LNMB PhD Course

Networks and Semidefinite Programming

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CHAPTER 1

GRAPH COLORING AND INDEPENDENT SETS

In this chapter we discuss how semidefinite programming can be used for constructing tractable bounds for two hard combinatorial problems: for finding maximum independent sets and minimum colorings in graphs.

We introduce the graph parameter $\vartheta(G)$, known as the *theta number* of the graph G. This parameter was introduced by L. Lovász in his seminal paper [7]. We present several equivalent formulations and explain how $\vartheta(G)$ can be used to compute maximum stable sets and minimum colorings in perfect graphs in polynomial time, whereas these problems are NP-hard for general graphs.

Here are some definitions that we use in this chapter. Let $G = (V, E)$ be a graph; often we let $V = [n] = \{1, \ldots, n\}$. Then, $\overline{E}$ denotes the set of pairs $\{i, j\}$ of distinct nodes that are not adjacent in G. The graph $\overline{G} = (V, \overline{E})$ is called the *complementary graph* of G. G is *self-complementary* if G and $\overline{G}$ are isomorphic graphs. Given a subset $S \subseteq V$, $G[S]$ denotes the *subgraph induced by* S: its node set is S and its edges are all pairs $\{i, j\} \in E$ with $i, j \in S$. The graph C_n is the circuit (or cycle) of length n, with node set $[n]$ and edges the pairs $\{i, i+1\}$ (for $i \in [n]$, indices taken modulo n). For a set $S \subseteq V$, its *characteristic vector* is the vector $\chi^S \in \{0, 1\}^V$, whose i-th entry is 1 if $i \in S$ and 0 otherwise. We let $e = (1, \ldots, 1)^{\mathsf{T}}$ denote the all-ones vector.

1.1 Preliminaries on graphs

1.1.1 Stability and chromatic numbers

A subset $S \subseteq V$ of nodes is said to be *stable* (or *independent*) if no two nodes of S are adjacent in G. Then the *stability number* of G is the parameter $\alpha(G)$ defined as the maximum cardinality of an independent set in G.

A subset $C \subseteq V$ of nodes is called a *clique* if every two distinct nodes in C are adjacent. The maximum cardinality of a clique in G is denoted $\omega(G)$, the *clique number* of G. Clearly,

$$\omega(G) = \alpha(\overline{G}).$$

Computing the stability number of a graph is a hard problem: Given a graph G and an integer k, deciding whether $\alpha(G) \geq k$ is an $\mathcal{N}P$-complete problem.

Given an integer $k \geq 1$, a k-*coloring* of G is an assignment of numbers (view them as *colors*) from $\{1, \cdots, k\}$ to the nodes in such a way that two adjacent nodes receive distinct colors. In other words, this corresponds to a partition of V into k stable sets: $V = S_1 \cup \cdots \cup S_k$, where S_i is the stable set consisting of all nodes that received the i-th color. The *coloring* (or *chromatic*) *number* is the smallest integer k for which G admits a k-coloring, it is denoted as $\chi(G)$.

Again it is an $\mathcal{N}P$-complete problem to decide whether a graph is k-colorable. In fact, it is $\mathcal{N}P$-complete to decide whether a planar graph is 3-colorable. On the other hand, it is known that every planar graph is 4-colorable – this is the celebrated 4-color theorem. Moreover, observe that one can decide in polynomial time whether a graph is 2-colorable, since one can check in polynomial time whether a graph is bipartite.

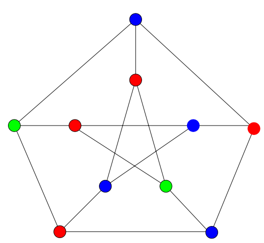

Figure 1.1: The Petersen graph has $\alpha(G) = 4$, $\omega(G) = 2$ and $\chi(G) = 3$

Clearly, any two nodes in a clique of G must receive distinct colors. Therefore, for any graph, the following inequality holds:

$$\omega(G) \leq \chi(G). \tag{1.1}$$

This inequality is strict, for example, when G is an odd circuit, i.e., a circuit of odd length at least 5, or its complement. Indeed, for an odd circuit C_{2n+1} ($n \geq 2$), $\omega(C_{2n+1}) = 2$ while $\chi(C_{2n+1}) = 3$. Moreover, for the complement

$G = \overline{C_{2n+1}}$, $\omega(G) = n$ while $\chi(G) = n + 1$. For an illustration see the cycle of length 7 and its complement in Figure 6.2.

