

Positive polynomials and optimization

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Part I: Representation of positive polynomials

Part II: Optimization of polynomials

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on compact basic closed semialgebraic sets

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- $g_0 := 1 \in \mathbb{R}[X]$ for convenience

n

f

$g_1, \dots, g_m$

S

g_0

Example. When

- $m = n$
- $g_1 := X_1, \dots, g_n := X_n$

then S is the nonnegative orthant $[0, \infty)^n$.

Example. When

- $m = n + 1$
- $g_1 := X_1, \dots, g_n := X_n$
- $g_{n+1} := 1 - (X_1 + \dots + X_n)$

then S is the unit simplex in $\mathbb{R}^n$.

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- $m = n + 2$
- $g_1 := X_1, \dots, g_n := X_n$
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- $g_{n+2} := X_1 + \dots + X_n - 1$

then S is the standard simplex

$$\Delta_n := \{x \in \mathbb{R}^n \mid x_1 \geq 0, \dots, x_n \geq 0, x_1 + \dots + x_n = 1\}.$$

Theorem of Pólya.

George Pólya: Über positive Darstellung von Polynomen
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Victoria Powers, Bruce Reznick: A new bound for Pólya's theorem with applications to polynomials positive on polyhedra

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$$(X_1 + \cdots + X_n)^k f = \sum_{|\alpha|=k+d} \frac{k!(k+d)^d}{\alpha_1! \cdots \alpha_n!} f \quad \underbrace{\left(\frac{\alpha}{k+d} \right)}_{\in \Delta_n} X^\alpha$$

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$$f_\varepsilon := \sum_{|\alpha|=d} a_\alpha (X_1)_{\varepsilon}^{\alpha_1} \cdots (X_n)_{\varepsilon}^{\alpha_n} \quad \text{where} \quad (X_i)_{\varepsilon}^{\alpha_i} := \prod_{j=0}^{\alpha_i-1} (X_i - j\varepsilon).$$

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But $f_\varepsilon \rightarrow f$ uniformly on Δ_n for $\varepsilon \rightarrow 0$.

Sums of squares and quadratic modules

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is a set of polynomials which are

for obvious reasons ≥ 0 on S .

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is the quadratic module generated by $g_1, \dots, g_m$.

Quadratic module: $0, 1 \in M$, $M + M \subseteq M$ and $\mathbb{R}[X]^2 M \subseteq M$.

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- Several **non-trivial** sufficient criteria for the hypothesis being satisfied due to **Schmüdgen** and **Jacobi & Prestel**.

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- Substitute back old coordinates.

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$$f - s \sum_{i=1}^m \underbrace{(1 - g_i)^{2k} g_i}_{> 0} > 0 \quad \text{on } C.$$

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$$f - s \sum_{i=1}^m \underbrace{(1 - g_i)^{2k} g_i}_{\in M} > 0 \quad \text{on } C.$$

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^n X_i^2 \in M$.

Then $f > 0$ on $S \implies f \in M$.

Proof outline.

- WLOG $f > 0$ on an arbitrary fixed compact set $C \subseteq \mathbb{R}^n$.

Find $h \in M$ such that $f - h > 0$ on C . If we show $f - h \in M$, then $f \in M$.

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Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 (N - \sum_{i=1}^n X_i^2)$.

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Choose n new coordinates lying in T

$$Y_i \mapsto \left(N + \frac{1}{4}\right) + X_i = \sum_{j \neq i} X_j^2 + \left(X_i + \frac{1}{2}\right)^2 + \left(N - \sum_{j=1}^n X_j^2\right) \in T$$

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and summing up to a constant.

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in the new coordinates:

$$f \left(\underbrace{\frac{Y_1 - Y_{n+1}}{2}}, \dots, \underbrace{\frac{Y_n - Y_{2n}}{2}} \right)$$

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and **summing up to $2n(N + \frac{1}{4})$** . Set $\Delta := \{y \mid \substack{y_1 \geq 0, \dots, y_{2n} \geq 0 \\ \sum y_i = 2n(N + \frac{1}{4})} \}$.

Apply Pólya on
$$\sum_{i=0}^d f_i \left(\underbrace{\frac{Y_1 - Y_{n+1}}{2}}_{\mapsto X_1}, \dots, \underbrace{\frac{Y_n - Y_{2n}}{2}}_{\mapsto X_n} \right) \left(\underbrace{\frac{Y_1 + \dots + Y_{2n}}{2n(N + \frac{1}{4})}}_{\mapsto 1} \right)^{d-i}$$

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and summing up to $2n(N + \frac{1}{4})$. Set $\Delta := \{y \mid \sum y_i = 2n(N + \frac{1}{4}), y_1 \geq 0, \dots, y_{2n} \geq 0\}$.

