LNMB - NETWORKS AND SEMIDEFINITE PROGRAMMING

Exercises for Lecture 5 - December 19, 2016

Exercise A. Let C be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Consider the programs:

$$p_1^* = \max_{X \in \mathcal{S}^n} \langle C, X \rangle \text{ such that } \operatorname{Tr}(X) = 1,\ X \succeq 0,\ I_n - X \succeq 0, \tag{1}$$

$$p_2^* = \max_{Y \in \mathbb{R}^{n \times k}} \langle C, YY^T \rangle \text{ such that } Y^T Y = I_k. \tag{2}$$

The goal of the exercise is to show Fan's theorem: The sum of the k largest eigenvalues of C is given by the semidefinite program (1):

$$\lambda_1 + \cdots + \lambda_k = p_1^*.$$

For this consider the following sets:

$$K_1 = \{X \in \mathcal{S}^n : \operatorname{Tr}(X) = k,\ X \succeq 0,\ I_n - X \succeq 0\},$$

$$K_2 = \{YY^T : Y \in \mathbb{R}^{n \times k},\ Y^T Y = I_k\},$$

$$P = \left\{ x \in \mathbb{R}^n : 0 \leq x_i \leq 1 \ (i \in [n]),\ \sum_{i=1}^{n} x_i = k \right\}.$$

(a) Let U be an orthogonal $n \times n$ matrix and $X \in \mathcal{S}^n$.
Show: $X \in K_1$ if and only if $UXU^T \in K_1$.
Show: X is an extreme point of K_1 if and only if UXU^T is an extreme point of K_1.

(b) Show: P is equal to the convex hull of the set $P \cap \{0,1\}^n$.

(c) Show: K_1 is compact and convex.

(d) Show: K_1 is equal to the convex hull of the set K_2.

Hint: You may use the fact that any compact convex set in some $\mathbb{R}^d$ is equal to the convex hull of its set of extreme points.

(e) Show: $\lambda_1 + \cdots + \lambda_k = p_1^* = p_2^*$.

(f) Show that the dual semidefinite program of (1) can be reformulated as the following semidefinite program:

$$d^* = \min_{z_0, z_1, \cdots, z_n \in \mathbb{R}} kz_0 + \sum_{i=1}^{n} z_i \text{ such that } z_0 I_n + \mathrm{Diag}(z) - C \succeq 0,\ z_1, \cdots, z_n \geq 0. \tag{3}$$

Is there a duality gap between the programs (1) and (3)?