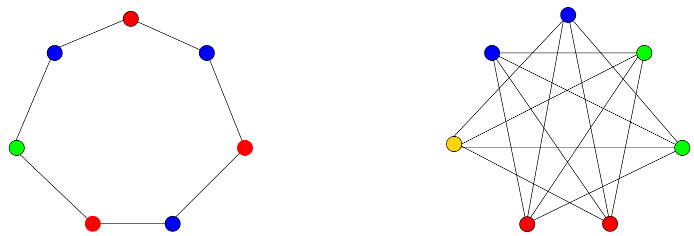

Figure 1.2: For C_7 and its complement $\overline{C_7}$: $\omega(C_7) = 2$, $\chi(C_7) = 3$, $\omega(\overline{C_7}) = \alpha(C_7) = 3$, $\chi(\overline{C_7}) = 4$

1.1.2 Perfect graphs

It is intriguing to understand for which graphs equality $\omega(G) = \chi(G)$ holds. Note that any graph G with $\omega(G) < \chi(G)$ can be embedded in a larger graph $\hat{G}$ with $\omega(\hat{G}) = \chi(\hat{G})$, simply by adding to G a clique of size $\chi(G)$ (disjoint from V). This justifies the following definition, introduced by C. Berge in the early sixties, which makes the problem well posed.

Definition 1.1.1. *A graph G is said to be* perfect *if equality*

$$\omega(H) = \chi(H)$$

holds for all induced subgraphs H of G (including $H = G$).

Here are some classes of perfect graphs. For each of them the relation $\omega(G) = \chi(G)$ gives a combinatorial min-max relation.

1. Bipartite graphs (the relation $\omega(G) = \chi(G) = 2$ is clear).

2. Line graphs of bipartite graphs (the min-max relation claims that the maximum cardinality of a matching is equal to the minimum cardinality of a vertex cover, which is König's theorem).

3. Comparability graphs (the min-max relation corresponds to Diilworth's theorem).

It follows from the definition and the above observation about odd circuits that if G is a perfect graph then it does not contain an odd circuit of length at least 5 or its complement as an induced subgraph. Berge already conjectured that *all* perfect graphs arise in this way. Resolving this conjecture has haunted generations of graph theorists. It was finally settled in 2002 by Chudnovsky, Robertson, Seymour and Thomas who proved the following result, known as the *strong perfect graph theorem*:

Theorem 1.1.2. (The strong perfect graph theorem)*[1] A graph G is perfect if and only if it does not contain an odd circuit of length at least 5 or its complement as an induced subgraph.*

This implies the following structural result about perfect graphs, known as the *perfect graph theorem*, already proved by Lovász in 1972.

Theorem 1.1.3. (The perfect graph theorem)*[6] If G is a perfect graph, then its complement $\overline{G}$ too is a perfect graph.*

We give a direct proof of Theorem 1.1.3 in the next section and we will mention later some other, more geometric, characterizations of perfect graphs (see, e.g., Theorem 1.2.5).

1.1.3 The perfect graph theorem

Lovász [6] proved the following result, which implies the perfect graph theorem (Theorem 1.1.3). The proof given below follows the elegant linear-algebraic argument of Gasparian [3].

Theorem 1.1.4. *A graph G is perfect if and only if $|V(G')| \leq \omega(G')\chi(G')$ for each induced subgraph G' of G.*

Proof. Necessity is easy: Assume that G is perfect and let G' be an induced subgraph of G. Then $\chi(G') = \omega(G')$ and thus $V(G')$ can be covered by $\omega(G')$ stable sets, which implies that $|V(G')| \leq \omega(G')\alpha(G')$.

To show sufficiency, assume for a contradiction that there exists a graph G which satisfies the condition but is not perfect; choose such a graph with $|V(G)|$ minimal. Then, $n \leq \alpha(G)\omega(G)$, $\omega(G) < \chi(G)$ and $\omega(G') = \chi(G')$ for each induced subgraph $G' \neq G$ of G. Set $\omega = \omega(G)$ and $\alpha = \alpha(G)$ for simplicity. Our first claim is:

Claim 1: There exist $\alpha\omega + 1$ stable sets $S_0, \ldots, S_{\alpha\omega}$ such that each vertex of G is covered by exactly α of them.

