**LNMB - NETWORKS AND SEMIDEFINITE PROGRAMMING**

# Exercises for Lecture 3 - December 5, 2017

**Exercise 1.** Let $G = (V, E)$ and let $\overline{G} = (V, \overline{E})$ denote its complementary graph.
Show: $G$ is an interval graph if and only if $G$ is chordal and $\overline{G}$ is a comparability graph.

**(1)** First show the 'only if' part.

**(2)** Then show the 'if part' through (2a)-(2d) below.

*Sketch of proof:* Assume that $G$ is chordal and $\overline{G}$ is a comparability graph. Hence there exists a partial order on $V$, denoted as $\leq$, such that for any distinct nodes $u, v$, $\{u, v\} \in \overline{E}$ if and only if $u \leq v$ or $v \leq u$. We are going to use this partial order on $V$ in order to define a total order on the set of maximal (with respect to inclusion) cliques of $G$, which is then used to construct an interval representation of $G$.

(2a) Let $C_1$ and $C_2$ be two maximal cliques of $G$.
Show: There exist nodes $u \in C_1$, $v \in C_2$ such that $\{u, v\} \in \overline{E}$.
Without loss of generality we may assume that $u \leq v$.
Show: If $u' \in C_1$, $v' \in C_2$ and $\{u', v'\} \in \overline{E}$ then $u' \leq v'$.
Then let us say that $C_1 \leq C_2$.

(2b) Let $C_1, C_2, C_3$ be three distinct maximal cliques of $G$.
Show: If $C_1 \leq C_2$ and $C_2 \leq C_3$ then $C_1 \leq C_3$.

(2c) Let $C_1, \ldots, C_q$ denote the maximal cliques of $G$ which can be labeled in such a way that $C_1 \leq C_2 \leq \ldots \leq C_q$ (in view of (2b)). For any node $u \in V$ define the set

$$I_u = \{i \in \{1, \ldots, q\} : u \in C_i\}.$$

Show: $I_u$ is an interval, i.e.,

$$i < k < j \text{ with } i, j \in I_u \Longrightarrow k \in I_u.$$

(2d) Show that the intervals $I_u$ $(u \in V)$ provide an interval representation of $G$, i.e., $\{u, v\} \in E$ if and only if $I_u \cap I_v \neq \emptyset$.

**Exercise 2.** Let $G = (V, E)$. A linear ordering $\pi = (v_1, \cdots, v_n)$ of the vertices of $V$ is said to be a *perfect elimination ordering* if, for any $1 \leq i < n$, the set of vertices

$$\tilde{N}(v_i) := \{v_j : i < j \text{ and } \{v_i, v_j\} \in E\}$$

(consisting of the vertices adjacent to $v_i$ that come after $v_i$ in the linear ordering $\pi$) is a clique in $G$. As you will show below this ordering is very useful to design efficient algorithms for computing $\omega(G)$, $\alpha(G)$ and $\chi(G)$ in chordal graphs.

We now assume that $G$ is a chordal graph.

(a) Show that $G$ has a perfect elimination ordering, give a polynomial time algorithm to find such an ordering $\pi$.

(b) Show that $G$ has at most $n$ maximal cliques and give a polynomial time algorithm permitting to enumerate the maximal cliques of $G$ and to compute $\omega(G)$.

(c) Give a polynomial time algorithm permitting to find a stable set in $G$ of maximum cardinality (and thus to compute $\alpha(G)$).

(d) Give a polynomial time algorithm permitting to find a vertex coloring of $G$ using $\omega(G)$ colors (which thus shows that $\chi(G) = \omega(G)$).

*Hint:* For (c) use a greedy approach starting from the first node in $\pi$ in order to construct a maximum cardinality stable set, and for (d) use a greedy approach starting from the last node in $\pi$ in order to construct a vertex coloring of $G$ using $\omega(G)$ colors.