LNMB - NETWORKS AND SEMIDEFINITE PROGRAMMING

Exercises for Lecture on Monday 6 February 2017

Exercise 1. Let $G = (V, E)$ be a graph and A_G be its adjacency matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Assume that G is k-regular (i.e., all vertices have degree k), so that $\lambda_1 = k$, with corresonding eigenvector the all-ones vector e.

(i) Show that $\vartheta(G) \leq \frac{-n\lambda_n}{\lambda_1 - \lambda_n}$.

(ii) Assume moreover that G is edge-transitive, i.e., for any two edges $\{i, j\}, \{i', j'\} \in E$ there exists an automorphism σ of G such that $\{i', j'\} = \{\sigma(i), \sigma(j)\}$.
Show that $\vartheta(G) = \frac{-n\lambda_n}{\lambda_1 - \lambda_n}$.

Hint: Use the formulation:

$$\vartheta(G) = \min\ t \ \text{ s.t. } tI - B \succeq 0,\ B_{ii} = 1\ (i \in V),\ B_{ij} = 1\ (\{i, j\} \in \overline{E}).$$

Remark: The cycle C_n is 2-regular and edge-transitive. For n odd and with $\omega_k := e^{2\mathbf{i}k\pi/n}$ for $1 \leq k \leq n$, the eigenvalues of A_{C_n} are $\omega_k + \omega_k^{-1} = 2\cos(2k\pi/n)$, with corresponding eigenvector $(1, \omega_k, \omega_k^2, \cdots, \omega_k^{n-1})$. From this you may deduce that $\lambda_1 = 2$ and $\lambda_n = -2\cos(\pi/n)$, so that you get

$$\vartheta(C_n) = \frac{n\cos(\pi/n)}{1 + \cos(\pi/n)}.$$

Exercise 2. Given a graph $G = (V, E)$, a symmetric matrix $B \in \mathcal{S}^n$ is said to *fit* G if it has non-zero diagonal entries and zero entries at positions of non-edges of G, i.e.,

$$B_{ii} \neq 0 \ \text{ for } i \in V,\ B_{ij} = 0 \ \text{ for all } \{i, j\} \in \overline{E}.$$

Consider the parameter $R(G)$ defined as the smallest rank of a matrix B which fits G.

(a) Show that $R(G) \leq \chi(\overline{G})$.

(b) Show that $R(G) \geq \alpha(G)$.

(c) Show that $R(G) \geq \Theta(G)$.

(This upper bound on the Shannon capacity is due to W. Haemers.)

Exercise 3. The goal of the exercise is to show a concise LP reformulation for the theta number of Hamming graphs (with only $n+1$ variables), also known as Delsarte LP bound.

Consider the Hammming graph $G(n,d)$ with vertex set $V = \{0,1\}^n$ and with an edge $\{u,v\}$ if $d_H(u,v) \leq d-1$. Here $d \geq 1$ is an integer and $d_H(u,v)$ is the Hamming distance between $u,v \in V$, defined as the number of positions i with $u_i \neq v_i$. We also set $|v| = d_H(v,0)$, the number of nonzero coordinates of $v \in V$.

Let $Sym(n)$ denote the set of permutations of $[n]$. Any $\sigma \in Sym(n)$ induces a permutation of V (again denoted σ for simplicity), obtained by permuting coordinates:

$$\sigma : V \to V \quad v \mapsto \sigma(v) = (v_{\sigma(1)}, \cdots, v_{\sigma(n)}).$$

Given $a \in V$, one can define the following permutation of V

$$s_a : V \to V \quad v \mapsto s_a(v) = a \oplus v,$$

where $a \oplus v$ is the element of V obtained by doing addition modulo 2 coordinate-wise. (For instance, $(1,0) \oplus (1,1) = (0,1)$ and $(1,1) \oplus (1,1) = (0,0)$.) Let $\mathcal{G}_n$ denote the subgroup of $Sym(V)$ generated by all such σ and s_a (for $a \in V$).

(a) Show that all permutations in $\mathcal{G}_n$ are automorphisms of $G(n,d)$ and that $G(n,d)$ is vertex-transitive under the action of $\mathcal{G}_n$.

(b) Show that if $u, v, u', v' \in V$ are such that $d_H(u,v) = d_H(u',v')$ then there exists $\pi \in \mathcal{G}_n$ such that $\pi(u) = u'$ and $\pi(v) = v'$.

(c) Show that if a matrix $X \in \mathcal{S}^V$ is invariant under action of $\mathcal{G}_n$ then it satisfies $X(u,v) = X(u',v')$ whenever $d_H(u,v) = d_H(u',v')$.

Let $\mathcal{B}_n$ denote the subspace of $\mathcal{S}^V$ consisting of all matrices that are invariant under action of $\mathcal{G}_n$. For $0 \leq k \leq n$ define the matrix $M_k \in \mathcal{S}^V$ with entries $M_k(u,v) = 1$ if $d_H(u,v) = k$ and zero otherwise. It follows from (c) that the matrices $M_0, M_1, \cdots, M_n$ form a linear basis of the vector space $\mathcal{B}_n$. We now indicate how to construct another basis of $\mathcal{B}_n$ (which will helpful to simultaneoulsy diagonalize all matrices in $\mathcal{B}_n$).

Given $a \in V$ define the vector in $\mathbb{R}^V$:

$$C_a = ((-1)^{a^T v})_{v \in V}$$

and define the matrices in $\mathcal{S}^V$:

$$B_k = \sum_{a \in V : |a| = k} C_a C_a^T \quad \text{for } 0 \leq k \leq n.$$

(d) What is B_0? What is the value of $\text{Tr}(B_k)$? Show that the matrices $B_0, B_1, \cdots, B_n$ are pairwise othogonal.

(e) For any $0 \leq k \leq n$ and $u, v \in V$, show that

$$B_k(u,v) = P_n^k(d_H(u,v)),$$

where $P_n^k(t)$ is defined by

$$P_n^k(t) := \sum_{i=0}^{k} (-1)^i \binom{t}{i} \binom{n-t}{k-i}$$

(known as the Krawtchouk polynomial).

(f) Show that $B_0, B_1, \cdots, B_n$ form a linear basis of $\mathcal{B}_n$.

(g) Let $X \in \mathcal{B}_n$, say $X = \sum_{k=0}^{n} x_k B_k$. Show that $X \succeq 0$ if and only if $x_0, \cdots, x_n \geq 0$.

(h) Show that the theta number of $G(n, d)$ can be reformulated as the following LP:

$$\vartheta(G(n,d)) = \max_{x_0, x_1, \cdots, x_n \in \mathbb{R}} 2^{2n} x_0 \text{ s.t. } \begin{array}{l} \sum_{k=0}^{n} x_k \binom{n}{k} = 2^{-n}, \\ \sum_{k=0}^{n} x_k P_n^k(t) = 0 \text{ for } t = 1, \cdots, d-1, \\ x_k \geq 0 \text{ for } k = 0, 1, \cdots, n. \end{array}$$