Exercise 1. Let $G = (V, E)$ be a graph and $A_G$ be its adjacency matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Assume that $G$ is $k$-regular (i.e., all vertices have degree $k$), so that $\lambda_1 = k$, with corresponding eigenvector the all-ones vector $e$.

(i) Show that $\vartheta(G) \leq -n\lambda_n - \lambda_n^{-1}$. 

(ii) Assume moreover that $G$ is edge-transitive, i.e., for any two edges $\{i,j\}, \{i',j'\} \in E$ there exists an automorphism $\sigma$ of $G$ such that $\{i',j'\} = \{\sigma(i), \sigma(j)\}$. Show that $\vartheta(G) = -n\lambda_n - \lambda_n^{-1}$.

Hint: Use the formulation: 
$$\vartheta(G) = \min_t \text{ s.t. } tI - B \succeq 0, \ B_{ii} = 1 \ (i \in V), \ B_{ij} = 1 \ (\{i,j\} \in E).$$

Remark: The cycle $C_n$ is 2-regular and edge-transitive. For $n$ odd and with $\omega_k := e^{2k\pi/n}$ for $1 \leq k \leq n$, the eigenvalues of $A_{C_n}$ are $\omega_k + \omega_k^{-1} = 2\cos(2k\pi/n)$, with corresponding eigenvector $(1, \omega_k, \omega_k^2, \cdots, \omega_k^{n-1})$. From this you may deduce that $\lambda_1 = 2$ and $\lambda_n = -2\cos(\pi/n)$, so that you get 
$$\vartheta(C_n) = \frac{n\cos(\pi/n)}{1 + \cos(\pi/n)}.$$

Exercise 2. Given a graph $G = (V, E)$, a symmetric matrix $B \in S^n$ is said to fit $G$ if it has non-zero diagonal entries and zero entries at positions of non-edges of $G$, i.e., 
$$B_{ii} \neq 0 \text{ for } i \in V, \ B_{ij} = 0 \text{ for all } \{i,j\} \in \overline{E}.$$ 
Consider the parameter $R(G)$ defined as the smallest rank of a matrix $B$ which fits $G$.

(a) Show that $R(G) \leq \chi(G)$. 

(b) Show that $R(G) \geq \alpha(G)$. 

(c) Show that $R(G) \geq \Theta(G)$. 

(This upper bound on the Shannon capacity is due to W. Haemers.)
Exercise 3. The goal of the exercise is to show a concise LP reformulation for the theta number of Hamming graphs (with only $n + 1$ variables), also known as Delsarte LP bound.

Consider the Hamming graph $G(n, d)$ with vertex set $V = \{0, 1\}^n$ and with an edge $\{u, v\}$ if $d_H(u, v) \leq d - 1$. Here $d \geq 1$ is an integer and $d_H(u, v)$ is the Hamming distance between $u, v \in V$, defined as the number of positions $i$ with $u_i \neq v_i$. We also set $|v| = d_H(v, 0)$, the number of nonzero coordinates of $v \in V$.

Let $\text{Sym}(n)$ denote the set of permutations of $[n]$. Any $\sigma \in \text{Sym}(n)$ induces a permutation of $V$ (again denoted $\sigma$ for simplicity), obtained by permuting coordinates:

$$\sigma : V \rightarrow V \quad v \mapsto \sigma(v) = (v_{\sigma(1)}, \ldots, v_{\sigma(n)})$$

Given $a \in V$, one can define the following permutation of $V$:

$$s_a : V \rightarrow V \quad v \mapsto s_a(v) = a \oplus v,$$

where $a \oplus v$ is the element of $V$ obtained by doing addition modulo 2 coordinate-wise. (For instance, $(1, 0) \oplus (1, 1) = (0, 1)$ and $(1, 1) \oplus (1, 1) = (0, 0).$)

Let $G_n$ denote the subgroup of $\text{Sym}(V)$ generated by all such $\sigma$ and $s_a$ (for $a \in V$).

(a) Show that all permutations in $G_n$ are automorphisms of $G(n, d)$ and that $G(n, d)$ is vertex-transitive under the action of $G_n$.

(b) Show that if $u, v, u', v' \in V$ are such that $d_H(u, v) = d_H(u', v')$ then there exists $\pi \in G_n$ such that $\pi(u) = u'$ and $\pi(v) = v'$.

(c) Show that if a matrix $X \in S^V$ is invariant under action of $G_n$ then it satisfies $X(u, v) = X(u', v')$ whenever $d_H(u, v) = d_H(u', v')$.

Let $B_n$ denote the subspace of $S^V$ consisting of all matrices that are invariant under action of $G_n$. For $0 \leq k \leq n$ define the matrix $M_k \in S^V$ with entries $M_k(u, v) = 1$ if $d_H(u, v) = k$ and zero otherwise. It follows from (c) that the matrices $M_0, M_1, \ldots, M_n$ form a linear basis of the vector space $B_n$. We now indicate how to construct another basis of $B_n$ (which will helpful to simultaneously diagonalize all matrices in $B_n$).

Given $a \in V$ define the vector in $\mathbb{R}^V$:

$$C_a = ((-1)^a)^Tv \in V$$

and define the matrices in $S^V$:

$$B_k = \sum_{a \in V: |a| = k} C_aC_a^T \quad \text{for } 0 \leq k \leq n.$$

(d) What is $B_0$? What is the value of $\text{Tr}(B_k)$? Show that the matrices $B_0, B_1, \ldots, B_n$ are pairwise orthogonal.

(e) For any $0 \leq k \leq n$ and $u, v \in V$, show that

$$B_k(u, v) = P^k_n(d_H(u, v)),$$
where \( P^k_n(t) \) is defined by

\[
P^k_n(t) := \sum_{i=0}^{k} (-1)^i \binom{t}{i} \binom{n-t}{k-i}
\]

(known as the Krawtchouk polynomial).

(f) Show that \( B_0, B_1, \ldots, B_n \) form a linear basis of \( B_n \).

(g) Let \( X \in B_n \), say \( X = \sum_{k=0}^{n} x_k B_k \). Show that \( X \succeq 0 \) if and only if \( x_0, \ldots, x_n \geq 0 \).

(h) Show that the theta number of \( G(n,d) \) can be reformulated as the following LP:

\[
\vartheta(G(n,d)) = \max_{x_0, x_1, \ldots, x_n \in \mathbb{R}^2} 2^{2n} x_0 \text{ s.t. } \begin{align*}
\sum_{k=0}^{n} x_k \binom{n}{k} &= 2^{-n}, \\
\sum_{k=0}^{n} x_k P^k_n(t) &= 0 \text{ for } t = 1, \ldots, d - 1, \\
x_k &\geq 0 \text{ for } k = 0, 1, \ldots, n.
\end{align*}
\]