LNMB - NETWORKS AND SEMIDEFINITE PROGRAMMING
LECTURE 1 - EXERCISES

Exercise 1. The goal is to show that the vertex-coloring problem is NP-complete. For this one gives a polynomial-time reduction from the stable set problem to the vertex-coloring problem.

Let $G = (V, E)$ be a graph and let k be an integer. We build a new graph H in the following way.
Let V' be a copy of V and let C be a set of cardinality k, where V, V' and C are disjoint. Then the vertex set of H is $V \cup V' \cup C$ and the edge set of H is defined as follows: Two vertices of V are adjacent in H if and only they are adjacent in G (that is, we have the graph G on V); the sets V' and C are cliques in H; each vertex of V is adjacent to each vertex of $V' \cup C$ except to its copy in V'; no vertex of V' is adjacent to a vertex of C.

Show: $\alpha(G) \geq k$ if and only if $\chi(H) \leq |V| + 1$.

Conclude that the vertex-coloring problem is NP-complete.

Exercise 2. Here we give a polynomial-time reduction from the vertex-coloring problem to the stable set problem.

Let $G = (V, E)$ be a graph and let k be an integer. We build a new graph H in the following way.
Say, $V = \{v_1, \ldots, v_n\}$. Let $V_1, \ldots, V_k$ be k disjoint copies of V, and set $V_h = \{v_{h,1}, \ldots, v_{h,n}\}$ for $h = 1, \ldots, k$. Then the vertex set of H is $V_1 \cup \ldots \cup V_k$ and its edges are defined as follows: Two vertices $v_{h,i}$ and $v_{h,j}$ of V_h are adjacent in H if and only v_i and v_j are adjacent in G (that is, we put a copy of G on each V_h); moreover, each set $\{v_{1,i}, v_{2,i}, \ldots, v_{k,i}\}$ (consisting of the k copies of v_i) is a clique for each $i = 1, \ldots, n$.

Show: $\chi(G) \leq k$ if and only if $\alpha(H) \geq n$.

Exercise 3. Let $G = (V, E)$ be a planar graph. The goal is to show that $\chi(G) \leq 5$.

1) Show: $2m \geq 3f$, where m is the number of edges and f is the number of faces in a plane drawing.

2) Show: G has at least one node u with degree at most 5.

Hint: Show that $m \geq 3n$ if every node has degree at least 6, where n is the number of nodes. And use Euler formula: $n + f = m + 2$. (For instance, for K_4, $n = 4$, $m = 6$, and $f = 4$).

3) Show: $\chi(G) \leq 5$ (using induction on n).

*Hint: Distinguish two cases: (i) u has degree at most 4; (ii) u has degree 5. In case (ii), pick two neighbors v_1 and v_2 of u that are not adjacent (***why do they exist?***) and consider the graph H obtained by contracting both edges $\{u, v_1\}$ and $\{u, v_2\}$, which is 5-colorable using induction.*

Exercise 4. (Exercise 7.4 in the Lecture Notes). Derive from König's edge cover theorem (Corollary 3.3a) that if G is the complement of a bipartite graph, then $\omega(G) = \chi(G)$.

(Recall that the symbol $\gamma(G)$ is used in the Lecture Notes for denoting the coloring number of G, which is denoted here as $\chi(G)$.)

Exercise 5. (Exercise 7.5 in the Lecture Notes). Derive König's edge cover theorem (Corollary 3.3a) from the strong perfect graph theorem.

Exercise 6. (Exercise 7.6 in the Lecture Notes). Let H be a bipartite graph and let G be the complement of the line-graph of H. Derive from König's matching theorem (Theorem 3.3) that $\omega(G) = \chi(G)$.