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> 0 on Δ when $f > 0$ on $\diamond := l(\Delta)$

Apply Pólya on

$$\sum_{i=0}^d f_i \left(\underbrace{\frac{Y_1 - Y_{n+1}}{2}}_{\mapsto X_1}, \dots, \underbrace{\frac{Y_n - Y_{2n}}{2}}_{\mapsto X_n} \right) \left(\overbrace{\frac{Y_1 + \dots + Y_{2n}}{2n(N + \frac{1}{4})}}^{=1 \text{ on } \Delta} \right)^{d-i} \underbrace{\hspace{1.5cm}}_{\mapsto 1}$$

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Choose $2n$ new coordinates lying in T

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 defines linear map $l : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$
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$$\begin{array}{c} \overbrace{\qquad\qquad\qquad}^{> 0 \text{ on } \Delta \text{ since } f > 0 \text{ on } \diamond := l(\Delta)} \\ \text{defines linear map } l : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n \\ \left(\frac{Y_1 + \dots + Y_{2n}}{2n(N + \frac{1}{4})}\right)^k \sum_{i=0}^d f_i \left(\underbrace{\frac{Y_1 - Y_{n+1}}{2}}_{\mapsto X_1}, \dots, \underbrace{\frac{Y_n - Y_{2n}}{2}}_{\mapsto X_n} \right) \left(\underbrace{\frac{Y_1 + \dots + Y_{2n}}{2n(N + \frac{1}{4})}}_{\mapsto 1} \right)^{d-i} \end{array}$$

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$$\underbrace{\left(\frac{Y_1 + \dots + Y_{2n}}{2n(N + \frac{1}{4})} \right)^k \sum_{i=0}^d f_i \left(\underbrace{\frac{Y_1 - Y_{n+1}}{2}}_{\mapsto X_1}, \dots, \underbrace{\frac{Y_n - Y_{2n}}{2}}_{\mapsto X_n} \right) \left(\underbrace{\frac{Y_1 + \dots + Y_{2n}}{2n(N + \frac{1}{4})}}_{\mapsto 1} \right)^{d-i}}_{\substack{> 0 \text{ on } \Delta \text{ since } f > 0 \text{ on } \diamond := l(\Delta) \\ \text{defines linear map } l : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n \\ = 1 \text{ on } \Delta}}$$

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^n X_i^2 \in M$.

Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := \diamond$.

Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 (N - \sum_{i=1}^n X_i^2)$.

$\rightsquigarrow$ Note $T \subseteq M$. Advantage: $TT \subseteq T$.

Choose $2n$ new coordinates lying in T

$$Y_{\substack{i \\ i+n}} \mapsto \left(N + \frac{1}{4}\right) \pm X_i = \sum_{j \neq i} X_j^2 + \left(X_i \pm \frac{1}{2}\right)^2 + \left(N - \sum_{j=1}^n X_j^2\right) \in T$$

and summing up to $2n(N + \frac{1}{4})$. Set $\Delta := \{y \mid \begin{smallmatrix} y_1 \geq 0, \dots, y_{2n} \geq 0 \\ \sum y_i = 2n(N + \frac{1}{4}) \end{smallmatrix}\}$.

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$M := \sum_{i=0}^m \sum \mathbb{R}[X]^2 g_i$ is the quadratic module generated by $g_1, \dots, g_m$.

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A preorder is a quadratic module closed under multiplication.

$$m = 1 \implies M = T$$

Krivine's Positivstellensatz. $f > 0$ on $S \iff \exists t \in T : \textcolor{red}{t}f \in 1 + T$

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Krivine's Positivstellensatz. $f > 0$ on $S \iff \exists t \in T : tf \in 1 + T$

- Proof is much harder.
- Is **not** suited for optimization problems
but only for feasibility problems.

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Schmüdgen's Positivstellensatz. Suppose S is compact.
Then $f > 0$ on $S \implies f \in T$.

Konrad Schmüdgen: The K -moment problem for compact
semi-algebraic sets
Math. Ann. **289**, No. 2, 203–206 (1991)

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- Using Krivine's Positivstellensatz, a **little trick** of Wörmann shows: S compact $\iff \exists N \in \mathbb{N} : N - \sum_{i=1}^n X_i^2 \in T$.
- Up to that our proof via Pólya works again.

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- Up to that our proof via Pólya works again.
- Up to that we have in this case another **more complicated** proof via Pólya **yielding** an interesting **bound** on the Pólya-exponent.