Proof of the claim: Let S_0 be a stable set of size α in G. For each node $v \in S_0$, as $G \backslash v$ is perfect (by the minimality assumption on G), $\chi(G \backslash v) = \omega(G \backslash v) \leq \omega$. Hence, $V \setminus \{v\}$ can be partitioned into ω stable sets. In this way we obtain a collection of $\alpha\omega$ stable sets which together with S_0 satisfy the claim. $\square$

Our next claim is:

Claim 2: For each $i = 0, 1, \ldots, \alpha\omega$, there exists a clique K_i of size ω such that $K_i \cap S_i = \emptyset$ and $K_i \cap S_j \neq \emptyset$ for $j \neq i$.

Proof of the claim: For each $i = 0, 1, \ldots, \alpha\omega$, as $G \setminus S_i$ is perfect we have that $\chi(G \backslash S_i) = \omega(S_i) \leq \omega$. This implies that $\chi(G \backslash S_i) = \omega$ since, if $\chi(G \backslash S_i) \leq \omega - 1$, then one could color G with ω colors, contradicting our assumption on G. Hence there exists a clique K_i disjoint from S_i and with $|K_i| = \omega$. Moreover K_i meets all the other $\alpha\omega$ stable sets S_j for $j \neq i$. This follows from the fact that each of the ω elements of K_i belongs to α stable sets among the S_j's (Claim 1) and these $\omega\alpha$ sets are pairwise distinct. $\square$

We can now conclude the proof. Define the matrices $M, N \in \mathbb{R}^{n\times(\alpha\omega+1)}$, whose columns are $\chi^{S_0}, \ldots, \chi^{S_{\alpha\omega}}$ (the incidence vectors of the stable sets S_i), and the vectors $\chi^{K_0}, \ldots, \chi^{\alpha\omega+1}$ (the incidence vectors of the cliques K_i), respectively. By Claim 2, we have that $M^{\mathsf{T}}N = J - I$ (where J is the all-ones matrix and I the identity). As $J - I$ is nonsingular, we obtain that that $\text{rank}(M^{\mathsf{T}}N) = \text{rank}(J - I) = \alpha\omega + 1$. On the other hand, $\text{rank}(M^{\mathsf{T}}N) \leq \text{rank}N \leq n$. Thus we obtain that $n \geq \alpha\omega + 1$, contradicting our assumption on G. $\square$

1.2 Linear programming bounds

1.2.1 Fractional stable sets and colorings

Let $\text{ST}(G)$ denote the polytope in $\mathbb{R}^V$ defined as the convex hull of the characteristic vectors of the stable sets of G:

$$\text{ST}(G) = \text{conv}\{\chi^S : S \subseteq V,\ S \text{ is a stable set in } G\},$$

called the *stable set polytope* of G. Hence, computing $\alpha(G)$ is linear optimization over the stable set polytope:

$$\alpha(G) = \max\{e^{\mathsf{T}}x : x \in \text{ST}(G)\}.$$

We have now defined the stable set polytope by listing explicitly its extreme points. Alternatively, it can also be represented by its hyperplanes representation, i.e., in the form

$$\text{ST}(G) = \{x \in \mathbb{R}^V : Ax \leq b\}$$

for some matrix A and some vector b. As computing the stability number is a hard problem one cannot hope to find the full linear inequality description of the stable set polytope (i.e., the explicit A and b). However some partial information is known: several classes of valid inequalities for the stable set polytope are known. For instance, if C is a clique of G, then the *clique inequality*

$$x(C) = \sum_{i\in C} x_i \leq 1 \tag{1.2}$$

is valid for $\text{ST}(G)$: any stable set can contain at most one vertex from the clique C. The clique inequalities define the polytope

$$\text{QST}(G) = \left\{x \in \mathbb{R}^V : x \geq 0,\ x(C) \leq 1\ \forall C \text{ clique of } G\right\} \tag{1.3}$$

and maximizing the linear function $e^{\mathsf{T}}x$ over it gives the parameter

$$\alpha^*(G) = \max\{e^{\mathsf{T}}x : x \in \text{QST}(G)\}, \tag{1.4}$$

known as the *fractional stability number* of G. Clearly, $\text{QST}(G)$ is a relaxation of the stable set polytope:

$$\text{ST}(G) \subseteq \text{QST}(G). \tag{1.5}$$

The parameter $\alpha^*(G)$ is nested between $\alpha(G)$ and $\chi(\overline{G})$, and it can also be interpreted in terms of *fractional colorings* of $\overline{G}$.

