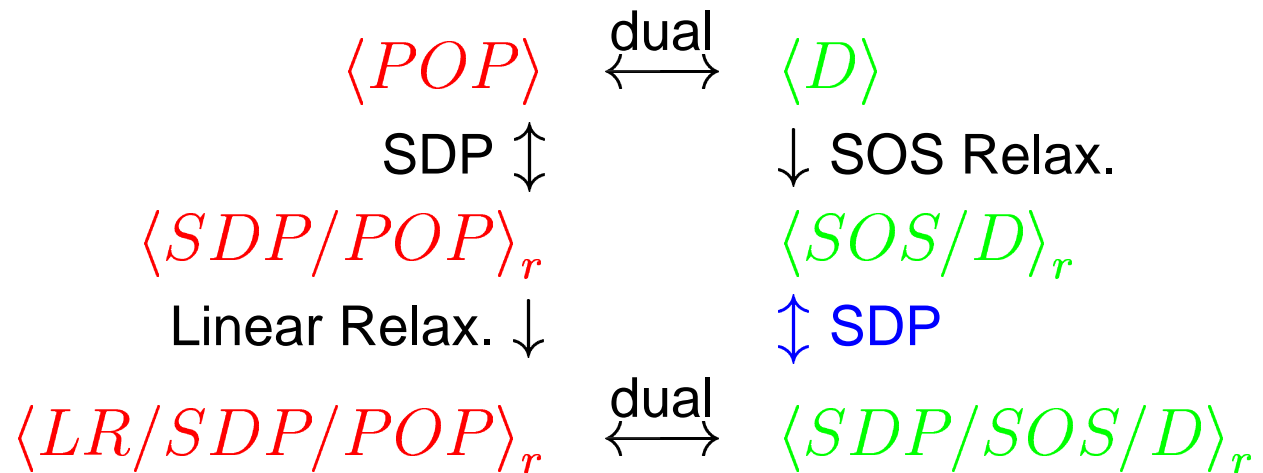
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- ⑥ \mathcal{D}^2 : Sums of Squares of polynomials in \mathcal{D} .

Theorem 2

$$\mathcal{D}^2 = \left\{ \sum_{f \in \mathcal{A}} \sum_{g \in \mathcal{A}} V_{fg} f \circ g \mid V \in \mathcal{S}_+^{|\mathcal{A}| \times |\mathcal{A}|} \right\}.$$

- ⑥ $\mathcal{S}_+^{|\mathcal{A}| \times |\mathcal{A}|}$: the set of $|\mathcal{A}| \times |\mathcal{A}|$ PSD matrices.

Overview of Our Results



- ⑥ $\text{optval} \langle SOS/D \rangle_r \rightarrow \text{optval} \langle D \rangle$ as $r \rightarrow \infty$.
- ⑥ $\text{optval} \langle LR/SDP/POP \rangle_r \rightarrow \text{optval} \langle POP \rangle$ as $r \rightarrow \infty$.

- ⑥ This work is motivated by Kojima [1] which deals with polynomial SDP. Some relationships between [1] and ours are also established in the paper [2].
- ⑥ In [2], a more detailed discussion is done to exploit sparsity of the polynomials to shrink the size of SDPs.

[1] M. Kojima : “Sums of Squares Relaxations of Polynomial Semidefinite Programs”, Research Report B-397, TIT.

[2] M. Kojima and M. Muramatsu : “An Extension of Sums of Squares Relaxations to Polynomial Optimization Problems over Symmetric Cones”, Research Report B-406, TIT.