Additionnal Exercises for the LNMB course CO1a
Monday September 13, 2010

Let $D = (V, A)$ be a digraph and let $\ell : A \to \Re$ be a length function on the arcs of D. Set $n = |V|$. Let $s, t \in V$.
A *directed circuit* C in D is given by a set of vertices $(v_1, \ldots, v_p)$ for which $(v_1, v_2), \ldots, (v_{p-1}, v_p), (v_p, v_1) \in A$. C has *negative length* if $\ell(C) < 0$, where $\ell(C) = \sum_{i=1}^{p-1} \ell_{(v_i, v_{i+1})} + \ell_{(v_p, v_1)}$ is the sum of the lengths of the arcs in C. The circuit C is said to be *reachable from* s if there exists a path from s to at least one node of C.

Exercise A: Assume that D does not contain a directed circuit of negative length reachable from s. Show that, if there exists an (s, t)-walk, then there exists a shortest (s, t)-walk which, moreover, is a path.

Exercise B. For $k = 0, 1, \ldots, n$, let f_k be the function defined in rel. (15) (p. 13) of the Lecture Notes. Moreover, define the function $g_k : V \to \Re \cup \{\infty\}$, where $g_k(v)$ is the minimum length of an (s, v)-path traversing at most k arcs (i.e. with cardinality at most k).
B1) Show that $f_k = g_k$ for $k = 0, 1, \ldots, n$. [Hint: Use induction on k.]
B2) Deduce that, if D does not contain a directed circuit of negative length reachable from s then, for all $v \in V$, $f_{n-1}(v)$ is equal to the length of a shortest path from s to v.

Exercise C. Show that there exists a directed circuit of negative length reachable from the vertex s if and only if $f_n \neq f_{n-1}$.