# LNMB - NETWORKS AND SEMIDEFINITE PROGRAMMING
# LECTURE 1 - EXERCISES

**Exercise 1.** The goal is to give a polynomial-time reduction from the vertex-coloring problem to the stable set problem.

Let $G = (V, E)$ be a graph and let $k$ be an integer. We build a new graph $H$ in the following way.
Say, $V = \{v_1, \ldots, v_n\}$. Let $V_1, \ldots, V_k$ be $k$ disjoint copies of $V$, and set $V_h = \{v_{h,1}, \ldots, v_{h,n}\}$ for $h = 1, \ldots, k$. Then the vertex set of $H$ is $V_1 \cup \ldots \cup V_k$ and its edges are defined as follows: Two vertices $v_{h,i}$ and $v_{h,j}$ of $V_h$ are adjacent in $H$ if and only $v_i$ and $v_j$ are adjacent in $G$ (that is, we put a copy of $G$ on each $V_h$); moreover, each set $\{v_{1,i}, v_{2,i}, \ldots, v_{k,i}\}$ (consisting of the $k$ copies of $v_i$) is a clique for each $i = 1, \ldots, n$.

Show: $\chi(G) \leq k$ if and only if $\alpha(H) \geq n$.

**Exercise 2.** Let $G = (V, E)$ be a connected planar graph with $n \geq 3$ nodes. The goal is to show that $\chi(G) \leq 5$.

We let $m$ denote the number of edges of $G$ and $f$ denote the number of faces in a plane drawing of $G$.

**1)** Show: $2m \geq 3f$.

**2)** Show: $m \leq 3n - 6$.

*Hint:* Use Euler formula: $n + f = m + 2$.
(For instance, for $K_4$, $n = 4$, $m = 6$, and $f = 4$).

**3)** Show: $G$ has at least one node $u$ with degree at most 5.

**3)** Show: $\chi(G) \leq 5$ (using induction on $n$).

*Hint: Distinguish two cases: (i) $u$ has degree at most 4; (ii) $u$ has degree 5. In case (ii), pick two neighbors $v_1$ and $v_2$ of $u$ that are not adjacent (**why do they exist?**) and consider the graph $H$ obtained by contracting both edges $\{u, v_1\}$ and $\{u, v_2\}$, which is 5-colorable using induction.*

**Exercise 3.** (Exercise 7.5 in the Lecture Notes). Derive König's edge cover theorem (Corollary 3.3a) from the Strong Perfect Graph Theorem.

**The Strong Perfect Graph Theorem** claims that if a graph $H$ does not contain an odd cycle or its complement as an induced subgraph then $\chi(H) = \omega(H)$.

**König's edge cover theorem** claims that, if $G$ is a bipartite graph with no isolated vertex, then $\alpha(G) = \rho(G)$, where $\rho(G)$ is the minimum number of edges needed to cover all vertices of $G$.

**Exercise 4.** Let $G = (V, E)$ be a connected graph such that no minor of $G$ is isomorphic to $K_4$. The goal is to show that $\chi(G) \leq 3$.

We recall that a graph $H$ is said to be a *minor* of $G$ if $H$ can be obtained from $G$ by applying a sequence of the following operations: (i) deleting a vertex (and its adjacent edges), (ii) deleting an edge, (iii) contracting an edge. (For instance, if $G$ contains a cycle of length at least 3 then $K_3$ is a minor of $G$.)

Call a set $S \subseteq V$ a *cutset* of $G$ if $G$ is connected but removing $S$ from $G$ disconnects the graph $G$.

**1)** Show: if $G$ is connected with no $K_4$-minor and $G$ has at least 4 nodes, then $G$ has a cutset $S$ of cardinality at most 2.

*Hint:* Observe first that the result holds if $G$ is a tree. So you may now assume that there exists a circuit $C$ contained in $G$. Deal first with the case (i) when $C$ contains all nodes of $G$ and (ii) when $C$ does not contain all nodes of $G$. In case (ii) use the fact that if a graph has no cutset of size at most 2 (i.e., $G$ is 3-connected) then, by Menger's theorem, there are three internally vertex-disjoint paths between any two distinct vertices.

**2)** Assume that $S$ is a cutset of $G$ of cardinality 2, say, $S = \{a, b\}$, and assume that $\{a, b\}$ is not an edge of $G$. Let $W_1, \cdots, W_k$ be the connected components of $G \setminus S$. For each $i \in [k]$, consider the graph $H_i$ obtained by adding the edge $\{a, b\}$ to $G[W_i \cup S]$ (the subgraph of $G$ induced by $S \cup W_i$). Show that $H_i$ does not contain $K_4$ as a minor.

**3)** Show: if $G$ is connected with no $K_4$-minor, then $\chi(G) \leq 3$.