Lemma 1.2.1. *For any graph G, we have*

$$\alpha(G) \leq \alpha^*(G) \leq \chi(\overline{G}). \tag{1.6}$$

Moreover, $\alpha^*(G)$ is equal to the optimal value of the linear program

$$\min \left\{ \sum_{C \text{ clique of } G} y_C : \sum_{C \text{ clique of } G} y_C \chi^C = e,\ y_C \geq 0\ \forall C \text{ clique of } G \right\}. \tag{1.7}$$

Proof. The inequality $\alpha(G) \leq \alpha^*(G)$ in (1.6) follows from the inclusion (1.5) and the inequality $\alpha^*(G) \leq \chi(\overline{G})$ follows from the definitions: If $x \in \text{QST}(G)$ and $V = C_1 \cup \cdots \cup C_k$ is a partition into k cliques, then

$$x^\mathsf{T} e = x^\mathsf{T} \left(\sum_{i=1}^k \chi^{C_i} \right) = \sum_{i=1}^k x(C_i) \leq \sum_{i=1}^k 1 = k.$$

We now show that the optimal value of (1.7) is equal to $\alpha^*(G)$ (which again gives the inequality $\alpha^*(G) \leq \chi(\overline{G})$). For this, we first observe that in the linear program (1.4) the condition $x \geq 0$ can be removed without changing the optimal value; that is,

$$\alpha^*(G) = \max\{e^\mathsf{T} x : x(C) \leq 1\ \forall C \text{ clique of } G\} \tag{1.8}$$

(check it). Now, it suffices to observe that the dual LP of the above linear program (1.8) coincides with the linear program (1.7). □

For instance, for an odd circuit C_{2n+1} ($n \geq 2$), $\alpha^*(C_{2n+1}) = \frac{2n+1}{2}$ (check it) lies strictly between $\alpha(C_{2n+1}) = n$ and $\chi(\overline{C_{2n+1}}) = n + 1$.

When G is a perfect graph, equality holds throughout in relation (1.6). As we see in the next section, there is a natural extension of this result to weighted graphs, which permits to show the equality $\text{ST}(G) = \text{QST}(G)$ when G is a perfect graph. Moreover, it turns out that this geometric property characterizes perfect graphs.

1.2.2 Polyhedral characterization of perfect graphs

For any graph G, the factional stable set polytope is a linear relaxation of the stable set polytope: $\text{ST}(G) \subseteq \text{QST}(G)$. Here we show a geometric characterization of perfect graphs: G is perfect if and only if both polytopes coincide: $\text{ST}(G) = \text{QST}(G)$.

The following operation of *duplicating a node* will be useful. Let $G = (V, E)$ be a graph and let $v \in V$. Add to G a new node, say v', which is adjacent to v and to all neighbours of v in G. In this way we obtain a new graph H, which we say is obtained from G by *duplicating* v. Repeated duplicating is called *replicating*.

Lemma 1.2.2. *Let H arise from G by duplicating a node. If G is perfect then H too is perfect.*

Proof. First we show that $\alpha(H) = \chi(\overline{H})$ if H arises from G by duplicating node v. Indeed, by construction, $\alpha(H) = \alpha(G)$, which is equal to $\chi(\overline{G})$ since G is perfect. Now, if $C_1, \ldots, C_t$ are cliques in G that cover V with (say) $v \in C_1$, then $C_1 \cup \{v'\}, \ldots, C_t$ are cliques in H covering $V(H)$. This shows that $\chi(\overline{G}) = \chi(\overline{H})$, which implies that $\alpha(H) = \chi(\overline{H})$.