An algorithmic approach to Schmüdgen's Positivstellensatz

Journal of Pure and Applied Algebra **166**, 307–319 (2002)

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$$\exists N \in \mathbb{N} : N - \sum_{i=1}^n X_i^2 \in \textcolor{red}{T} \iff S \text{ compact}$$

follows either from Schmüdgen's Positivstellensatz

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From this follows that $\exists N \in \mathbb{N} : N - \sum_{i=1}^n X_i^2 \in \textcolor{red}{M}$ is for example equivalent to the compactness of $\{x \in \mathbb{R}^n \mid h(x) \geq 0\}$ for some $h \in M$.

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Have we cheated?

Jacobi and Prestel: Deeper criteria.

Thomas Jacobi, Alexander Prestel:

Distinguished representations of strictly positive polynomials

J. Reine Angew. Math. **532**, 223–235 (2001)

Part II

Optimization of polynomials

on compact basic closed semialgebraic sets

Notation for the whole talk

- $X := (X_1, \dots, X_n)$ variables
- $\mathbb{R}[X] := \mathbb{R}[X_1, \dots, X_n]$ polynomial ring
- $f \in \mathbb{R}[X]$ an arbitrary polynomial
- $g_1, \dots, g_m \in \mathbb{R}[X]$ polynomials defining ...
- ... the set $S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$
- $g_0 := 1 \in \mathbb{R}[X]$ for convenience
- $M := \sum_{i=0}^m \sum \mathbb{R}[X]^2 g_i = \{ \sum_{i=0}^m \sigma_i g_i \mid \sigma_i \in \sum \mathbb{R}[X]^2 \}$
the quadratic module generated by $g_1, \dots, g_m$

n

f

$g_1, \dots, g_m$

S

g_0

M

For the rest of the talk, **assume** that

$$N - \sum_{i=1}^n X_i^2 \in M$$

for some $N \in \mathbb{N}$.

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In particular, S is compact.

Optimization

We consider the problem of minimizing f on S .

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$$f^* := \inf\{f(x) \mid x \in S\} \in \mathbb{R} \cup \{\infty\}$$

Optimization

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$$f^* := \inf\{f(x) \mid x \in S\} \in \mathbb{R} \cup \{\infty\}$$

and, if possible, a **minimizer**, i.e., an element of the set

$$S^* := \{x^* \in S \mid \forall x \in S : f(x^*) \leq f(x)\}.$$

Convexification

Convexify the problem by brute force.

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- Take a **dual standpoint**:

$$f^* = \sup\{a \in \mathbb{R} \mid f - a \geq 0 \text{ on } S\} = \sup\{a \in \mathbb{R} \mid f - a > 0 \text{ on } S\}$$

Describing measures and positive polynomials

Putinar's solution to the moment problem. For every map $L : \mathbb{R}[X] \rightarrow \mathbb{R}$ are equivalent:

- (1) L is linear, $L(1) = 1$ and $L(M) \subseteq [0, \infty)$
- (2) $\exists \mu \in \mathcal{M}^1(S) : \forall p \in \mathbb{R}[X] : L(p) = \int p d\mu$

Mihai Putinar: Positive polynomials on compact semi-algebraic sets
Indiana Univ. Math. J. **42**, No. 3, 969–984 (1993)

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Stone-Weierstrass Approximation $\uparrow$ Riesz Representation

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Putinar's $\Downarrow$ Positivstellensatz

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$$\mathbb{R}[X]$$

polynomial ring

$$M := \sum_{i=0}^m \sum \mathbb{R}[X]^2 g_i$$

quadratic module

$$= \left\{ \sum_{i=0}^m \sigma_i g_i \mid \sigma_i \in \sum \mathbb{R}[X]^2 \right\}$$

Introduce finite-dimensional analogues $M_{\textcolor{red}{k}} \subseteq \mathbb{R}[X]_{\textcolor{red}{k}}$ of $M \subseteq \mathbb{R}[X]$.

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$$\mathbb{R}[X]_{\textcolor{red}{k}} := \{p \mid p \in \mathbb{R}[X], \deg p \leq \textcolor{red}{k}\} \quad \textcolor{blue}{\text{real vector space}}$$

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for arbitrary

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Here $d_i := \max\{e \in \mathbb{N} \mid 2e + \deg g_i \leq k\}$.

Warning: Never confuse M_k with $M \cap \mathbb{R}[X]_k \supsetneq M_k$.

