LNMB - NETWORKS AND SEMIDEFINITE PROGRAMMING

Hints for Exercise 7.29 (Lecture 3 - December 10, 2018)

Exercise 7.29. Let $G = (V, E)$ and let $\overline{G} = (V, \overline{E})$ denote its complementary graph. Show: G is an interval graph if and only if G is chordal and $\overline{G}$ is a comparability graph.

(1) First show the 'only if' part.

(2) Then show the 'if part' through (2a)-(2d) below.

Sketch of proof: Assume that G is chordal and $\overline{G}$ is a comparability graph. Hence there exists a partial order on V, denoted as $\leq$, such that for any distinct nodes u, v, $\{u, v\} \in \overline{E}$ if and only if $u \leq v$ or $v \leq u$. We are going to use this partial order on V in order to define a total order on the set of maximal (with respect to inclusion) cliques of G, which is then used to construct an interval representation of G.

(2a) Let C_1 and C_2 be two maximal cliques of G.
Show: There exist nodes $u \in C_1$, $v \in C_2$ such that $\{u, v\} \in \overline{E}$.
Without loss of generality we may assume that $u \leq v$.
Show: If $u' \in C_1$, $v' \in C_2$ and $\{u', v'\} \in \overline{E}$ then $u' \leq v'$.
Then let us say that $C_1 \leq C_2$.

(2b) Let C_1, C_2, C_3 be three distinct maximal cliques of G.
Show: If $C_1 \leq C_2$ and $C_2 \leq C_3$ then $C_1 \leq C_3$.

(2c) Let $C_1, \ldots, C_q$ denote the maximal cliques of G which can be labeled in such a way that $C_1 \leq C_2 \leq \ldots \leq C_q$ (in view of (2b)). For any node $u \in V$ define the set

$$I_u = \{i \in \{1, \ldots, q\} : u \in C_i\}.$$

Show: I_u is an interval, i.e.,

$$i < k < j \ \text{ with } i, j \in I_u \Longrightarrow k \in I_u.$$

(2d) Show that the intervals I_u $(u \in V)$ provide an interval representation of G, i.e., $\{u, v\} \in E$ if and only if $I_u \cap I_v \neq \emptyset$.