From this we can conclude that, if H arises from G by duplicating a node v, then $\alpha(H') = \chi(\overline{H'})$ for any induced subgraph H' of H, using induction on the number of nodes of G. Indeed, either H' is an induced subgraph of G (if H' does not contain both v and v'), or H' is obtained by duplicating v in an induced subgraph of G; in both cases we have that $\alpha(H') = \chi(\overline{H'})$.

Hence, if H arises by duplicating a node in a perfect graph G, then $\overline{H}$ is perfect which, by Theorem 1.1.3, implies that H is perfect. $\square$

Given node weights $w \in \mathbb{R}^V_+$, we define the following weighted analogues of the (fractional) stability and chromatic numbers:

$$\alpha(G, w) = \max_{x \in \text{ST}(G)} w^{\mathsf{T}} x,$$

$$\alpha^*(G, w) = \max_{x \in \text{QST}(G)} w^{\mathsf{T}} x$$

$$= \min_y \left\{ \sum_{C \text{ clique of } G} y_C : \sum_{C \text{ clique of } G} y_C \chi^C = w,\ y_C \geq 0\ \forall C \text{ clique of } G \right\},$$

$$\chi(G, w) = \min_y \left\{ \sum_{C \text{ clique of } G} y_C : \sum_{C \text{ clique of } G} y_C \chi^C = w,\ y_C \in \mathbb{Z},\ y_C \geq 0\ \forall C \text{ clique of } G \right\}.$$

When w is the all-ones weight function, we find again $\alpha(G)$, $\alpha^*(G)$ and $\chi(G)$, respectively. The following analogue of (1.6) holds for arbitrary node weights:

$$\alpha(G, w) \leq \alpha^*(G, w) \leq \chi(\overline{G}, w). \tag{1.9}$$

Lemma 1.2.3. *Let G be a perfect graph and let $w \in \mathbb{Z}^V_{\geq 0}$ be nonnegative integer node weights. Then, $\alpha(\overline{G}, w) = \chi(G, w)$.*

Proof. Let H denote the graph obtained from G by duplicating node i w_i times if $w_i \geq 1$ and deleting node i if $w_i = 0$. Then, by construction, $\alpha(\overline{G}, w) = \omega(H)$, which is equal to $\chi(H)$ since H is perfect (by Lemma 1.2.2). Say, $\tilde{S}_1, \ldots, \tilde{S}_t$ are $t = \chi(H)$ stable sets in H partitioning $V(H)$. Each stable set $\tilde{S}_k$ corresponds to a stable set S_k in G (since $\tilde{S}_k$ contains at most one of the w_i copies of each node i of G). Now, these stable sets $S_1, \dots, S_t$ have the property that each node i of G belongs to exactly w_i of them, which shows that $\chi(G, w) \leq t = \chi(H)$. This implies that $\chi(G, w) \leq \chi(H) = \alpha(\overline{G}, w)$, giving equality $\chi(G, w) = \alpha(\overline{G}, w)$. □

We will also use the following geometric property of down-monotone polytopes. A polytope $P \subseteq \mathbb{R}^n_{\geq 0}$ is said to be *down-monotone* if $x \in P$ and $0 \leq y \leq x$ (coordinate-wise) implies $y \in P$.

Lemma 1.2.4. *Let $P, Q \subseteq \mathbb{R}^n$ be polytopes such that $P \subseteq Q$.*

(i) $P = Q$ if and only if the following equality holds for all weights $w \in \mathbb{R}^n$:

$$\max_{x \in P} w^{\mathsf{T}} x = \max_{x \in Q} w^{\mathsf{T}} x. \tag{1.10}$$

(ii) Assume that $P \subseteq Q \subseteq \mathbb{R}^n_{\geq 0}$ are down-monotone. Then $P = Q$ if and only if (1.10) holds for all nonnegative weights $w \in \mathbb{R}^n_{\geq 0}$.

Moreover, in (i) and (ii) it suffices to show that (1.10) holds for integer weights w.