We saw that

$$f^* = \inf\{L(f) \mid L : \mathbb{R}[X] \rightarrow \mathbb{R} \text{ is linear, } L(1) = 1, L(M) \subseteq [0, \infty)\} \quad \text{and}$$

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In analogy to this, we set

$$P_k^* = \inf\{L(f) \mid L : \mathbb{R}[X]_{\textcolor{red}{k}} \rightarrow \mathbb{R} \text{ is linear, } L(1) = 1, L(M_{\textcolor{red}{k}}) \subseteq [0, \infty)\} \quad \text{and}$$

$$D_k^* = \sup\{a \in \mathbb{R} \mid f - a \in M_{\textcolor{red}{k}}\}$$

for every $k \in \mathcal{N}$.

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for every $k \in \mathcal{N}$.

$P_k^* \in \mathbb{R} \cup \{\pm\infty\}$ and $D_k^* \in \mathbb{R} \cup \{\pm\infty\}$ are the optimal values of the following pair of optimization problems ...

$$\begin{aligned}
 (P_k) \quad & \text{minimize} \quad L(f) \quad \text{subject to} \quad L : \mathbb{R}[X]_k \rightarrow \mathbb{R} \text{ is linear,} \\
 & L(1) = 1 \text{ and} \\
 & L(M_k) \subseteq [0, \infty)
 \end{aligned}$$

$$\begin{aligned}
 (D_k) \quad & \text{maximize} \quad a \quad \text{subject to} \quad a \in \mathbb{R} \text{ and} \\
 & f - a \in M_k
 \end{aligned}$$

$$\begin{aligned}
(P_k) \quad & \text{minimize} \quad L(f) \quad \text{subject to} \quad L : \mathbb{R}[X]_k \rightarrow \mathbb{R} \text{ is linear,} \\
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& f - a \in M_k
\end{aligned}$$

Theorem (Lasserre). $(D_k^*)_{k \in \mathcal{N}}$ and $(P_k^*)_{k \in \mathcal{N}}$ are increasing sequences that converge to f^* and satisfy $D_k^* \leq P_k^* \leq f^*$.

Proof.

$$\begin{aligned}
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Jean Lasserre: Global optimization with polynomials and the problem of moments
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Theorem. Suppose $m = 1$ and $g := g_1$. Then there exists $C \in \mathbb{N}$ depending on f and g and $c \in \mathbb{N}$ depending on g such that

$$f^* - D_k^* \leq \frac{C}{\sqrt[c]{k}} \quad \text{for big } k.$$

On the complexity of Schmüdgen's Positivstellensatz
Journal of Complexity **20**, No. 4, 529—543 (2004)

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Dependence on f can be made explicit. Proof hints to make dependence on g explicit for concrete g .

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In practice: Convergence usually very fast,
often $D_k^* = P_k^* = f^*$ for small k .

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What can we know from Putinar's solution to the moment
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A priori nothing! But with additional compactness arguments involving Tychonoff's Theorem, the following ...

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Theorem. Suppose that L_k solves (P_k) nearly to optimality ($k \in \mathcal{N}$).

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Optimization of polynomials on compact semialgebraic sets
preprint

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Theorem (Lasserre). If S has nonempty interior, then $D_k^* = P_k^*$.

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Optimization of polynomials on compact semialgebraic sets
preprint

Murray Marshall: Optimization of polynomial functions
Canad. Math. Bull. **46**, 575–587 (2003)

Jean Lasserre: Global optimization with polynomials and the
problem of moments
SIAM J. Optim. **11**, No. 3, 796–817 (2001)

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- b does not depend polynomially on $(\mathcal{D}, k)$.

Further properties of the method

- Feasible solutions of the semidefinite program corresponding to (D_k) give rise to a lower bound a of f^* together with a certificate (advantage) in form of a representation of $f - a$ proving $f - a \in M_k$.

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- Method converges to unique minimizers. **Disadvantage:** Possibly **from outside** the set.
- If there is a unique minimizer and it lies in the interior of S , then the method produces a sequence of intervals containing f^* whose endpoints converge to f^* .

Advertissement

Optimization of polynomials on basic closed semialgebraic sets
preprint

<http://www.mathe.uni-konstanz.de/schweigh/>

Announcement

Positive Polynomials

March 14–18, 2005, CIRM, Luminy (near Marseille)

organized by Marie-Françoise Roy and Markus Schweighofer
sponsored by European network RAAG and CIRM

Real Algebraic Geometry (Sums of squares)

+

Functional Analysis (Moment problems)

+

Optimization (Interior Point Methods)

<http://www.mathe.uni-konstanz.de/schweigh/luminy2005/>

Contact: Markus.Schweighofer@uni-konstanz.de