Proof. (i) The 'only if' part is clear. The 'if part' follows using the 'hyperplane separation' theorem: Assume that $P \subset Q$ and that there exists $z \in Q \setminus P$. Then there exists a hyperplane separating z from P, i.e., there exists a nonzero vector $w \in \mathbb{R}^n$ and a scalar $w_0 \in \mathbb{R}$ such that $w^{\mathsf{T}} z > w_0$ and $w^{\mathsf{T}} x \leq w_0$ for all $x \in P$. These two facts contradict the condition (1.10).

(ii) The 'only if' part is clear. For the 'if part', it suffices to show that the equality (1.10) holds for all weights w if it holds for all nonnegative weights w'. This follows from the following claim (applied to both P and Q).

Claim: Let $P \subseteq \mathbb{R}^n_{\geq 0}$ be a down-monotone polytope, let $w \in \mathbb{R}^n$ and define the nonnegative vector $w' \in \mathbb{R}^n_{\geq 0}$ by $w'_i = \max\{w_i, 0\}$ for $i \in [n]$. Then, $\max_{x \in P} w^{\mathsf{T}} x = \max_{x \in P} (w')^{\mathsf{T}} x$.

Proof of the claim: Suppose $x \in P$ maximizes $w^{\mathsf{T}} x$ over P; we claim that $x_i = 0$ at all positions i for which $w_i < 0$. Indeed, if $x_i > 0$ and $w_i < 0$ then, by setting $y_i = 0$ and $y_j = x_j$ for $j \neq i$, one obtains another point $y \in P$ (since $0 \leq y \leq x$ and P is down-monotone) with $w^{\mathsf{T}} y > w^{\mathsf{T}} x$. Therefore, $w^{\mathsf{T}} x = (w')^{\mathsf{T}} x$ and thus x maximizes w' over P. □

The last part of the lemma follows using a continuity argument (if (1.10) holds for all integer weights w, it holds for all rational weights (by scaling) and thus for all real weights (taking limits)). □

We can now show the following geometric characterization of perfect graphs, due to Chvátal [2].

Theorem 1.2.5. *[2] A graph G is perfect if and only if* ST$(G) =$ QST(G).

Proof. First assume that G is perfect, we show that $\mathrm{ST}(G) = \mathrm{QST}(G)$. As $\mathrm{ST}(G)$ and $\mathrm{QST}(G)$ are down-monotone in $\mathbb{R}^V_{\geq 0}$, we can apply Lemma 1.2.4. Hence, it suffices to show that, for any $w \in \mathbb{Z}^V_{\geq 0}$, $\alpha(G,w) = \max_{x \in \mathrm{ST}(G)} w^{\mathsf{T}} x$ is equal to $\alpha^*(G,w) = \max_{x \in \mathrm{QST}(G)} w^{\mathsf{T}} x$, which follows from Lemma 1.2.3 (applied to $\overline{G}$).

Conversely, assume that $\mathrm{ST}(G) = \mathrm{QST}(G)$ and that G is not perfect. Pick a minimal subset $U \subseteq V$ for which the subgraph G' of G induced by U satisfies $\alpha(G') < \chi(\overline{G'})$. Setting $w = \chi^U$, we have that $\alpha(G') = \alpha(G,w)$ which, by assumption, is equal to $\max_{x \in \mathrm{QST}(G)} w^{\mathsf{T}} x = \alpha^*(G,w)$. Consider the dual of the linear program defining $\alpha^*(G,w)$ with an optimal solution $y = (y_C)$. Pick a clique C of G for which $y_C > 0$. Using complementary slackness, we deduce that $x(C) = 1$ for any optimal solution $x \in \mathrm{QST}(G)$ and thus $|C \cap S| = 1$ for any maximum cardinality stable set $S \subseteq U$. Let G'' denote the subgraph of G induced by $U \setminus C$. Then, $\alpha(G'') \leq \alpha(G') - 1 < \chi(\overline{G'}) - 1 \leq \chi(\overline{G''})$, which contradicts the minimality assumption made on U. □

When G is a perfect graph, an explicit linear inequality description is known for its stable set polytope, given by the clique inequalities. However, it is not clear how to use this information in order to give an efficient algorithm for optimizing over the stable set polytope of a perfect graph. As we see later in Section **??** there is yet another description of $\mathrm{ST}(G)$ – in terms of semidefinite programming, using the theta body $\mathrm{TH}(G)$ – that will allow to give an efficient algorithm.

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